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STRONGER BOUNDS FOR GENERALIZED DEGREES AND MENGER PATH SYSTEMS

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Abstract

For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G, there are m internally vertex disjoint paths of length at most d. For a positive integer t a graph G satisfies the minimum generalized degree condition $\delta_t(G) \geq s$ if the cardinality of the union of the neighborhoods of each set of t vertices of G is at least s. Generalized degree conditions that ensure that $P_{d,m}(G)$ is satisfied have been investigated. In particular, it has been shown, for fixed positive integers $t \geq 5$, $d \geq 5t^2$, and m, that if an m-connected graph G of order n satisfies the generalized degree condition $\delta_t(G) > (t/(t+1))(5n/(d+2)) + (m-1)d + 3t^2$, then for n sufficiently large G has property $P_{d,m}(G)$. In this note, this result will be improved by obtaining corresponding results on property $P_{d,m}(G)$ using a generalized degree condition $\delta_t(G)$, except that the restriction $d \geq 5t^2$ will be replaced by the weaker restriction $d \geq \max\{5t+28, t+77\}$. Also, it will be shown, just as in the original result, that if the order of magnitude of $\delta_t(G)$ is decreased, then $P_{d,m}(G)$ will not, in general, hold; so the result is sharp in terms of the order of magnitude of $\delta_t(G)$.

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1. Introduction

For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G there are at least m internally disjoint paths, each of length at most d. In [O] property $P_{d,m}(G)$ and its application to computer networks and distributed processing was introduced. Extremal results for $P_{d,m}(G)$ were investigated in [FJOST]. These results were extended in [FGS] where various combinations of connectivity, minimum degree, sum of degree properties, and neighborhood conditions implying $P_{d,m}(G)$ were studied.

The *neighborhood* of a vertex v of a graph G is the collection of vertices adjacent to v, and will be denoted by $N_G(v)$. For any set S of vertices of G the neighborhood of v in S, which is $N_G(v) \cap S$, will be denoted $N_S(v)$. For a fixed positive integer t and a graph G, we write $\delta_t(G) \geq s$ if the cardinality of the union of the neighborhoods of each set of t vertices of G is at least s. In [FGL] conditions that imply $P_{d,m}(G)$ based on generalized minimum degrees were investigated. In particular, the following was proved.

Theorem A. *Let $t \geq 5$, $d \geq 5t^2$, and $m \geq 2$ be fixed integers. If G is an m-connected graph of order n with $\delta_t(G) \geq (t/(t+1))(5n/(d+2)) + (m-1)d + 3t^2$, then for n sufficiently large $P_{d,m}(G)$ is satisfied.*

We will improve on Theorem A by proving the following result.

Theorem 1. *Let $t \geq 5$ and $m \geq 2$ be fixed integers with s an integer satisfying $0 \leq s \leq t-2$. If $d \geq \max\{5t+28, t+77\}$ and G is an m-connected graph of order n with*

$$\delta_t(G) \geq \max_s \left(\frac{t-s}{t+1-s}\right)\left(\frac{5n}{d+2-s}\right) + (m-1)d + 3t^2,$$

then for n sufficiently large, $P_{d,m}(G)$ is satisfied.

It can be shown that for $d \geq t^2 + t - 2$, the generalized degree condition in Theorem 1 is just $\delta_t(G) \geq (t/(t+1))(5n/(d+2)) + (m-1)d + 3t^2$, (i.e. when $s = 0$), so Theorem 1 does imply Theorem A and more.

2. Results

Notation and standard definitions in the paper will generally follow that found in [CL]. Any special notation will be described as needed. For vertices

x and y of a graph G let $P_{d,m}(x,y)$ denote the property that there are m internally vertex-disjoint $x-y$ paths in G, each of length at most d. A collection of such paths is called a *Menger Path System* for x and y. The following technical lemma will be needed in the proof of Theorem 1. The proof can be found in [FGL], but since it is short, for completeness it is included here.

Lemma 2. *Let d, m, t and $k > 1$ be fixed positive integers. Futhermore, let G be an m-connected graph of order n with $\delta_t(G) \geq n/k$. If there exists a pair $\{x, y\}$ of vertices of G such that G does not satisfy $P_{d,m}(x,y)$, but $G+uv$ does satisfy $P_{d,m}(x,y)$ for each pair u, v of nonadjacent vertices, then for n sufficiently large (say, $n \geq (kt)^3 md/3$) G satisfies $P_{max\{d+1,2k+t\},m}(x,y)$.*

Proof. Suppose that G contains a pair u, v of nonadjacent vertices such that $|N_G(u) \cap N_G(v)| > md$. By assumption, $G+uv$ contains m internally disjoint $x-y$ paths $P_1, P_2, \cdots, P_m$, each of length at most d, with $uv \in E(P_1)$. Since $|N_G(u) \cap N_G(v)| > md$, there is a vertex z of G such that $uz, zv \in E(G)$ and $z \notin V(P_i)$, $i = 1, 2, \cdots, m$. Clearly then, G contains the desired system of $x-y$ paths. Thus, we may assume that for every pair u, v of nonadjacent vertices of G, $|N_G(u) \cap N_G(v)| \leq md$.

Since $\delta_t(G) \geq n/k$, there are at most $t-1$ vertices of G of degree less than n/kt. Let $A = \{v \in V(G) \mid deg_G\ v \geq n/kt\}$. Construct a sequence $v_1, v_2, \cdots, v_\ell$ of vertices of G as follows. Let v_1 be a vertex of minimum degree in A with $A_1 = N_A(v_1) \cup \{v_1\}$. Let v_2 be a vertex of minimum degree in $A - A_1$ with $A_2 = N_A(v_2) \cup \{v_2\}$. In general, let v_i be a vertex of minimum degree in $A - \cup_{j=1}^{i-1} A_j$ with $A_i = N_A(v_i) \cup \{v_i\}$. Then for some ℓ, $A = \cup_{j=1}^{\ell} A_j$. For $i \neq j$, the vertices v_i and v_j are nonadjacent and, consequently, $|A_i \cap A_j| \leq md$. Futhermore, since $deg\ v \geq n/kt$ for each $v \in A$ and n is sufficiently large, we have that $\ell \leq kt + 1$.

For $i = 1, 2, \cdots, \ell$, let $B_i = A_i - \cup_{j \neq i} A_j$. Let $u, v \in B_i$ for some $1 \leq i \leq \ell$ and suppose $uv \notin E(G)$. Since $uv \notin E(G)$, we have that $|N_G(u) \cap N_G(v)| \leq md$. Thus, one of u and v, say u, has at most $(|B_i| + md)/2$ adjacencies in B_i. Futhermore, u has at most ℓmd adjacencies in $A - B_i$. Thus, for n sufficiently large,

$$deg_A\ u \leq (|B_i| + md)/2 + \ell md < deg_{(A - \cup_{j=1}^{i-1} A_j)}\ v_i,$$

contradicting the choice of v_i. Thus, the subgraph $< B_i >_G$ induced by B_i in G is complete for $i = 1, 2, \cdots, \ell$.

Next, consider an arbitrary $w \in A - \cup_{i=1}^{\ell} B_i$. Since $deg_G\, w \geq n/tk$, for some $1 \leq i \leq \ell$ we have $|N_G(w) \cap B_i| > md$. Consequently, w is adjacent to each vertex of B_i. Thus G contains disjoint sets $C_1, C_2, \cdots, C_\ell$ of vertices, each with approximately n/kt or more vertices, such that $|\cup_{i=1}^{\ell} C_i| \geq n - t + 1$ and $< C_i >_G$ is complete for $i = 1, 2, \cdots, \ell$.

If for some i, $|C_i| < n/k - (md\ell)t$, then, since $\delta_t \geq n/k$, each vertex of C_i, with at most $t-1$ exceptions, will be adjacent to at least md vertices of some C_j. However, if a vertex is adjacent to md vertices of C_j, it is adjacent to all vertices of C_j. Thus, for some $j \neq i$, each of the vertices of C_j is adjacent to each of the vertices of C_i. Hence, we can assume that $|C_i| \geq n/k - (md\ell)t$ for $i = 1, 2, \cdots, \ell$, and $\ell \leq k$.

Let $P_1, P_2, \cdots, P_m$ be a collection of m internally disjoint $x - y$ paths, the sum of whose lengths is minimum. Such P_i exist since G is m-connected. Then each path P_i contains at most 2 vertices of C_i for $i = 1, 2, \cdots, \ell$ and $\ell \leq k$. Thus each P_i has length at most $2k + t$. This completes the proof of Lemma 2. ■

Before giving a proof of Theorem 1, we describe some examples which indicate the nature of extremal examples for property $P_{d,m}$ and show that the result of Theorem 1 is asymptotically sharp. "Generalized Wheel type" graphs give important information on the extremal properties related to $P_{d,m}(G)$. We start with the wheel graph $W_b = K_1 + C_b$ that has b spokes and b vertices on the rim. Replace each vertex of W_b with a complete graph, and make each vertex of the corresponding complete graph adjacent to the vertices in the neighborhood of the replaced vertex. The graphs obtained by this expansion of vertices of a wheel form a family of "generalized wheels". More precisely, order the vertices of W_b starting with the center and followed by the vertices on the rim in a natural order around the cycle. For positive integers $p(i)$ $(0 \leq i \leq b)$, the generalized wheel obtained from W_b by replacing the ith vertex with a complete graph $K_{p(i)}$ will be denoted by $W(p(0), p(1), \cdots, p(b))$.

In the cases of interest to us, most of the $p(i)$'s in the generalized wheel will follow some pattern, so we will adopt the more compact notation of representing the sequence $(p(j), \cdots, p(k))$ by $(k - j + 1; p)$ when $p = p(j) = \cdots = p(k)$. Thus, $W(1, r; 1) = W_r$. Also, if the pattern $(p(1), p(2), \cdots, p(r))$ is repeated s times, we will represent this by $(s; (p(1), p(2), \cdots, p(r)))$. Hence the generalized wheel $W(m, s; (1, p, p, 1))$ has m vertices in the center and along the rim there is an alternating pattern of two single vertices followed by two complete graphs with p vertices.

With this notation we can now describe the examples that illustrate that the result in Theorem 1 has the correct order of magnitude. Let m, d, s and t be positive integers with $0 \leq s \leq t-2$. Let n be a positive integer such that

$$r = \frac{5n - 5m + 6 - 3s - 2d}{(d+2-s)(t-s+1)}$$

is an integer. For $d+2-s$ divisible by 5, consider the graph

$$H_s = W(m-2, s; 1, \frac{d+2-s}{5}; (1, r, (t-s-1)r, r, 1)).$$

This generalized wheel H_s has n vertices, is m-connected, and does not satisfy $P_{d,m}$. Also,

$$\delta_t(H_s) = \left(\frac{t-s}{t-s+1}\right)\left(\frac{5n-5m+6-3s-2d}{d+2-s}\right) + m - 1.$$

By the previous examples, if $\delta_t(G) \geq f$ is a generalized degree condition implying that $P_{d,m}(G)$ is satisfied, then $f \geq \delta_t(H_s)$ for each s $(0 \leq s \leq t-2)$. However, if m and t are fixed and $d \geq t$ is considered as a variable, then each function $\delta_t(H_s)$ (of d) will dominate the remaining $\delta_t(H_i)$ over some subinterval of $[t, \infty)$. Note for n sufficiently large, that

$$\delta_t(H_s) \leq \delta_t(H_{s+1})$$

is equivalent to

$$\frac{t-s}{(t-s+1)(d+2-s)} \leq \frac{t-s-1}{(t-s)(d+1-s)},$$

and this is equivalent to

$$d \leq (t-s)^2 + s - 2.$$

Therefore, if we let

$$I_0 = [t^2 - 2, \infty), \text{ and}$$

$$I_s = [(t-s)^2 + s - 2, (t-s+1)^2 + s - 3] \text{ for } 0 < s \leq t-2,$$

then for $d \in I_s$, we have $\delta_t(H_s) \geq \delta_t(H_i)$ for all i $(0 \leq i \leq t-2)$.

The previous discussion motivates the following statement of Theorem 1.

Theorem 1′. *Let $t \geq 5$, $d \geq \max\{5t+28, t+77\}$, and $m \geq 2$ be fixed integers. If G is an m-connected graph of order n with*

$$\delta_t(G) \geq \left(\frac{t-s}{t-s+1}\right)\left(\frac{5n}{d+2}\right) + (m-1)d + 3t^2 \text{ for } d \in I_s,$$

then for n sufficiently large, $P_{d,m}(G)$ is satisfied.

Proof. Assume, to the contrary, for some d with $(t-s)^2+s-2 \leq d \leq (t+1-s)^2+s-3$, that G is an m-connected graph with $\delta_t(G) \geq (t-s)/(t+1-s))(5n/(d+2-s)) + (m-1)d + 3t^2$ that does not satisfy $P_{d,m}(G)$ but $G+uv$ does satisfy $P_{d,m}(G)$ for each pair u, v of nonadjacent vertices of G. Since G does not satisfy $P_{d,m}(G)$, there are vertices x and y of G for which G does not satisfy $P_{d,m}(x,y)$. By Lemma 2, G contains a collection of m internally disjoint $x-y$ paths, each of length at most $\max\{d+1, 2\lfloor((t+1-s)/(t-s))(d+2)/5)\rfloor + t\} = d+1$. Among all such collections let $P_1, P_2, \cdots, P_m$ be one, the sum of whose lengths is minimum.

Assume now that P_1 has length $d+1$, say $P_1 : x = x_1, x_2, \cdots, x_{d+2} = y$. Let $N = V(G) - \cup_{j=2}^{m} V(P_j)$. Observe that if $v \in N$, then v can be adjacent to at most 3 vertices of P_1, and for any i, vx_i and vx_j for $j \geq i+3$ are not simultaneously in $E(G)$. Let $d_N(u,v)$ denote the distance between u and v in the graph induced by the vertices in N. For $i = 1, 2, \cdots, d+2$ define N_i as follows:

$$N_1 = \{x_1\},$$

$$N_i = \{v \in N \mid d_N(x,v) = i-1\} \text{ for } 2 \leq i < d+2, \text{ and}$$

$$N_{d+2} = N - \cup_{i=1}^{d+1} N_i.$$

For each i, $x_i \in N_i$, the N_i's form a partition of N. Note that if $S \subseteq N_i$, then $N_G(S) \subseteq \cup_{j=2}^{m} V(P_j) \cup N_{i-1} \cup N_i \cup N_{i+1}$. Let $\overline{N}(S)$ denote $N_G(S) - \cup_{j=2}^{m} V(P_j) \subseteq N_{i-1} \cup N_i \cup N_{i+1}$. Then if S has at least t vertices, $|\overline{N}(S)| \geq (\frac{t-s}{t+1-s})(\frac{5n}{d+2-s}) + 3t^2$. Define a *strong block* of G to be a sequence $N_i, N_{i+1}, \cdots, N_j$ that satisfies the following conditions:

(1) no two consecutive terms N_ℓ and $N_{\ell+1}$ have $|N_\ell| < t$ and $|N_{\ell+1}| < t$;
(2) if $i \neq 1$, then $|N_{i-1}| < t$ and $|N_i| < t$; and
(3) if $j \neq d+2$, then $|N_j| < t$ and $|N_{j+1}| < t$.

The strong blocks partition the vertices of N and also form a partition of the N_j's. The *length* of the strong block $N_i, N_{i+1}, \cdots, N_j$ is defined to be $j-i+1$; the *left endpoint* is x_i and the *right endpoint* is x_j. Each strong block of length $m \geq 6$ will be further partitioned into smaller pieces, with

each piece containing either 2, 3, 4, or 5 consecutive N_j's. In fact if $m \not\equiv 1 \mod 4$, then each of these pieces will have either 3 or 4 consecutive N_j's, and there will be at most 2 pieces with 3 terms that will contain endpoints of the strong block. If $m \equiv 1 \mod 4$, then each of these pieces will have length 3, 4, or 5; however, the number of pieces of length 5 will be at most 1 and will contain an endpoint of the strong block. The *blocks* of G will be the strong blocks of G of length at most 5 or the smaller pieces of length 2, 3, 4, or 5 that partition the strong blocks of length at least 6. Thus, the blocks of G also partition the vertices of N and form a partition of the N_j's.

Each block contains either $1, 2, 3, 4$ or 5 consecutive N_j's, and all blocks of length $1, 2$, or 5 are either strong blocks or contain an endpoint of a strong block. Note also, that any (strong) block of length 2 must be N_{d+1}, N_{d+2} and that $|N_{d+2}| \geq t$. If N_i, N_{i+1}, N_{i+2} is a block of length 3, then $|N_{i+1}| \geq t$, if $N_i, N_{i+1}, N_{i+2}, N_{i+3}$ is a block of length 4, then $|N_{i+1}| \geq t$ or $|N_{i+2}| \geq t$, and if $N_i, N_{i+1}, N_{i+2}, N_{i+3}, N_{i+4}$ is a block of length 5, then both $|N_{i+1}| \geq t$ and $|N_{i+3}| \geq t$. These latter properties are a result of properly chosing the partitions of the long blocks.

Observe first that G contains at most $t-1$ blocks of length 1; otherwise, suppose $N_{i_1}, N_{i_2}, \cdots, N_{i_t}$ were t blocks of length 1. Then if $S = \{x_{i_1}, x_{i_2}, \cdots, x_{i_t}\}$, we have $|N_G(S)| < 3t^2 + (m-1)d$, contradicting $\delta_t(G) \geq ((t-1)/(t+1-s))(5n/(d+2-s)) + (m-1)d + 3t^2$. Assume, then, that G has k blocks of length 1 where $0 \leq k \leq t-1$.

We first show that $\delta_t(G) \geq ((t-s)/(t+1-s))(5n/(d+2-s)) + (m-1)d + 3t^2$ for $d \in I_s$ implies that $\delta_t(G) \geq ((t-k)/(t+1-k))(5n/(d+2-k)) + (m-1)d + 3t^2$ in the range $0 \leq k \leq t-1$. The required inequality is equivalent to showing that

$$\frac{t-s}{(t+1-s)(d+2-s)} \geq \frac{t-k}{(t+1-k)(d+2-k)}.$$

for $d \in I_s$. To verify the previous inequality it is sufficient to show that for each $0 \leq \ell \leq t-2$

$$\frac{t-\ell}{(t+1-\ell)(d+2-\ell)} \geq \frac{t-\ell-1}{(t-\ell)(d+1-\ell)}.$$

is equivalent to

$$d \geq (t-\ell)^2 + \ell - 2.$$

This can be verified in a straightforward way. Thus, $(t-s)/((t+1-s)(d+2-s))$ dominates $(t-k)((t+1-k)(d+2-k))$ for all k if $d \in I_s$.

To complete the proof it is sufficient to reach a contradiction by showing that the generalized degree condition $\delta_t(G) \geq ((t-k)/(t+1-k))(5n/(d+2-k)) + (m-1)d + 3t^2$ implies that G has more than n vertices. Those blocks of length 1 have at most $t-1$ vertices, but the remaining blocks have some positive fraction of n vertices. We will show that, on the average, a block of length $r \geq 1$ will have at least $rn/(d+2-k)$ vertices. This implies that each of the $d+2-k$ different N_j's not in a block of length 1 have, on the average, $n/(d+2-k)$ vertices, and this implies G has more than n vertices.

We will first consider the case when $t-k \geq 5$. Each block of length s for $s = 2, 3,$ or 4 will be shown to have more than $sn/(d+2-k)$ vertices. If N_i, N_{i+1} is a block of length 2, then necessarily $i+1 = d+2$ and $|N_{d+2}| \geq t$. Let $S \subseteq N_{d+2}$, where $|S| = t$. Then $\overline{N}(S) \subseteq N_{d+1} \cup N_{d+2}$ and so

$$|N_{d+1} \cup N_{d+2}| \geq |\overline{N}(S)| > \left(\frac{t-k}{t-k+1}\right)\left(\frac{5n}{d+2-k}\right) > \left(\frac{2n}{d+2-k}\right),$$

since $t-k \geq 5$. Moreover, the block of length 2 will have more than the required average. In fact, each N_j in such a block will have, on the average, an excess of at least

$$\frac{1}{2}\left(\left(\frac{t-k}{t-k+1}\right)\left(\frac{5n}{d+2-k}\right) - \left(\frac{2n}{d+2-k}\right)\right) = \frac{(3(t-k)-2)n}{2(t+1-k)(d+2-k)}.$$

If N_i, N_{i+1}, N_{i+2} is a block of length 3, then $|N_{i+1}| \geq t$ and so, as above, we have

$$|N_i \cup N_{i+1} \cup N_{i+2}| > \left(\frac{t-k}{t-k+1}\right)\left(\frac{5n}{d+2-k}\right) > \left(\frac{3n}{d+2-k}\right).$$

In this case, the average excess of each N_j in a block of length 3 is at least

$$\frac{(2(t-k)-3)n}{3(t+1-k)(d+2-k)}.$$

If $N_i, N_{i+1}, N_{i+2}, N_{i+3}$ is a block of length 4, then $|N_{i+1}| \geq t$ or $|N_{i+2}| \geq t$, and again we have

$$|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3}| > \left(\frac{t-k}{t-k+1}\right)\left(\frac{5n}{d+2-k}\right) > \left(\frac{4n}{d+2-k}\right).$$

In this case, the average excess of each N_j in a block of length 4 is at least

$$\frac{((t-k)-4)n}{4(t+1-k)(d+2-k)}.$$

Therefore, for those N_j that are in a block of length $2, 3$ or 4, the average excess over the required $n/(d+2-k)$ is at least $((t-k)-4)n/(4(t+1-k)(d+2-k))$, which is the minimum excess for those N_j in blocks of length 4.

If $N_i, N_{i+1}, N_{i+2}, N_{i+3}, N_{i+4}$ is a block of length 5, then $|N_{i+1}| \geq t$ and $|N_{i+3}| \geq t$, and again we have

$$|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3} \cup N_{i+4}| > \left(\frac{t-k}{t-k+1}\right)\left(\frac{5n}{d+2-k}\right).$$

However, in this case the required $5n/(d+2-k)$ is not ensured, and in fact there is a deficit in the entire block of at most $5n/((t+1-k)(d+2-k))$, or an average deficit on each N_j of at most $n/((t+1-k)(d+2-k))$. If

$$|N_{i+1}| \text{ or } |N_{i+3}| \geq \left(\frac{1}{t-k+1}\right)\left(\frac{5n}{d+2-k}\right),$$

then a stronger statement

$$|N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3} \cup N_{i+4}| > \left(\frac{5n}{d+2-k}\right)$$

is true. Blocks of length 5 with $|N_{i+1}|$ or $|N_{i+3}| \geq 5n/((t-k+1)(d+2-k))$ will be called *positive blocks*, and the remaining blocks of length 5 will be called *negative* blocks.

If G contains no blocks of length 5, then the average number of vertices in those $d+2-k$ distinct N_j's in blocks of length at least 2 is at least $n/(d+2-k)$, and so

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > (d+2-k)\left(\frac{n}{d+2-k}\right) = n,$$

and we arrive at a contradiction. Thus, we may assume that G contains $r \geq 1$ blocks of length 5. If there are as many as $t-k$ blocks of length 5, then consider a set S of t vertices in N obtained by selecting one vertex from each of the k distinct blocks of length 1, and the left endpoint (or right endpoint) of each of the $t-k$ blocks of length 5. Then $|\overline{N}(S)| \geq 5(t-k)n/((t-k+1)(d+2-k)) + 3t^2$, and thus the corresponding "N_{i+1}" in one of the $t-k$ blocks of length 5 with endpoint x_i must have at least $5n/((t-k+1)(d+2-k))$ vertices and so the block is positive. Also, on the average, the $t-k$ blocks of length 5 have at least $5n/(d+2-k)$ vertices. We can conclude from this that the number of negative blocks is less than $t-k$, and if there are as many as $t-k$ blocks of length 5, then on the average the blocks of length 5 have at least $5n/(d+2-k)$ vertices.

Consequently, if there are as many as $t-k$ blocks of length 5, then each block of length $s \geq 2$ has, on the average $sn/(d+2-k)$ vertices. Hence,

$$|N_1 \cup N_2 \cup \cdots \cup N_{d+2}| > (d+2-k)\left(\frac{n}{d+2+\ell-t}\right) = n,$$

a contradiction. Thus, we can conclude that the number r of blocks of length 5 is less than $t-k$. The deficit from all the blocks of length 5 is at most

$$\frac{5n(t-k-1)}{(t+1-k)(d+2-k)},$$

and the excess from the remaining blocks is at least

$$\frac{n(d+3-t)(t-k-4)}{4(t+1-k)(d+2-k)}.$$

If this excess is greater than the deficit above, then a contradiction to the number of vertices in G is again reached. Thus, to complete this case, it is sufficient to show that

$$\frac{(d+3-t)(t-k-4)}{4} \geq 5(t-k-1),$$

or equivalently

$$d \geq \frac{20(t-k-1)}{t-k-4} + t - 3.$$

However, this last inequality is true for $d \geq t+77$, as $t-k \geq 5$.

We are now left with the case $t-k \leq 4$. Note that since $d \geq 5t+28$, s and t must satisfy the inequalities

$$5t+28 \leq d \leq (t-s+1)^2 + s - 3 \leq t^2 + 2t - 2.$$

It follows immediately that $t \geq 8$, and therefore from the first two inequalities we can also conclude $t-s \geq 8$.

Just as in the previous case, the average number of vertices in those N_j that are in a block of length $2, 3$, or 4 is at least

$$M := \left(\frac{1}{4}\right)\left(\frac{t-s}{t-s+1}\right)\left(\frac{5n}{d+2-s}\right).$$

This follows from the fact that blocks of length 4 give the minimum value. Since $t-s \geq 8$, the average number of vertices in these N_j is at least $\frac{10}{9}n/(d+2-s)$.

Suppose G contains two blocks $N_i, N_{i+1}, \ldots, N_{i+4}$ and $N_j, N_{j+1}, \ldots, N_{j+4}$ of length 5. Let S be a set of t vertices consisting of $x_i, x_{i+4}, x_j, x_{j+4}$ and $t-4 \leq k$ vertices chosen from the blocks of length 1. Since the set S has at most $(m-1)d$ adjacencies on the other paths P_i for $i \neq 1$, $|\overline{N}(S)| \geq ((t-s)/(t+1-s))(5n/(d+2-s)) + 3t^2$. Note also that there are at most $3t^2$ vertices in $\overline{N}(S)$ that are not in one of the 2 blocks of length 5. It follows that one of $N_{i+1}, N_{i+3}, N_{j+1}, N_{j+3}$ contains at least $(1/4)((t-s)/(t+1-s))(5n/(d+2-s))$ vertices, so that one of the two blocks of length 5 contains at least

$$\left(\frac{t-s}{t+1-s}\right)\left(\frac{5n}{d+2-s}\right) + \left(\frac{1}{4}\right)\left(\frac{t-s}{t+1-s}\right)\left(\frac{5n}{d+2-s}\right) = 5M$$

vertices. Moreover, when $k \geq t-2$, a similar argument applies for the last block of length 5 as well. In the latter case all blocks but those of length 1 contain at least M vertices in their parts N_j on the average. Therefore the number of vertices is greater than $(d+2-k)M \geq (d+3-t)M$. It can be observed that M, when viewed as a function of s, takes its smallest value when $s=0$ or when s is as close to t as possible (i.e., $s=t-8$). In either case, $(d+3-t)M \geq n$ can be verified as $d \geq 5t+28$ and $d \geq t+60$, respectively.

In the case $k \leq t-3$, an even stronger inequality is true, because the last block of length 5 contains at least $4M$ vertices. Therefore, G must contain at least $(d+4-t)M \geq n$ vertices, a final contradiction. ■

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