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SOME MAXIMUM MULTIGRAPHS AND EDGE/VERTEX DISTANCE COLOURINGS

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Abstract

Shannon–Vizing–type problems concerning the upper bound for a distance chromatic index of multigraphs G in terms of the maximum degree $\Delta(G)$ are studied. Conjectures generalizing those related to the strong chromatic index are presented. The chromatic d–index and chromatic d–number of paths, cycles, trees and *some* hypercubes are determined. Among hypercubes, however, the exact order of their growth is found.

Keywords: (strong) chromatic index, chromatic number, matching, hypercube, error–correcting code, asymptotics

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1. Introduction

Investigations presented in this paper have been inspired by Andersen's talk on the strong chromatic index, sq, of cubic graphs [1] at the Kiel '90 conference in Germany. In the plenary discussion that followed the talk the present author identified sq with the chromatic 2–index, $q^{(d)}$ with $d = 2$, as defined below. Since 1991 early versions of the following Conjectures 1–3 have been presented at some conferences held in Slovakia or Poland, including Zemplinska Širava '91 (now Slovakia) and Lubiatów '92 and '93 (Poland). Moreover, they were presented with some additions in Enschede '93, see [12]. Conjectures 2 and 4 have been corrected in what follows.

Only recently the author has learned that Conjectures 1 and 2 strengthen two conjectures of Faudree et al. [5] on corresponding inequalities; moreover, Conjecture 3 extends one of [5] from $d = 2$ to any natural d so that the case $d = 1$ gives the known Shannon's result [10]. On the other hand, Conjecture 4 on equality in Conjecture 3 extends Vizing's refinement [13] upon Shannon's result.

Let k, d and D be positive integers. The letter G stands for a multigraph. Let $L(G)$ denote the line graph of G. It is assumed that the line graph operator L transforms a loop into a vertex with an incident loop. The edge distance of distinct edges e and f of G is defined to be the number of internal vertices in a shortest walk joining e and f. In general, the *edge distance* of edges e, f in G equals the distance of vertices e, f in $L(G)$, in symbols,

$$\mathrm{dist}_{E(G)}(e, f) = d_{L(G)}(e, f).$$

Recall that a set of vertices is called *independent* if the set induces no edge. In general, we call a set W of vertices to be d^+ *independent* if no loop is incident to a vertex from W and any two distinct vertices in W are at distance larger than d. Similarly, a d^+ *independent set of edges* (in other words, a d^+ *matching*) in G is a set of non–loop edges such that any two of them are at edge distance greater than d. Assume that *d–independent* means d^+ independent. Thus, a set of edges is called d–independent if it is independent as a subset of vertices in the dth power $L(G)^d$ of the line graph $L(G)$ of G. Hence, 1^+ *independent* (or 1–independent) means independent; a 2^+ *matching* means an induced matching. Graphs without large 2^+ matchings are proved in Wagon [14] to have the chromatic number bounded by a function of the maximum clique order.

The *chromatic d–index*, $q^{(d)}(G)$, of G is the smallest cardinality among partitions of the edge set $E(G)$ into d^+ matchings. Note that chromatic 2–index (with $d = 2$) coincides with the strong chromatic index [4, 3]. Moreover, $q^{(d)}(G)$ exists iff G has no loops.

In what follows G is assumed to be a *loopless* multigraph. Then

$$q^{(d)}(G) = \chi(L(G)^d),$$

the chromatic number of the dth power of the line graph $L(G)$ of G. In what follows, by an *edge d^+ colouring* of G we mean an edge colouring whose each colour class, by definition, is a d^+ matching of the multigraph.

Thus an edge distance colouring is a vertex distance colouring of the line graph. Define the *chromatic d–number* of G, $\chi^{(d)}(G)$, to be the smallest cardinality among partitions of the vertex set $V(G)$ of G into d^+ independent subsets of vertices, cf. definition in [8, 7]. Hence,

$$\chi^{(d)}(G) = \chi(G^d),$$

G^d being the dth power of G. Moreover,

$$q^{(d)}(G) = \chi^{(d)}(L(G)).$$

Shannon–Vizing–type problems concerning the upper bound for a chromatic d–index of multigraphs G in terms of the maximum degree $\Delta(G)$ are studied. Conjectures generalizing those related to the strong chromatic index are stated. Chromatic d–index and d–number of paths, cycles, trees and some hypercubes are determined.

2. Bounds on chromatic d–index

There are two standard lower bounds for a chromatic parameter like $q^{(d)}(G)$. One involves a d^+ *matching number*, $\mu_d(G)$, which is the sharp upper bound on the size among colour classes. Clearly, $\mu_d(G)$ is defined to be the largest cardinality among d^+ matchings in G. Then

$$q^{(d)}(G) \geq |E(G)|/\mu_d(G). \tag{1}$$

Another bound for $q^{(d)}(G)$ is simply the density (in other words, the clique number) of $L(G)^d$. Let the pre–image in G of a clique in $L(G)^d$ be called a $d\hat{}$ cluster (or *diameter–d cluster*) in G. Thus, a $d\hat{}$ *cluster* is defined to be a set of edges such that each pair of them are at the edge distance at most d. Let $\omega_1^d(G)$, called the $d\hat{}$ *cluster number* of G, be the largest cardinality among $d\hat{}$ clusters in G. Consequently,

$$q^{(d)}(G) \geq \omega_1^d(G). \tag{2}$$

If $d = 1$ then a $d\hat{}$ cluster in G comprises edges either with a common vertex or induced by vertices of a triangle C_3 in G. Hence,

$$\omega_1^1(G) = \max\{\Delta(G), \max_{V'=V(C_3)} |E(<V'>_G)|\}.$$

In general, it is possible to give a lower bound for $\omega_1^d(G)$ which is attainable for some G. To this end, let E^{d*} be an edge of G if d is even, else let E^{d*} be a set of edges with a common vertex. Given a set E_1 of edges [or $E_1 = e$, an edge] of G, let $N^i(E_1)$, the *distance–i neighbourhood* of E_1, be the set of all edges whose minimum edge distance from an element of E_1 [from e] is exactly i. Let

$$\hat{N}^i(E_1) = \bigcup_{j=0}^{i} N^j(E_1).$$

It is clear that $\hat{N}^i(E_1)$ is a $d\hat{}$ cluster in G if $i = \lfloor d/2 \rfloor$ and $E_1 = E^{d*}$. Let

$$\nu_1^d(G) = \max_{E_1} |\hat{N}^{\lfloor d/2 \rfloor}(E_1)|$$

with maximum over all $E_1 = E^{d*}$ in G. Hence,

$$\omega_1^d(G) \geq \nu_1^d(G), \tag{3}$$

the equality being true if G has no cycle of length c for $3 \leq c \leq 3d$.
The following result on a standard upper bound comes from Brooks' theorem.

$$q^{(d)}(G) \leq 1 + \Delta(L(G)^d)$$

with equality for a connected multigraph G iff $d = 1$ and G is an odd cycle or $E(G)$ is a $d\hat{}$ cluster. Hence,

$$\begin{aligned} q^{(d)}(G) &\leq 1 + \max_{e \in E(G)} \sum_{i=1}^{d} |N^i(e)| \\ &\leq 1 + 2\sum_{i=1}^{d} (\Delta(G) - 1)^i, \end{aligned} \tag{4}$$

the bound being too large if $d > 1$.

3. Maximum multigraphs

The following constructions are involved in our conjectures as well as in supporting results presented in what follows.

For $D \geq 2$, denote by $\tilde{G}$, $\tilde{G} = C_{2d+1}(D)$, a multigraph obtained from the cycle C_{2d+1} by multiplying each edge if $d = 1$ or each vertex if $d \geq 2$. The multiplication factor is exactly $(D/2)^{d-1}$ if D is even. If D is odd and $d = 1$, 2, the factor is $(D \pm 1)/2$, factor $(D + 1)/2$

being applied once if $d = 1$, else twice and to adjacent vertices of C_5 (if $d = 2$). If D is odd and $d \geq 2$ then one vertex, say x_0, of C_{2d+1} is multiplied by $((\Delta-1)/2)^{d-1}$ and every two vertices at distance i from x_0 are multiplied by $(\Delta+1)^{i-1}(\Delta-1)^{d-i}/2^{d-1}$, $i = 1, 2, \ldots, d$. The factors become multiplicities of edges if $d = 1$. If $d \geq 2$ and $D > 2$, new vertices are joined by some edges corresponding to those of C_{2d+1} in such a way that all vertices are of degree D with the exception that, for odd D, all vertices obtained by multyplying the vertex x_0 are of degree $D-1$. Moreover, distance $d-1$ between pairs of vertices on C_{2d+1} is to be preserved under multiplication, i.e., under $C_{2d+1} \mapsto \tilde{G}$. Thus the multigraph $\tilde{G}$ is uniquely determined for $d \leq 2$. Namely, if $d = 2$, edges of C_5 correspond to complete bipartite subgraphs of the resulting graph $\tilde{G}$.

We conjecture that such a $\tilde{G}$ exists and is unique for $D > 2$ and $d > 2$. A $C_7(4)$ has been found. Next, a $C_7(3)$ has been found in cooperation with Mrs. E. Sidorowicz [11].

Conjecture 1. *The maximum size, $f(k,d,D)$ $(= |E|)$, among multigraphs with maximum degree $\Delta \leq D$ and with d^+ matching number $\mu_d \leq k$ is $kf(1,d,D)$.*

Conjecture 1 is true if $D = 1$ or $d = 1$. Moreover, it is supported by a result on related maximum bipartite graphs [4] for $d = 2$.

Conjecture 2. *For $k = 1$ and $D \geq 2$, the corresponding extremal multigraph with $f(1,d,D)$ edges, the maximum degree $\Delta \leq D$, the line diameter at most d* (i.e., without any two d^+ independent edges) *and without isolated vertices is $C_{2d+1}(D)$ and is unique. Moreover,*

$$f(1,d,D) = \begin{cases} (2d+1)D^d/2^d & \text{if } D \text{ is even and } d \geq 2, \\ \lfloor 3D/2 \rfloor & \text{if } D = 1 \text{ or } d = 1, \\ (D(D+1)^d - (D-1)^{d+1})/2^d & \text{otherwise,} \end{cases}$$

whence $f(1,d,D) = [(2d+1)D^d - dD^{d-1} + \ldots + (-1)^d]/2^d$ for odd $D \geq 3$ and $d \geq 2$.

Note that loops (each of which contributes 2 to the degree of the incident vertex) do not appear in the multigraph. Furthermore, the multigraph is a simple graph if $d \geq 2$. Conjecture 2 is clearly true if $d = 1$. If $d = 2$ then Conjecture 2 for graphs together with its specification in Bermond et al. [2] involving the graph $C_5(D)$ for even D is proved in Chung et al. [3].

4. Sharp upper bound

Conjecture 3. *$q^{(d)}(G) \le f(1,d,D)$ when G is a loopless multigraph with the maximum degree $\Delta \le D$.*

Conjecture 3 for $d=2$ actually coincides with Conjecture 1 of [5] and, for odd D and $d=2$, is a strengthening of a conjecture of Erdös and Nešetřil (1985, cf. [4]). Conjecture 3 is true for $d=1$ (Shannon [10]) or $d=2$ and $D=3$ (Andersen [1] and Horák et al. [6]). We are going to show its truth for $D \le 2$ and any d.

In fact, we prove a bit more. Given a positive integer p and a simple graph G, let ${}^pG$ denote the union of p edge–disjoint copies of G, all copies on the fixed vertex set $V = V(G)$.

Theorem 1. *If $G = P_n$, a path, or $G = C_n$ $(n \ge 3)$, a cycle, both on n vertices, then*

$$\begin{aligned} q^{(d)}({}^pP_n) &= p\,q^{(d)}(P_n) \quad \textit{where} \\ q^{(d)}(P_n) &= \min\{n-1,\ d+1\}, \\ q^{(d)}({}^pC_n) &= |E({}^pC_n)| = pn \qquad \textit{if } n \le 2d+1, \\ &= p(d+1) + \lceil pr/j \rceil \ \ \textit{if } n = j(d+1) + r \ (\ge d+1) \end{aligned}$$

where j and r are integers, $j \ge 1$ and $0 \le r \le d$.

Proof. Assume $n \ge d+1$ whence $j \ge 1$. Let $a = \lfloor r/j \rfloor$. Then

$$n = j(d+1+a) + s, \quad 0 \le s < j.$$

Since edges of any path P_{d+2} on $d+1$ edges must have distinct colours,

$$\begin{aligned} \mu_d(C_n) &= \max\{1,\ \lfloor n/(d+1) \rfloor\}, \quad n \ge 3 \\ &= \mu_d({}^pC_n). \end{aligned}$$

Hence, by (1),

$$q^{(d)}({}^pC_n) \ge \lceil pn/j \rceil = p(d+1+a) + \lceil ps/j \rceil.$$

To prove that equality holds, we find an edge d^+ colouring using a set $\mathcal{C}$ of $|\mathcal{C}| = p(d+1+a)$ colours together with $k_1 := \lceil ps/j \rceil$ additional colours (if $s > 0$).

First we colour ps edges with all additional colours if $s > 0$. To this end, fix an orientation of the underlying cycle C_n and let $1, 2, \ldots, n$ be positions of edges on the cycle. Let $kj + l = ps$ where $k = \lfloor ps/j \rfloor$ whence $0 \leq l < j$. Mark the following j positions of edges on C_n: $1 + i(d+1)$ for $i = 0, 1, \ldots, j-1$. Choose k_1 disjoint sets $\tilde{E}$ of edges such that each set comprises nonadjacent edges from the marked positions, k sets being of cardinality j and, for $l > 0$, one set of cardinality l. Use each of the k_1 additional colours to colour all edges in a set $\tilde{E}$ so that edges in distinct sets are assigned distinct colours.

Thus, if $s > 0$ then ps edges have been coloured with the given additional colours. Let ${}^pC_n = \bigcup_{i=1}^{p} C_n^{(i)}$ where each $C_n^{(i)}$ is a simple n–cycle, includes exactly s coloured edges and has a fixed orientation. Partition the colour set $\mathcal{C}$ into p subsets $\mathcal{C}^{(i)}$ each of which is assumed to be cyclically ordered and has cardinality $d + 1 + a$, $i = 1, \ldots, p$. Use consecutive colours from $\mathcal{C}^{(i)}$ to colour all uncoloured edges of $C_n^{(i)}$ while passing along the oriented cycle $C_n^{(i)}$, $i = 1, \ldots, p$.

The remaining cases are obvious. ■

Remark 1.

$$q^{(d)}({}^pC_n) < p\, q^{(d)}(C_n)$$

exactly when $p > 1$ and $j \not| n$ (or $s > 0$).

The following result is well–known if $d = 1 = p$.

Corollary 2 *If* $d + 1 \mid n$ *(or* $r = 0$*),*

$$\begin{aligned} q^{(d)}({}^pC_n) &= p(d+1), \quad \textit{else} \\ &= p(d+1) + 1 \end{aligned}$$

if $n \geq pd(d+1)$ *(or* $j \geq pd$*) and* $r > 0$. ■

Remark 2. Theorem 1 shows that the bound (1) is sharp for each order n of G, $n \geq 2$. Moreover, as the line graph $L(P_n) = P_{n-1}$ for $n \geq 2$ and $L(C_n) = C_n$ for $n \geq 3$,

$$\begin{aligned} \chi^{(d)}(P_n) &= \min\{n,\ d+1\}, \\ \chi^{(d)}(C_n) &= q^{(d)}(C_n), \end{aligned}$$

cf. Theorem 1, which agrees with F. Kramer [7].

Claim 3. *The equality in Conjecture* 3 *holds only if* G *contains the extremal multigraph* $C_{2d+1}(D)$ *as a submultigraph* (which is false if $d = 1$ and $D = 2, 3$).

Conjecture 4. *Claim* 3 *is true if* $d \geq 2$ *and* $D \geq 2$ *unless possibly* G *is one of a few exceptions.*

Conjecture 4 is proved above for $D = 2$ and any $d \geq 2$. It has been prompted by a result of Vizing [13] which says that Claim 3 is true for $d = 1$ and $D \geq 4$. The 8–gon C_8 with all four diagonals is an exception [6] for $d = 2$ and $D = 3$.
It is natural to consider the above problems restricted to connected (if $k \geq 2$), bipartite, or planar multigraphs; for graphs and $d = 2$, cf. results and problems of [4, 5].

5. Bounds on chromatic d–number

Let $\alpha_d(G)$ denote the d^+ *independence number* of G,

$$\alpha_d(G) = \alpha(G^d),$$

$\alpha(G)$ being the independence number of G. Define a $d\hat{}$ *clique* of G to be a subset of vertices whose diameter in G is at most d. Let $\omega_0^d(G)$, called the $d\hat{}$ *clique number* of G, be the largest cardinality among $d\hat{}$ cliques in G. Hence, $1\hat{}$ clique is a clique, $\omega_0^1(G)$ $(= \omega_0(G))$ is the clique number of G. Using notation similar to that in Sect. 2, we define $\nu_0^d(G)$, the lower bound for $\omega_0^d(G)$, as follows. Let V^{d*} be a vertex if d is even, else let V^{d*} be a nontrivial clique. Let

$$\nu_0^d(G) = \max_{V_1} |\hat{N}^{\lfloor d/2 \rfloor}(V_1)|$$

with maximum over all $V_1 = V^{d*}$ in G. The following formulae are counterparts of the formulae (1)–(4).

$$\chi^{(d)}(G) \geq |V(G)|/\alpha_d(G), \tag{5}$$

$$\chi^{(d)}(G) \geq \omega_0^d(G) \tag{6}$$

$$\geq \nu_0^d(G). \tag{7}$$

$$\chi^{(d)}(G) \leq 1 + \Delta(G^d) \leq 1 + \max_{v \in V(G)} \sum_{i=1}^{d} |N^i(v)|$$

$$\leq 1 + \Delta(G) \sum_{i=1}^{d} (\Delta(G) - 1)^{i-1}, \tag{8}$$

the bound being too large if $d > 1$.

6. Distance chromaticity of trees

We are going to show that the bound (2) is sharp for each nontrivial order of G. To this end, we show that trees G realize equality in (2) and obviously in (3) too, which is known if $d = 1$ $(q(G) = \Delta(G))$ or $d = 2$ ([5]).

Theorem 4. *If G is a tree then*

$$\begin{aligned} q^{(d)}(G) &= \nu_1^d(G), \\ \chi^{(d)}(G) &= \nu_0^d(G). \end{aligned}$$

Proof. Consider $q^{(d)}$ first. Due to (2) and (3) it is enough to show the existence of an edge d^+ colouring of G with $\nu_1^d(G)$ colours. Proceed by induction on the order $|V(G)|$ of G. Put $T(0) = G$. Call a vertex to be a *leaf* if it has at most one neighbour. Put $T(i+1) = T(i) - A_i$ where A_i is the set of leaves of $T(i)$. If $A_{\lfloor d/2\rfloor+1} = \emptyset$ then $E(G)$ is a $d\hat{}$ cluster of cardinality $\nu_1^d(G) = q^{(d)}(G)$, as required. Otherwise, let E_0 be an $A_{\lfloor d/2\rfloor}$–$A_{\lfloor d/2\rfloor+1}$ edge if d is even, else let E_0 be the set of all edges in G incident to a fixed vertex in $A_{\lfloor d/2\rfloor+1}$ Thus $E_0 = E^{d*}$. An A_0–A_1 edge e belonging to the $d\hat{}$ cluster $\hat{E} := \hat{N}^{\lfloor d/2\rfloor}(E_0)$ can be chosen so that $\hat{E}$ comprises all edges of G whose edge distance from e is at most d. Let G_1 be a subtree obtained from G by deleting e together with the leaf incident to e. By the induction hypothesis, G_1 has an edge d^+ colouring with $\nu_1^d(G_1)$ colours where $|\hat{E}| - 1 \le \nu_1^d(G_1) \le \nu_1^d(G)$. Hence the number of colours available for e is $\nu_1^d(G) - |\hat{E}| + 1 \ge 1$.
The second formula can be proved analogously. ■

7. Distance matchings in hypercubes

Error–detecting binary codes will be involved. Refer therefore to [9] for coding theory.

Recall that a t–dimensional cube, Q^t, is a t–regular bipartite graph with $t\,2^{t-1}$ edges and with the vertex set $V = \{0, 1\}^t$ of cardinality 2^t. Moreover, the edge diameter and diameter of Q^t both equal t. Vertices of Q^t are binary t–vectors or binary codewords of length t. Note that distance of two vertices in Q^t is their Hamming distance d_H, d_H being the number of positions in which coordinates differ.

We are going to prove the following result on distance chromaticity of hypercubes. Recall that in asymptotic notation Θ specifies the exact order of growth.

Theorem 5. *For any natural constant* d*, if* $t \to \infty$ *then*

$$\begin{aligned} q^{(d)}(Q^t) &= \Theta(t^{\lfloor (d+1)/2 \rfloor}), \\ \chi^{(d)}(Q^t) &= \Theta(t^{\lfloor d/2 \rfloor}), \end{aligned}$$

orders of growth being equal for each even d.

We use formulas (1) and (5) to estimate the chromatic parameters in question from below. Upper bounds come from properties of large binary linear codes. These are primitive narrow–sense BCH codes and their shortened or extended versions.

Recall that maximum cardinality among 0–1 codes of length m and minimum (Hamming) distance d (between codewords) is denoted $A(m, d)$. The corresponding maximum size in the subclass of *linear* codes is denoted $B(m, d)$. Therefore $B(m, d)$ is a power of 2 and

$$B(m, d) \leq A(m, d).$$

Moreover, because $A(m, d)$ is decreasing in d, that is,

$$A(m, d) \geq A(m, d + 1),$$

the d^+ independence number of Q^m is $A(m, d + 1)$,

$$\alpha_{d-1}(Q^m) = A(m, d). \tag{9}$$

We are going to determine the d^+ matching number of Q^t. Namely,

$$\mu_d(Q^t) = A(t - 1,\, d). \tag{10}$$

It is easily seen that each diametrical matching, a maximum $(t-1)^+$ matching, of Q^t consists of two parallel edges. These are edges parallel to any of t coordinate axes. This follows from the following observation. If e, f are two edges in Q^t parallel to ith and jth coordinate axes Ox_i and Ox_j, respectively, let $\widehat{e, f}$ stand for the pair of codewords which are obtained by removal of both the ith and jth coordinates from endvertices of e and f, respectively. Then the length of the resulting codewords is $t-1$ or $t-2$ according as $i=j$ or not. Given an edge e, the codeword $\hat{e}$ comprising $t-1$ common coordinates of endvertices of e is called a *codeword* of e. In particular, $\widehat{e,f} = \{\hat{e}, \hat{f}\}$ if $i=j$. Moreover,

Lemma 6.

$$\mathrm{dist}_E(e,\ f) = 1 + d_H(\widehat{e,f})$$

for any distinct edges $e,\ f$ *of the hypercube* Q^t. ■

Write $M \parallel Ox_i$ if M is a set of edges e such that $e \parallel Ox_i$. For any fixed integer i, $1 \le i \le t$, let $\gamma_i : e \mapsto \hat{e}$ move any edge e, $e \parallel Ox_i$, to the codeword $\hat{e}$ of e. Thus γ_i is a bijection onto the vertex set of Q^{t-1}. Due to Lemma 6, we call γ_i to be a *co–isometry.* Thus there is a co–isometric correspondence (a restriction of γ_i) from any set M onto some set $\mathcal{C}$ and conversely (use ${\gamma_i}^{-1}$) from any $\mathcal{C}$ onto some M; M being a d^+ matching of Q^t and $M \parallel Ox_i$, $\mathcal{C}$ being a binary code of length $t-1$ and minimum distance at least d. Hence,

$$\mu_d(Q^t) \ge A(t-1,\ d).$$

The converse inequality follows from the next lemma.

Note in this context that, for $t \ge 3$, a maximum matching (1^+ matching) of Q^t need not consist of parallel edges. Nevertheless, any distance matching of maximum size can comprise parallel edges only. We are going to show even more. Namely, we present a bijective transformation of a distance matching onto a close one and parallel to a prescribed coordinate axis. Given two matchings M and M' of Q^t, a bijection $\varphi : M \to M'$ is called a *step–bijection* if, for each $e \in M$, e and $\varphi(e)$ are mutually equal or adjacent.

Lemma 7. *For any* d^+ *matching* M *of the cube* Q^t *there is a step–bijection* $\varphi : M \to M'$ *onto a* d^+ *matching* M' *of* Q^t, $M' = M'^i$, *such that* $M' \parallel Ox_i$ *for any prescribed* i.

Proof. Contrary, suppose there are integers $t,\ d,\ i$ and a d^+ matching M of Q^t such that $M \not\parallel Ox_i$ and each $M'^i = M$. Let S_j be a half–cube which is the intersection of Q^t with the hyperplane $x_i = j$; $j = 0,\ 1$. The following remark on d^+ matching M will be useful.

($\star$) *Any two vertices of any half–cube* S_j *which belong to distinct edges in* M *are at distance not less than* d.

Define (initial values of) sets, say M'', M_0 and M_1, by assuming that $M = M'' \cup M_0 \cup M_1$ where $M'' \parallel Ox_i$ and $M_j = M \cap E(S_j)$, $j = 0$, 1. Hence $M_0 \cup M_1 \neq \emptyset$. Due to symmetry, assume $M_0 \neq \emptyset$. Construct a step–transformation φ of M. Assume first that φ is the identity on the present M''.

Let k, $\neg k \in \{0, 1\}$ and $\neg k \neq k$. Continue constructing sets E_k, E'_k and $E_{\neg k}$ until $E_{\neg k} = \emptyset$. The sets are to comprise edges and to meet the following requirements:

$$E_k \subseteq M_k, \quad E'_k \parallel Ox_i, \quad E_{\neg k} \subseteq M_{\neg k}.$$

Moreover, each e in E_k is to have exactly one endvertex which is considered admissible in a sense to be made precise in what follows. Begin by putting $k = 0$ and $E_0 = \{e\}$ for any edge $e \in M_0$ and assume that either endvertex of e is considered *admissible*. Given $\emptyset \neq E_k \subseteq M_k$ and any $e \in E_k$, let P_0 be the admissible endvertex of e. There is the unique vertex P_1 such that the edge $P_0P_1 \parallel Ox_i$ in Q^t. Define $\varphi(e) := P_0P_1$. Thus φ restricted to E_k is a step–transformation. Put E'_k to be $\varphi[E_k]$, the image of E_k under φ. Perform updating

$$M'' \leftarrow M'' \cup E'_k \quad \text{and} \quad M_k \leftarrow M_k - E_k.$$

Hence, due to ($\star$), M'' is a d^+ matching parallel to Ox_i. Consider $M_{\neg k}$, the remaining M_j, and its half–cube $S_{\neg k}$. Let V''_k be the set of vertices in which edges $e \in E'_k$ intersect the half–cube $S_{\neg k}$. Define $E_{\neg k}$ to comprise all edges $PR \in M_{\neg k}$ such that one endvertex, say R, is at distance, say δ, less than d from a vertex, x_R, in the set V''_k. Assume that the remaining endvertex, P, is considered *admissible*. Note that $\delta = d - 1$ (if PR exists) since M is a d^+ matching and due to definition of E'_k. Hence, $d_H(x_R, P) = d$ because otherwise Q^t would have an odd cycle.

Claim 8. *All admissible vertices of any* E_k *are unicoloured in any proper vertex bicolouring of* Q^t.

Proof (by induction). Claim is true for the first E_0, a singleton. Assume Claim is true for an E_k. Then the set E'_k is uniquely determined and the vertex subset V''_k is clearly unicoloured. So are all admissible vertices P

of edges in the next set E, $E = E_{\neg k}$, since each P is at the distance d from a vertex in V_k''. ■

Consequently, admissible vertices in our $E_{\neg k}$ are uniquely defined. Constructing sets E (as well as M'', M_0, M_1) stops because Q^t is a finite graph. Now, let $M' = M'' \cup M_0 \cup M_1$. Hence $M' \neq M$ and M' is clearly a d^+ matching. Extend φ to the whole M by assuming that φ is the identity on the final value of the set $M_0 \cup M_1$. Thus φ is a step–bijection of M onto M', $M' \neq M$, a contradiction, which ends the proof of Lemma. ■

Corollary 9. *For any natural d, there is a maximum d^+ matching of Q^t which comprises edges parallel to any fixed coordinate axis.* ■

Hence we get the formula (10), $\mu_d(Q^t) = A(t-1,\, d)$.

Corollary 10.

$$q^{(d)}(Q^t) \leq t\,\chi^{(d-1)}(Q^{t-1}).$$

■

The equality above is true for the chromatic index $q = q^{(1)}$ (with $d = 1$) provided that $\chi^{(0)} := 1$; and follows for $d = 2$ from $sq(Q^t) = 2t$ which is proved in Faudree et al. [5].

8. CHROMATICITY OF HYPERCUBES

Note that useful distance colourings can be defined by using a linear binary code. It is so because all translates of a linear code are known to form a partition of the vertex set of the hypercube whose dimension is the length the code.

Lemma 11. *For positive integers $t > 1$, d and $\bar{d} \geq d$,*

$$\begin{array}{lclcl} 1/A(t-1,d) & \leq & q^{(d)}(Q^t)/(t\,2^{t-1}) & \leq & 1/B(t-1,\bar{d}), \\ 1/A(t,d+1) & \leq & \chi^{(d)}(Q^t)/2^t & \leq & 1/B(t,\bar{d}+1). \end{array}$$

Proof. Lower bounds follow from (1), (10) and (5), (9), respectively. On the other hand, let $\mathcal{C}$ be a linear binary code of length m, minimum distance δ and the largest possible size $B(m,\delta)$. Then all translates of $\mathcal{C}$

form a partition $\mathcal{P}$ of $V(Q^m)$. This $\mathcal{P}$ is therefore a uniform vertex d^+ colouring of Q^t if $m = t$ and $\delta = \bar{d} + 1$ for any $\bar{d} \geq d$. Thus the very last inequality is proved. Similarly, for $m = t - 1$ and $\delta = \bar{d}$, the images under co–isometries ${\gamma_i}^{-1}$ of members of the partition $\mathcal{P}$ are d^+ matchings (colour classes of edges) in Q^t. ■

***Proof of Theorem* 5.** For all d and t, we can get the following lower bounds whose order of growth is as stated in Theorem 5.

$$q^{(d)}(Q^t) \;\geq\; \epsilon(d)\, t \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{t - \epsilon(d)}{i} \tag{11}$$

where $\epsilon(d) = 2 - (d \bmod 2) \in \{1, 2\}$;

$$\chi^{(d)}(Q^t) \;\geq\; \epsilon^*(d) \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{t - (t \bmod 2)}{i} \tag{12}$$

where $\epsilon^*(d) = 1 + (d \bmod 2)$.

In fact, the inequalities follow from the preceding lemma due to the sphere–packing bound

$$A(m, 2r+1)(1 + \binom{m}{1} + \ldots + \binom{m}{r}) \leq 2^m$$

for odd $d = 2r + 1 \geq 1$ and the well–known equality

$$A(n, 2r) = A(n-1, 2r-1)$$

where r is a natural number. Using the Johnson bound instead of the sphere–packing bound can improve on coefficients in lower order terms.

In order to get appropriate upper bounds, note that shortening a linear binary code which includes a codeword with $x_i = 1$ is known to consist in taking a cross–section of the code at ith coordinate, i. e. , in taking all codewords with $x_i = 0$ and deleting the x_i coordinate. The resulting code is linear, has length one smaller, halved size, and the minimum distance unchanged. Recall that primitive narrow–sense BCH binary code of length $\bar{n} = 2^s - 1$ and designed distance $2j$ coincides with that of designed distance $d = 2j + 1$, which is linear, has dimension $\geq \bar{n} - sj$ and minimum distance $\bar{d} \geq d$. Hence $B(\bar{n}, \bar{d}) \geq 2^{\bar{n}-sj}$. Shortening the code to the length n gives

$$B(n, \bar{d}) \geq 2^{n-sj} \text{ for } 2^{s-1} \leq n \leq 2^s - 1 \text{ and some } \bar{d} \geq 2j + 1.$$

Hence, by the preceding lemma,

$$
\begin{aligned}
&(13) \qquad & q^{(2j+1)}(Q^t) &\leq t\,2^{sj} \quad (\text{for } t = n+1),\\
&(14) \qquad & \chi^{(2j)}(Q^t) &\leq 2^{sj} \quad (\text{for } t = n),
\end{aligned}
$$

where

$$
\begin{aligned}
2^{sj} &= (t+\alpha)^j && \text{if } t = 2^s - \alpha,\\
&= [2(t-a)]^j && \text{if } t = 2^{s-1} + a \text{ for appropriate constants } \alpha \text{ and } a.
\end{aligned}
$$

Therefore the orders of the upper bounds are as stated in Theorem 5.

Consider the remaining values of d. It is known that

$$B(n, 2j) = B(n-1, 2j-1).$$

For $\bar{n}$ and n as above, using a BCH code of length $\bar{n}$ and designed distance $\delta = 2j-1$ (smaller than before) can give $B(n, \bar{\delta}) \geq 2^{n-s(j-1)}$ where $\bar{\delta}$ is the true minimum distance of the code, $\bar{\delta} \geq \delta$. For s (and n) large enough, however, $\bar{\delta} = \delta = 2j-1$ by the Farr result [9, p. 259]. Then

$$B(n, 2j) \geq 2^{n-1-s(j-1)}.$$

Hence, due to Lemma 11, for s and t large enough,

$$
\begin{aligned}
&(15) \qquad & q^{(2j)}(Q^t) &\leq 2t\,2^{s(j-1)} \quad (\text{for } t = n+1),\\
&(16) \qquad & \chi^{(2j-1)}(Q^t) &\leq 2\,2^{s(j-1)} \quad (\text{for } t = n).
\end{aligned}
$$

As above, one can see that orders of the upper bounds are as stated in Theorem 5. ■

Remark. In some cases linear binary codes which are larger than shortened primitive BCH codes used above are known. These are, for instance, nonprimitive BCH codes (of distance 6 or 5) or codes Y1 and Y4 obtaibable by a specialized shortening a code, or HS codes (after Helgert and Stinaff). Using such codes improves on (the highest coefficient in) upper bounds presented above.

There are binary nonlinear codes $\mathcal{C}$ of length m whose certain translates make up a partition of $V(Q^m)$. Such is the punctured Preparata code $\mathcal{P}(2r)^*$ for each integer $r \geq 2$. On the other hand, all translates of any binary linear code constitute such a partition. This gives rise to the following corollary of Lemma 11.

Corollary 12. *If the vertex set $V(Q^m)$ admits a partition into translates of a binary code $\mathcal{C}$ of length m, minimum distance d and size $A(m,d)$ then equality holds in Corollary 10 for $t = m+1$. Moreover, the chromatic d–index and chromatic δ–number of respective hypercubes are*

$$q^{(d)}(Q^t) = t\,2^{t-1}/A(t-1,d) \quad (t = m+1), \tag{17}$$

$$\chi^{(\delta)}(Q^m) = 2^m/A(m,\delta+1) \quad (\delta = d-1). \tag{18}$$

■

This result enables us to determine some distance chromatic parameters. Note that $A(n,1) = 2^n = B(n,1)$. Hence $2^n = A(n+1,2) = B(n+1,2)$. This gives results mentioned at the very end of Sect. 7. It is known that, for $d = 3$ and $n = 2^s - \alpha$ where $\alpha = 1, 2, 3, 4$ and s is a natural number such that $n \geq s$, the Hamming code ($\alpha = 1$) or else the shortened Hamming code has the largest possible size $A(n,3) = B(n,3)$, which, moreover, is the same as $A(n+1,4) = B(n+1,4) = 2^{n-s}$. Therefore
$\chi^{(1)}(Q^t) = 2$ for all t,
$\chi^{(2)}(Q^t) = 2^s = t + \alpha$ for $t = 2^s - \alpha$, $\alpha = 1, 2, 3, 4$ such that $t \geq s \geq 1$.

On the other hand, for $t = 2^s - \beta$, $\beta = 0, 1, 2, 3$ such that $t \geq 1 + s \geq 1$,

$$\begin{aligned}\chi^{(3)}(Q^t) &= 2\,2^s = 2(t+\beta);\\ &= 2\chi^{(2)}(Q^t) \quad \text{for } t = 2^s - \gamma,\ \gamma = 1,2,3 \text{ such that } t \geq s+1 \geq 3.\end{aligned}$$

Similarly, for all t, $q^{(1)}(Q^t) = t$ and $q^{(2)}(Q^t) = 2t$.

For $t = 2^s - \beta$, $\beta = 0, 1, 2, 3$ such that $t \geq 1 + s \geq 1$,
$q^{(3)}(Q^t) = t\,2^s = t(t+\beta)$;
$q^{(4)}(Q^t) = 2t\,2^s = 2t(t-1+\beta)$ for $t = 2^s + 1 - \beta$ such that $t \geq 2 + s \geq 2$.

Hence,
$q^{(4)}(Q^t) = 2q^{(3)}(Q^t)$ for $t = 2^s - \gamma \geq 4$, $\gamma = 0, 1, 2$.

Thus the smallest dimensions t, $t = t_i(d)$, among hypercubes whose chromatic d–number ($i = 0$) or chromatic d–index ($i = 1$) are not determined above are $t_0(2) = 8$, $t_0(3) = 9 = t_1(3)$ and $t_1(4) = 10$. Then bounds can be obtained from Lemma 11, using $A(8,3) = 20$ and $B(8,3) = 16$ or their extensions to arguments (9, 4).

Using the nonlinear punctured Preparata code $\mathcal{P}(2r)^*$ mentioned above, which is of length $n := 4^r - 1$, minimum distance 5 and the largest possible size $A(n,5) = 2^{n+1-4r}$ where $r \geq 2$, one can get
$\chi^{(4)}(Q^n) = (n+1)^2/2$,
$q^{(5)}(Q^t) = t^3/2$ for $t = n+1$.

At the other extreme, when d/n is bounded away from zero, there are known binary linear codes whose sizes are the largest possible among all binary codes. So is the binary simplex code (i.e., the dual of the Hamming code) of length $n = 2^s - 1$, minimum distance $d = 2^{s-1}$ and size s, that is, then $A(n,d) = s = B(n,d) = A(n-1,d-1) = B(n-1,d-1)$. Now, the exact values of chromatic parameters we can get from the preceding Corollary grow exponentially as dimension tends to infinity. It so in general then.

Theorem 13. *$q^{(d)}(Q^t)$ and $\chi^{(d)}(Q^t)$ grow exponentially as $t \to \infty$ and $d/t \geq \lambda > 0$.*

Proof. This follows from lower bounds (11) and (12) due to the following inequality on a sum of binomial coefficients (cf. [9, p. 310]).

$$\sum_{i=0}^{\mu n} \binom{n}{i} \geq 2^{nH_2(\mu)} / \sqrt{8n\mu(1-\mu)}$$

for any constant μ such that $0 < \mu < \frac{1}{2}$, where $H_2(x)$ is the entropy function,

$$H_2(x) = x \log_2 1/x + (1-x) \log_2 1/(1-x).$$

■

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