THE EDGE DOMINATION PROBLEM[1]

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Abstract

An *edge dominating set* of a graph is a set D of edges such that every edge not in D is adjacent to at least one edge in D. In this paper we present a linear time algorithm for finding a minimum edge dominating set of a block graph.

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1. Introduction

In this paper all graphs are simple, i.e., finite, undirected, loopless, and without parallel edges. The concept of domination arises naturally from location problems in operations research. The *domination problem* in a graph is to find a minimum sized vertex set D such that every vertex not in D is adjacent to at least one vertex in D. Domination and its variants have been extensively studied during the past two decades. This paper considers one variant, edge domination, from an algorithmic point of view. An *edge dominating set* of a graph is a set D of edges such that every edge not in D is adjacent to at least one edge in D. An *independent edge dominating set* is an edge dominating set in which no two distinct edges adjacent. The

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(independent) edge domination problem is to find a minimum (independent) edge dominating set of a graph. Mitchell and Hedetniemi [5] presented a linear time algorithm for the edge domination problem in trees. Yannakakis and Garvil [7] also gave a linear time algorithm for the problem in trees and proved that it is NP–complete in planar graphs or bipartite graphs of maximum degree 3 (see also page 190 of [3]). They also mentioned that in any graph the size of a minimum edge dominating set is equal to the size of a minimum independent edge dominating set.

The purpose of this paper is to present a linear time algorithm for the edge domination problem in block graphs which include trees. For technical reasons, we actually consider a slightly more general problem introduced in [5]. Suppose the vertex set of a graph G is partitioned into three sets, B, C, and R, where B consists of *bound* vertices, C consists of *covered* vertices and R consists of *required* vertices. A *mixed edge dominating set* of G *(with respect to* B, C, R) is a set of edges D such that

(MED1) for every edge (u, v) incident to a bound vertex u, either v is covered, or $(u, v) \in D$, or (u, v) is adjacent to an edge in D;

(MED2) every required vertex is incident to an edge in D.

Note that the edge domination problem is just the mixed edge domination with $B = V$ and $C = R = \emptyset$. This generalization can be viewed as a lebeling algorithm in which a vertex has a label "*bound*" or "*covered*" or "*required*" if it is in B or C or R, respectively. The idea of a lebeling algorithm was first introduced by Cockayne, Goodman, and Hedetniemi for solving the domination problem in trees in 1975. It is a natural but powerfull tool when we use an induction to treat a tree from leaves toward to the center.

In the above definition, a graph has no mixed edge dominating set when it contains a required vertex of degree zero. In order to resolve this difficulty and simplify some of the arguments, we allow D to contain *pseudo–edges* (x, x) even if G is loopless. If a mixed edge dominating set D contains a pseudo–edge (x, x) and x is incident to an edge e, then $(D - \{(x, x)\}) \cup \{e\}$ is cleary a mixed edge dominating set of size at most $|D|$.

As a mixed edge dominating set of a graph $G = (V, E)$ is indeed an edge dominating set of G when $V = B$, in order to find a minimum edge dominating set, we only have to label all vertices "*bound*" and find a minimum mixed edge dominating set, which we call an *MMED–set* for short.

We close this section by giving a brief review of block graphs and the block–cut–vertex structure of a graph. In a graph G, a vertex x is a *cut–vertex* if deleting x and all edges incident to it increases the number of connected components. A *block* is a maximal connected subgraph without a cut–vertex. The intersection of two distinct blocks contains at most one vertex; and a vertex is a cut–vertex if and only if it is the intersection of two or more blocks. Consequently, a graph with one or more cut–vertices has at least two blocks. A *block graph* is a graph whose blocks are complete graphs.

For any two vertices u and v in a graph G, any path from u to v must pass through a unique sequence of blocks $B_1, ..., B_r$, where B_i and B_{i+1} have a common cut–vertex that is a vertex of the path for $i = 1, ..., r-1$. Moreover, for any graph G containing m blocks $B_1, ..., B_m$ and n cut–vertices $c_1, ..., c_n$, consider the graph $G^* = (V^*, E^*)$ where

$$V^* = \{B_1, ..., \ B_m, \ c_1, ... \ c_n\}$$

and

$$E^* = \{(B_i, c_j) : c_j \in B_i, \ 1 \le i \le m, \ 1 \le j \le n\}.$$

Then G^* is a forest whose leaves are precisely the blocks with exactly one cut–vertex in G, which we call *end blocks*, and whose isolated vertices are exactly those blocks without cut–vertices in G. The *block–cut–vertex* structure G^* of a graph G can be found by a depth first search in linear time (see [1]).

2. A linear time algorithm in block graphs

In this section we first set forth three lemmas which provide the justification for an algorithm for the mixed edge domination problem in block graphs. For any subset S of V in a graph $G = (V, E)$, let $G - S$ denote the graph with vertex set $V - S$ and edge set $\{(x, y) \in E : x$ and y are in $V - S\}$. $G - w$ stands for $G - \{w\}$. In the following lemmas, if the partition (B, C, R) of G is given, then $G - S$ is considered with the partition $(B - S, \ C - S, \ R - S)$ with some specific modification, as in Lemma 3.

Lemma 1. *Suppose w is a covered vertex in a graph G. Any MMED–set D' of $G - w$ is also an MMED–set of G.*

Proof. D' is clearly a mixed edge dominating set of G. It suffices to prove that $|D'| \le |D|$ for any MMED–set D of G. Let $(w, x_1), ..., (w, x_r)$ be the set of edges in D which are incident to w. Consider $D'' =$

$D - \{(x,x_1), ...\ (w,x_r)\} \cup \{(x_1,x_1), ..., (x_r,x_r)\}$. It is clear that D'' is a mixed edge dominating set of $G - w$ and $|D''| \leq |D|$. Thus $|D'| \leq |D''| \leq |D|$. ∎

Our algorithm for the mixed edge domination problem of a block graph works from an end block inward to the center of the graph. So we need lemmas which handle an end block of a graph. For the final stage, we also need lemmas for a block without a cut–vertex. To save space, we combine these two sets of lemmas together, as follows:

Lemma 2. *Suppose G is a block graph and A is either an end block of G with a cut–vertex x or a block without a cut–vertex and x is any vertex of A. If D' is an MMED–set of $G - \{u,v\}$, where u and v are two distinct required vertices in $A - x$, then $D' \cup \{(u,v)\}$ is an MMED–set of G.*

Proof. Note that (u,v) is an edge since A is a complete graph. $D' \cup \{(u,v)\}$ is clearly a mixed edge dominating set of G. So it suffices to prove that $|D' \cup \{(u,v)\}| \leq |D|$ for any MMED–set D of G. Since every required vertex is incident to an edge in D, we may assume that $(u,u_1), ..., (u,u_r)$ with $r \geq 1$ (respectively, $(v,v_1), ..., (v,v_s)$ with $s \geq 1$) are all edges in D which are incident to u (respectively, v). Since u and v are not cut–vertices of A, $u_1, ..., u_r, v_1, ..., v_s$ are all in A and are thus pairwise adjacent. Let $D_1 = D - \{(u,u_1), ..., (u,u_r), (v,v_1), ..., (v,v_s)\} \cup \{(u_2,u_2), ..., (u_r,u_r), (v_2,v_2), ..., (v_s,v_s)\}$. And let $D_2 = D_1 \cup \{(u_1,v_1)\}$ when $(u,v) \notin D$; otherwise, say $(u,v) = (u,u_1) = (v,v_1)$, let $D_2 = D_1$. It is straightforward to check that D_2 is a mixed edge dominating set of $G - \{u,v\}$ and $|D_2| \leq |D| - 1$. Thus $|D'| \leq |D_2|$ and so $|D' \cup \{(u,v)\}| = |D'| + 1 \leq |D_2| + 1 \leq |D|$. ∎

Once we have Lemmas 1 and 2, it remains only to consider the case when $V(A - x) = \{v_1, ..., v_n\}$ with $v_1 \in B \cup R$ and $v_2, ..., v_n \in B$.

Lemma 3. *Suppose G is a block graph and A is either an end block of G with a cut–vertex x or a block without a cut–vertex and x is any vertex of A. Suppose $V(A - x) = \{v_1, ..., v_n\}$ with $v_1 \in B \cup R$ and $v_2, ...\ v_n \in B$.*

(1) *Suppose n is even or $n = 1$ with $v_1 \in R$. If D' is an MMED–set of $G' \equiv G - V(A - x)$ with x relabeled by "covered", then $D^* \equiv D' \cup \{(x,v_1), (v_2,v_3), ..., (v_{n-2},v_{n-1})\}$ is an MMED–set of G.*

(2) *Suppose $n \geq 3$ is odd or $n = 1$ with $v_1 \in B$, i.e., n is odd and $v_n \in B$. If D' is an MMED–set of $G' \equiv G - V(A - x)$*

with x relabeled by "covered" (respectively, "required") when x is covered (respectively, "bound" or "required") in G, then $D^* \equiv D' \cup \{(v_1, v_2), ..., (v_{n-2}, v_{n-1})\}$ is an MMED–set of G.

Proof. (1) D^* is cleary a mixed edge dominating set of G. So we only have to prove that $|D^*| \leq |D|$ for any MMED–set D of G. Note that $|D^*| = |D'| + \lceil n/2 \rceil$. Since n is even with $v_1, ..., v_n \in B \cup R$ or $n = 1$ with $v_1 \in R$, D has at least $\lceil n/2 \rceil$ edges in A, for each v_i is only adjacent to edges in A. By the fact that x is relabeled by"covered" in G', $D - E(A)$ is a mixed edge dominating set of G'. Therefore, $|D'| \leq |D - E(A)| \leq |D| - \lceil n/2 \rceil$ and so $|D^*| = |D'| + \lceil n/2 \rceil \leq |D|$.

(2) D^* is a mixed edge dominating set of G by a straightforward check. So we only have to show that $|D^*| \leq |D|$ for any MMED–set D of G. Note that $|D^*| = D' + (n-1)/2$. Since $v_1, ..., v_n \in B \cup R$ and n is odd, D has at least $(n-1)/2$ edges both of whose ends are in $\{v_1, ..., v_n\}$. Let D'' be the result of replacing (x, y) by (x, x) in $D - E(A - x)$ when $y \in V(A - x)$. Then D'' is a mixed edge dominating set of G' and $|D''| \leq |D| - (n-1)/2$. Thus, $|D'| \leq |D''|$ and so $|D^*| = |D'| + (n-1)/2 \leq |D''| + (n-1)/2 \leq |D|$. ■

From the above lemmas, we obtain the following algorithm for the mixed edge dominating problem in block graphs:

Algorithm ED. Find a minimum mixed edge dominating set of a block graph.

Input: A block graph $G = (V, E)$ whose vertex set V is partitioned into B, C, and R.

Output: A minimum mixed edge dominating set D of G.

Begin
label(y)=b for all $y \in B$;
label(y)=c for all $y \in C$;
label(y)=r for all $y \in R$;
$G' \leftarrow G$;
$D \leftarrow \emptyset$;
do while $(G' \neq \emptyset)$
choose a block A with at most one cut–vertex in G';
Case 1. A has only one vertex x.
if label(x)= r **then**

if $(x, y) \in E(G)$ **then** $D \leftarrow D \cup \{(x, y)\}$
else $D \leftarrow D \cup \{(x, x)\}$;
$G' \leftarrow G' - x$;
Case 2. A has a cut–vertex x, or A has at least two vertices but no cut–vertex (choose any $x \in A$ in this case).
$A' \leftarrow V(A - x)$;
asumme $A' = \{w_1, ..., w_r, u_1, ..., u_s, y_1, ..., y_t\}$
where all label$(w_i) = c$, all label$(u_j) = r$, all label$(y_k) = b$;
$D \leftarrow D \cup \{(u_1, u_2), ..., (u_{m-1}, u_m)\}$ where $m = 2\lfloor s/2 \rfloor$;
$A' \leftarrow A' - \{w_1, ..., w_r, u_1, ..., u_m\}$;
asumme $A' = \{v_1, ..., v_n\}$ where label$(v_1) = r/b$ and all other label$(v_i) = b$;
if n is even or $n = 1$ with label$(v_1) = r$
then $[D \leftarrow D \cup \{(x, v_1), (v_2, v_3), ..., (v_{n-2}, v_{n-1})\}$; label$(x) \leftarrow c$;]
else $[D \leftarrow D \cup \{(x_1, v_2), (v_3, v_4), ..., (v_{n-2}, v_{n-1})\}$;
if label$(x) = b$ **then** label$(x) \leftarrow r$;]
$G' \leftarrow G' - A'$;
end while;
End.

Note that after handling a block with at least two vertices but no cut–vertex, as in *Case* 2, a block of only one vertex is created, which is going to be handled by *Case* 1.

Theorem 4. *Algorithm ED finds a minimum mixed edge dominating set of a block graph in linear time.*

Proof. The correctness of the algorithm follows from Lemmas 1 to 3. All steps of the algorithm obviously run in linear time except the part of choosing a block with at most one cut–vertex. However, a depth first search (see [1]) provides a linear algorithm for constructing the block–cut–vertex structure G^* of G. A vertex of degree at most one in G^* corresponds to a block with at most one cut–vertex in G. We can, of course, merge two algorithms together without actually producing G^*. ■

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