# A NOTE ON CAREFUL PACKING OF A GRAPH 

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#### Abstract

Let $G$ be a simple graph of order $n$ and size $e(G)$. It is well known that if $e(G) \leq n-2$, then there is an edge-disjoint placement of two copies of $G$ into $K_{n}$. We prove that with the same condition on size of $G$ we have actually (with few exceptions) a careful packing of $G$, that is an edge-disjoint placement of two copies of $G$ into $K_{n} \backslash C_{n}$.


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## 1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n=|V(G)|$ and size $e(G)=|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

For graphs $G$ and $H$ we denote by $G \cup H$ the vertex disjoint union of graphs $G$ and $H$ and $k G$ stands for the disjoint union of $k$ copies of the graph $G$.

Suppose $G_{1}, \ldots, G_{k}$ are graphs of order $n$. We say that there is a packing of $G_{1}, \ldots, G_{k}$ (into the complete graph $K_{n}$ ) if there exist injections $\alpha_{i}: V\left(G_{i}\right) \longrightarrow V\left(K_{n}\right), i=1, \ldots, k$, such that $\alpha_{i}^{*}\left(E\left(G_{i}\right)\right) \cap \alpha_{j}^{*}\left(E\left(G_{j}\right)\right)=$ $\emptyset$ for $i \neq j$, where the map $\alpha_{i}^{*}: E\left(G_{i}\right) \longrightarrow E\left(K_{n}\right)$ is induced by $\alpha_{i}$.

A packing of $k$ copies of a graph $G$ will be called a $k$-placement of $G$. A packing of two copies of $G$ i.e. a $2-$ placement is an embedding of $G$ (in its complement $\bar{G}$ ). So, an embedding of a graph $G$ is a permutation

[^0]$\sigma$ on $V(G)$ such that if an edge $x y$ belongs to $E(G)$ then $\sigma(x) \sigma(y)$ does not belong to $E(G)$.

A careful packing of a graph $G$ is a packing of $C_{n}$ and two copies of $G$ into the complete graph. In others words this is an edge-disjoint placement of two copies of $G$ into $K_{n} \backslash C_{n}$. Geometrically speaking, if we identify the cycle $C_{n}$ with a convex $n$-gon on the plane, the careful packing of $G$ means the possibility to draw (edge-disjointly) two copies of $G$ using only the internal edges.

The following theorem was proved, independently, in [2], [4] and [7].
Theorem 1. Let $G=(V, E)$ be a graph of order n. If $|E(G)| \leq n-2$, then $G$ can be embedded in its complement $\bar{G}$.

The example of the star $K_{1, n-1}$ shows that Theorem 1 cannot be improved by increasing the size of $G$.

This result have been improved in many ways. For instance, the following theorem completely characterizes those graphs with $n$ vertices and $n-1$ edges which are embeddable ([5], [6].

Theorem 2. Let $G=(V, E)$ be a graph of order n. If $|E(G)| \leq n-1$, then either $G$ is embeddable or $G$ is isomorphic to one of the following graphs : $K_{1, n-1}, K_{1, n-4} \cup K_{3} \quad$ for $n \geq 8, K_{1} \cup 2 K_{3}, K_{1} \cup C_{4}, K_{1} \cup K_{3}$ and $K_{2} \cup K_{3}$.

Remark. For other generalization and improvements of Theorem 2 see for instance [8], [9] or [10]. The general references here are [11] and [1] (see also [12]).

Our purpose is to prove the following

Theorem 3. Let $G$ be a graph of order $n, \quad n \geq 6$. If $e(G) \leq n-2$, then there exists a careful packing of $G$ except for two graphs of order 6 : $K_{3} \cup K_{2} \cup K_{1}$ and $C_{4} \cup 2 K_{1}$, and for two families of graphs: $K_{1, n-2} \cup K_{1}$ and $K_{1, n-3} \cup K_{2}$.

The proof the theorem is given in the next section.

Corollary 4. Let $G$ be a graph of order $n, \quad n \geq 3$. If $e(G) \leq n-3$, then there exists a careful packing of $G$.

Proof. The corollary is evident for $n=3$ and 4 and easy to verify for $n=5$. For $n \geq 6$ it follows from Theorem 3 .

We finish this section with some remarks.
Observe first that if we want to pack two copies of a graph $G$ together with the cycle $C_{n}$, then the following necessary condition must hold:

$$
\Delta(G)+\delta(G) \leq n-3
$$

For, the vertex $u$ with $d(u)=\Delta(G)$ must be placed with another vertex of $G$ and with a vertex of $C_{n}$ of degree 2 . Another evident, necessary condition is determined by the number of edges in the complete graph $K_{n}$. We must have $2(n-2)+n \leq\binom{ n}{2}$ which implies $n \geq 6$.

So, from this point of view, there are only two "small" exceptional graphs in Theorem 3.

Since it is very easy to find a 2 -placement for exceptional graphs of Theorem 3, so this theorem is an improvement of Theorem 1. On the other hand, Corollary 4 can also be considered as an improvement of the following theorem of Ore (cf.[3]).

Theorem 5. If $G$ is a simple graph of order $n \geq 3$ and $e(G)>\binom{n-1}{2}+1$, then $G$ is Hamiltonian.

Indeed, restated in terms of packing, Theorem 5 states that if $G$ is a graph of order $n, \quad n \geq 3$, and $e(G) \leq n-3$, then there is a packing of $G$ into $K_{n} \backslash C_{n}$, whereas Corollary 4 ensures a packing of two copies of $G$ into $K_{n} \backslash C_{n}$.

## 2. Proof

We start with some simple observations formulated as lemmas.
Lemma 6. Let $G$ be a graph composed of the cycle $C_{k}$ and one vertex, say $u$, not on the cycle. Denote by $\left|N\left(u, C_{k}\right)\right|$ the number of edges connecting $u$ with $C_{k}$. If $\left|N\left(u, C_{k}\right)\right|>\frac{k}{2}$, then the cycle $C_{k}$ can be extended to a cycle of length $k+1$ passing through $u$.

Lemma 7. Let $G$ be a graph composed of the cycle $C_{k}$ and two vertices, say $u, v$, not on the cycle. If

1. $u v \in E(G)$,
2. $\left|N\left(u, C_{k}\right)\right| \geq 1,\left|N\left(v, C_{k}\right)\right| \geq 1$,
3. $\left|N\left(u, C_{k}\right)\right|+\left|N\left(v, C_{k}\right)\right| \geq k+1$,
then the cycle $C_{k}$ can be extended to a cycle of length $k+2$ passing through $u$ and $v$.

Proof. It is easy to see that at least one of the neighbours of the vertex $v$ on the cycle $C_{k}$ has as its neighbour on the cycle $C_{k}$, a vertex connected by an edge with the vertex $u$. The possibility to extend the cycle $C_{k}$ to the cycle $C_{k+2}$ is now evident.

Lemma 8. If the graph $G$ has an end-vertex, say $x$, adjacent to the vertex, say $y$, of degree $d(y) \geq \frac{n-1}{2}$ and there is a careful packing of $G^{\prime}=G \backslash\{x\}$, then there is a careful packing of the graph $G$.

Proof. Observe first that in the careful packing of $G^{\prime}$ the image of $y$ is distinct from $y$. Indeed, otherwise we would have too many edges adjacent to $y$ in $K_{n-1}$ (two edges of $C_{n-1}$ and at least $n-2$ edges belonging to two copies of $G^{\prime}$ ).

Thus it is easy to extend the packing of $G^{\prime}$ (by putting $x$ on $x$ ) and then to extend $C_{n-1}$ by applying Lemma 6 to the complement of the graph $G$.

Proof of Theorem 3. In the remainder of this section we adopt the following convention: Given a careful packing of a graph $G$, we say that an edge $e$ of $K_{n}$ is black or blue if it belongs to the first or second copy of $G$, respectively, and that an edge $e$ of $K_{n}$ is red if it belongs to the corresponding cycle $C_{n}$.

The proof is by induction on $n$. Without loss of generality we may assume that all the graphs under consideration are of maximum size $n-2$. Let us start with small values of $n$ i.e. $n=6$ and $n=7$. It is easy to see that there are five graphs of order 6 and size 4 which are not exceptional: $K_{1} \cup P_{5}, K_{1} \cup S_{5}^{\prime}, K_{2} \cup P_{4}, 2 P_{3}$ and $2 K_{1} \cup\left(S_{3}+e\right)$. The careful packings of these graphs are depicted in Figure 1 (the edges of $C_{6}$ are not marked). Observe that they can be used to obtain the careful packings of ( $n, n-2$ )graphs for $n=7$. We can also use Lemma 8. The details are left to the reader.


Figure 1. Carefull packing of graphs of order 6

Suppose now that the theorem is true for all $n^{\prime}<n$ and let $G$ be an $(n, n-2)$-graph. Assume also that $G$ is not one of the exceptional graphs. We shall consider two main cases.

Case 1. $G$ has two independent end-edges.
Denote the independent end-edges of $G$ by $u u^{\prime}$ and $v v^{\prime}, u, v$ being the corresponding end-vertices of $G$. Consider now the graph $G^{\prime}=G \backslash\{u, v\}$. Suppose that there exists a careful packing for $G^{\prime}$, say $\sigma^{\prime}$. It is easy to extend the bijection $\sigma^{\prime}$ to a packing of $G$. Moreover, since the edge $u v$ is neither black nor blue, we can consider it as a red one. We assign the red colour also to $n-4$ edges connecting $u$ with $C_{n-2}$ and to $n-4$ edges connecting $v$ with $C_{n-2}$. By Lemma 7 (with $k=n-2$ ) the careful packing of $G$ exists. The case where $G^{\prime}$ is an exceptional graph will be considered below as Case 3 .

Case 2. $G$ has not two independent end-edges.
Since $G$ has at least two tree components, the above condition implies that at least one of them is trivial and the other is a star. Let $u$ be an isolated vertex of $G$ and let $x$ be a vertex defined by

$$
d_{G}(x)=\min \left\{d_{G}(y): y \in V(G), d_{G}(y) \geq 2\right\}
$$

We consider the graph $G^{\prime}=G \backslash\{u, x\}$. Suppose that $G^{\prime}$ is not one of the exceptional graphs; other cases are considered below as Case 3. Then there exists a careful packing for $G^{\prime}$, say $\sigma^{\prime}$. It is evident that by putting $x$ on $u$ and $u$ on $x$ we extend $\sigma^{\prime}$ to a packing of $G$. We may assume that the vertices $x$ and $u$ send $n-2-d(x)$ red edges to the red cycle $C_{n-2}$ contained in $G^{\prime}$. We can apply Lemma 7 and obtain a careful packing of $G$ if $2\left(n-2-d_{G}(x)\right) \geq n-1$. Hence $n-3 \geq 2 d_{G}(x)$.

Thus, we may assume that

$$
\begin{equation*}
d_{G}(x) \geq \frac{n-2}{2} \tag{*}
\end{equation*}
$$

So, for $n \geq 7, d_{G}(x) \geq 3$. Consider first the case where $G$ has two trivial components.

Case 2 (a) $G$ has two isolated vertices, say $u, v$.
Consider first the case $n=8$. The case by case examination shows that: either $G$ contains an end-vertex such that we can apply Lemma 8 , or $G$ is such that the graph $G^{\prime}=G \backslash\{u, x\}$ is exceptional (see Case 3). So, we may assume that $n \geq 9$. Consider now the graph $G_{1}=G \backslash\{u, v, x\}$. If $G_{1}$ is not one of the exceptional graphs, we can apply the induction hypothesis. Let $\sigma_{1}$ be a careful packing of $G_{1}$. Denote by $y_{1}$ a vertex of $G_{1}$ non adjacent to $x$ (such a vertex exists by the definition of $x$ ). Without loss of generality we may assume that $y_{1}$ is the first vertex on the red cycle $y_{1}, y_{2}, \ldots, y_{n-3}$ corresponding to the careful packing of $G_{1}$. Then the cycle $x y_{1} y_{2} \ldots y_{n-3} u v x$ can be considered as a red cycle of the careful packing of $G$, say $\sigma$, obtained from $\sigma^{\prime}$ by putting $\sigma(x)=v, \sigma(v)=x, \sigma(u)=u$ and $\sigma(w)=\sigma^{\prime}(w)$ for $w \in V(G) \backslash\{u, v, x\}$.

Case 2 (b) $G$ has only one isolated vertex.
Hence $G$ is of the form $K_{1} \cup K_{1, r} \cup R$ where $r \geq 1$ and the graph $R$ has no isolated vertices. Moreover, since by Case $1, R$ contains no end-vertices we may assume, by $\left({ }^{*}\right)$, that either all vertices of $R$ are of a degree greater than or equal to $\frac{n-2}{2}$, or $R$ is empty. In the first case, for $n>6$, this contradicts the fact that the average degree of $R$ is equal to 2 . In the second case $G$ is exceptional, a contradiction.

Case 3. $G^{\prime}$ is one of the exceptional graphs, where $G^{\prime}$ denotes one of the graphs defined in Cases 1 or $2(n \geq 8)$.

We shall need some additional notations. Namely, by $S_{p}^{\prime}$ we denote a tree of order $p$ obtained by subdividing one of the edges of the star $K_{1, p-2}$ and by ( $K_{1, p-1}+e$ ) we denote, as usually, the graph of order $p$ obtained by adding one edge to the edge-set of the star $K_{1, p-1}$.


Figure 2. Carefull packing of $K_{2} \cup P_{3} \cup C_{3}$ and $K_{1} \cup K_{1,3} \cup C_{3}$

Without loss of generality we may assume that every other choice of two or three (for $n \geq 9$ ) vertices in a way described in Cases 1 and 2 leads also to one of the exceptional graphs. Of course, we can proceed as in Case 2 also in the case where the graph $G$ has two independent end-edges.

Recall that $G$ itself is not an exceptional graph.
The case by case examination shows that then $G$ belongs to one of the following families of graphs: $P_{3} \cup K_{1, n-4}, K_{1} \cup S_{n-1}^{\prime}, 2 K_{1} \cup\left(K_{1, n-3}+e\right)$, $K_{1} \cup K_{3} \cup K_{1, n-5}$, or $n=8$ and $G$ is isomorphic to $4 K_{1} \cup K_{4}, 2 K_{1} \cup 2 K_{3}$ $2 K_{2} \cup C_{4}, K_{2} \cup P_{3} \cup C_{3}$ or $3 K_{1} \cup K_{2,3}$.

Observe that in all graphs belonging to the above mentioned families, except for $K_{1} \cup K_{3} \cup K_{1,3}$, there is a vertex of a degree greater than or equal to $n-4$, so we can apply Lemma 8 (since $n \geq 8$ ).

The careful packings of $4 K_{1} \cup K_{4}, 2 K_{2} \cup C_{4}$ or $2 K_{1} \cup 2 K_{3}$ are very symmetric and easy to find.

The careful packing of $K_{2} \cup P_{3} \cup C_{3}$ as well as the careful packing of $K_{1} \cup K_{3} \cup K_{1,3}$ are depicted in Fig. 2.

Finally, the careful packing of $3 K_{1} \cup K_{2,3}$ can be easily obtained from the careful packing of $2 K_{1} \cup K_{1,3}$ into $K_{6}$.

This completes the proof of the theorem.

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