A NOTE ON CAREFUL PACKING OF A GRAPH

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Abstract

Let G be a simple graph of order n and size $e(G)$. It is well known that if $e(G) \leq n-2$, then there is an edge–disjoint placement of two copies of G into K_n. We prove that with the same condition on size of G we have actually (with few exceptions) a careful packing of G, that is an edge–disjoint placement of two copies of G into $K_n \setminus C_n$.

Keywords: pucking of graphs

1991 Mathematics Subject Classification: 05C70

1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

For graphs G and H we denote by $G \cup H$ the *vertex disjoint union* of graphs G and H and kG stands for the disjoint union of k copies of the graph G.

Suppose $G_1, \ldots, G_k$ are graphs of order n. We say that there is a *packing* of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist injections $\alpha_i : V(G_i) \longrightarrow V(K_n)$, $i = 1, \ldots, k$, such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \longrightarrow E(K_n)$ is induced by α_i.

A packing of k copies of a graph G will be called a *k–placement* of G. A packing of two copies of G i.e. a 2–placement is an *embedding* of G (in its complement $\overline{G}$). So, an embedding of a graph G is a permutation

[1]This paper was partially supported by Polish Research Grant KBN 2 P 301 050 031

σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

A *careful packing* of a graph G is a packing of C_n and two copies of G into the complete graph. In others words this is an edge–disjoint placement of two copies of G into $K_n \setminus C_n$. Geometrically speaking, if we identify the cycle C_n with a convex n–gon on the plane, the careful packing of G means the possibility to draw (edge-disjointly) two copies of G using only the internal edges.

The following theorem was proved, independently, in [2], [4] and [7].

Theorem 1. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$, then G can be embedded in its complement $\overline{G}$.*

The example of the star $K_{1,n-1}$ shows that Theorem 1 cannot be improved by increasing the size of G.

This result have been improved in many ways. For instance, the following theorem completely characterizes those graphs with n vertices and $n-1$ edges which are embeddable ([5], [6].

Theorem 2. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 1$, then either G is embeddable or G is isomorphic to one of the following graphs : $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ for $n \geq 8$, $K_1 \cup 2K_3$, $K_1 \cup C_4$, $K_1 \cup K_3$ and $K_2 \cup K_3$.*

Remark . For other generalization and improvements of Theorem 2 see for instance [8], [9] or [10]. The general references here are [11] and [1] (see also [12]).

Our purpose is to prove the following

Theorem 3. *Let G be a graph of order n, $n \geq 6$. If $e(G) \leq n - 2$, then there exists a careful packing of G except for two graphs of order 6: $K_3 \cup K_2 \cup K_1$ and $C_4 \cup 2K_1$, and for two families of graphs: $K_{1,n-2} \cup K_1$ and $K_{1,n-3} \cup K_2$.*

The proof the theorem is given in the next section.

Corollary 4 . *Let G be a graph of order n, $n \geq 3$. If $e(G) \leq n - 3$, then there exists a careful packing of G.*

Proof. The corollary is evident for $n = 3$ and 4 and easy to verify for $n = 5$. For $n \geq 6$ it follows from Theorem 3. ■

We finish this section with some remarks.

Observe first that if we want to pack two copies of a graph G together with the cycle C_n, then the following necessary condition must hold:

$$\Delta(G) + \delta(G) \leq n - 3.$$

For, the vertex u with $d(u) = \Delta(G)$ must be placed with another vertex of G and with a vertex of C_n of degree 2. Another evident, necessary condition is determined by the number of edges in the complete graph K_n. We must have $2(n-2) + n \leq \binom{n}{2}$ which implies $n \geq 6$.

So, from this point of view, there are only two "small" exceptional graphs in Theorem 3.

Since it is very easy to find a 2–placement for exceptional graphs of Theorem 3, so this theorem is an improvement of Theorem 1. On the other hand, Corollary 4 can also be considered as an improvement of the following theorem of Ore (cf.[3]).

Theorem 5. *If G is a simple graph of order $n \geq 3$ and $e(G) > \binom{n-1}{2} + 1$, then G is Hamiltonian.*

Indeed, restated in terms of packing, Theorem 5 states that if G is a graph of order n, $n \geq 3$, and $e(G) \leq n - 3$, then there is a packing of G into $K_n \setminus C_n$, whereas Corollary 4 ensures a packing of two copies of G into $K_n \setminus C_n$.

2. Proof

We start with some simple observations formulated as lemmas.

Lemma 6. *Let G be a graph composed of the cycle C_k and one vertex, say u, not on the cycle. Denote by $|N(u, C_k)|$ the number of edges connecting u with C_k. If $|N(u, C_k)| > \frac{k}{2}$, then the cycle C_k can be extended to a cycle of length $k + 1$ passing through u.* ■

Lemma 7. *Let G be a graph composed of the cycle C_k and two vertices, say u, v, not on the cycle. If*

1. $uv \in E(G)$,
2. $|N(u, C_k)| \geq 1, \ |N(v, C_k)| \geq 1$,
3. $|N(u, C_k)| + |N(v, C_k)| \geq k + 1$,

then the cycle C_k *can be extended to a cycle of length* $k+2$ *passing through* u *and* v.

Proof. It is easy to see that at least one of the neighbours of the vertex v on the cycle C_k has as its neighbour on the cycle C_k, a vertex connected by an edge with the vertex u. The possibility to extend the cycle C_k to the cycle C_{k+2} is now evident. ■

Lemma 8. *If the graph* G *has an end-vertex, say* x, *adjacent to the vertex, say* y, *of degree* $d(y) \geq \frac{n-1}{2}$ *and there is a careful packing of* $G' = G \setminus \{x\}$, *then there is a careful packing of the graph* G.

Proof. Observe first that in the careful packing of G' the image of y is distinct from y. Indeed, otherwise we would have too many edges adjacent to y in K_{n-1} (two edges of C_{n-1} and at least $n-2$ edges belonging to two copies of G').

Thus it is easy to extend the packing of G' (by putting x on x) and then to extend C_{n-1} by applying Lemma 6 to the complement of the graph G. ■

Proof of Theorem 3. In the remainder of this section we adopt the following convention: Given a careful packing of a graph G, we say that an edge e of K_n is *black* or *blue* if it belongs to the first or second copy of G, respectively, and that an edge e of K_n is *red* if it belongs to the corresponding cycle C_n.

The proof is by induction on n. Without loss of generality we may assume that all the graphs under consideration are of maximum size $n-2$. Let us start with small values of n i.e. $n = 6$ and $n = 7$. It is easy to see that there are five graphs of order 6 and size 4 which are not exceptional: $K_1 \cup P_5$, $K_1 \cup S_5'$, $K_2 \cup P_4$, $2P_3$ and $2K_1 \cup (S_3 + e)$. The careful packings of these graphs are depicted in Figure 1 (the edges of C_6 are not marked). Observe that they can be used to obtain the careful packings of $(n, n-2)$–graphs for $n = 7$. We can also use Lemma 8. The details are left to the reader.

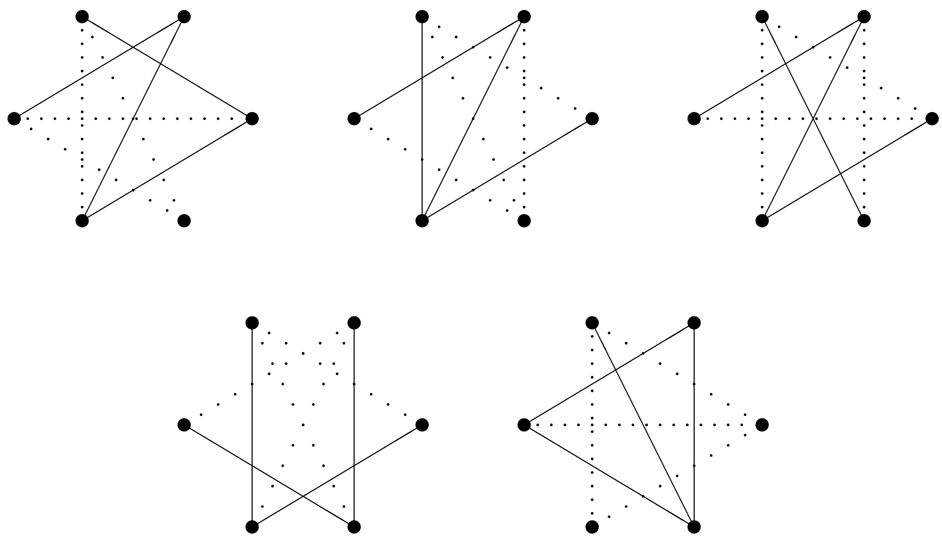

Figure 1. Carefull packing of graphs of order 6

Suppose now that the theorem is true for all $n' < n$ and let G be an $(n, n-2)$–graph. Assume also that G is not one of the exceptional graphs. We shall consider two main cases.

Case 1. G has two independent end-edges.
Denote the independent end-edges of G by uu' and vv', u, v being the corresponding end-vertices of G. Consider now the graph $G' = G \setminus \{u, v\}$. Suppose that there exists a careful packing for G', say σ'. It is easy to extend the bijection σ' to a packing of G. Moreover, since the edge uv is neither black nor blue, we can consider it as a red one. We assign the red colour also to $n-4$ edges connecting u with C_{n-2} and to $n-4$ edges connecting v with C_{n-2}. By Lemma 7 (with $k = n-2$) the careful packing of G exists. The case where G' is an exceptional graph will be considered below as *Case* 3.

Case 2. G has not two independent end-edges.
Since G has at least two tree components, the above condition implies that at least one of them is trivial and the other is a star. Let u be an isolated vertex of G and let x be a vertex defined by

$$d_G(x) = \min\{d_G(y) : y \in V(G)\ ,\ d_G(y) \geq 2\}$$

We consider the graph $G' = G \setminus \{u, x\}$. Suppose that G' is not one of the exceptional graphs; other cases are considered below as *Case* 3. Then there exists a careful packing for G', say σ'. It is evident that by putting x on u and u on x we extend σ' to a packing of G. We may assume that the vertices x and u send $n-2-d(x)$ red edges to the red cycle C_{n-2} contained in G'. We can apply Lemma 7 and obtain a careful packing of G if $2(n-2-d_G(x)) \geq n-1$. Hence $n-3 \geq 2d_G(x)$.

Thus, we may assume that

$$(*) \qquad d_G(x) \geq \frac{n-2}{2}$$

So, for $n \geq 7, d_G(x) \geq 3$. Consider first the case where G has two trivial components.

Case 2 (a) G has two isolated vertices, say u, v.

Consider first the case $n = 8$. The case by case examination shows that: either G contains an end-vertex such that we can apply Lemma 8, or G is such that the graph $G' = G \setminus \{u, x\}$ is exceptional (see *Case* 3). So, we may assume that $n \geq 9$. Consider now the graph $G_1 = G \setminus \{u, v, x\}$. If G_1 is not one of the exceptional graphs, we can apply the induction hypothesis. Let σ_1 be a careful packing of G_1. Denote by y_1 a vertex of G_1 non adjacent to x (such a vertex exists by the definition of x). Without loss of generality we may assume that y_1 is the first vertex on the red cycle $y_1, y_2, \ldots, y_{n-3}$ corresponding to the careful packing of G_1. Then the cycle $xy_1y_2 \ldots y_{n-3}uvx$ can be considered as a red cycle of the careful packing of G, say σ, obtained from σ' by putting $\sigma(x) = v$, $\sigma(v) = x$, $\sigma(u) = u$ and $\sigma(w) = \sigma'(w)$ for $w \in V(G) \setminus \{u, v, x\}$.

Case 2 (b) G has only one isolated vertex.

Hence G is of the form $K_1 \cup K_{1,r} \cup R$ where $r \geq 1$ and the graph R has no isolated vertices. Moreover, since by *Case* 1, R contains no end-vertices we may assume, by (*), that either all vertices of R are of a degree greater than or equal to $\frac{n-2}{2}$, or R is empty. In the first case, for $n > 6$, this contradicts the fact that the average degree of R is equal to 2. In the second case G is exceptional, a contradiction.

Case 3. G' is one of the exceptional graphs, where G' denotes one of the graphs defined in *Cases* 1 or 2 ($n \geq 8$).

We shall need some additional notations. Namely, by S'_p we denote a tree of order p obtained by subdividing one of the edges of the star $K_{1,p-2}$ and by $(K_{1,p-1} + e)$ we denote, as usually, the graph of order p obtained by adding one edge to the edge-set of the star $K_{1,p-1}$.

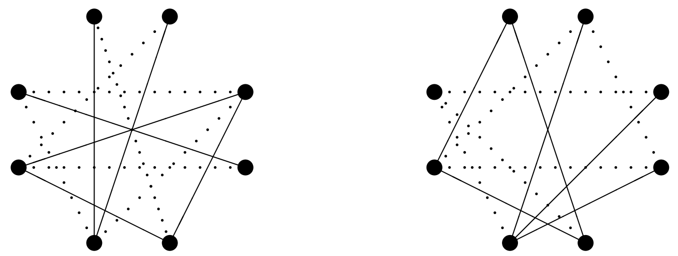

Figure 2. Carefull packing of $K_2 \cup P_3 \cup C_3$ and $K_1 \cup K_{1,3} \cup C_3$

Without loss of generality we may assume that every other choice of two or three (for $n \geq 9$) vertices in a way described in *Cases* 1 and 2 leads also to one of the exceptional graphs. Of course, we can proceed as in *Case* 2 also in the case where the graph G has two independent end-edges.

Recall that G itself is not an exceptional graph.

The case by case examination shows that then G belongs to one of the following families of graphs: $P_3 \cup K_{1,n-4}$, $K_1 \cup S'_{n-1}$, $2K_1 \cup (K_{1,n-3} + e)$, $K_1 \cup K_3 \cup K_{1,n-5}$, or $n = 8$ and G is isomorphic to $4K_1 \cup K_4$, $2K_1 \cup 2K_3$ $2K_2 \cup C_4$, $K_2 \cup P_3 \cup C_3$ or $3K_1 \cup K_{2,3}$.

Observe that in all graphs belonging to the above mentioned families, except for $K_1 \cup K_3 \cup K_{1,3}$, there is a vertex of a degree greater than or equal to $n - 4$, so we can apply Lemma 8 (since $n \geq 8$).

The careful packings of $4K_1 \cup K_4$, $2K_2 \cup C_4$ or $2K_1 \cup 2K_3$ are very symmetric and easy to find.

The careful packing of $K_2 \cup P_3 \cup C_3$ as well as the careful packing of $K_1 \cup K_3 \cup K_{1,3}$ are depicted in Fig. 2.

Finally, the careful packing of $3K_1 \cup K_{2,3}$ can be easily obtained from the careful packing of $2K_1 \cup K_{1,3}$ into K_6.

This completes the proof of the theorem. ■

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Received 26 April 1994
Revised 14 July 1994