

## PROBLEMS REMAINING NP-COMPLETE FOR SPARSE OR DENSE GRAPHS

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### Abstract

For each fixed pair  $\alpha, c > 0$  let INDEPENDENT SET ( $m \leq cn^\alpha$ ) and INDEPENDENT SET ( $m \geq \binom{n}{2} - cn^\alpha$ ) be the problem INDEPENDENT SET restricted to graphs on  $n$  vertices with  $m \leq cn^\alpha$  or  $m \geq \binom{n}{2} - cn^\alpha$  edges, respectively. Analogously, HAMILTONIAN CIRCUIT ( $m \leq n + cn^\alpha$ ) and HAMILTONIAN PATH ( $m \leq n + cn^\alpha$ ) are the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with  $m \leq n + cn^\alpha$  edges. For each  $\epsilon > 0$  let HAMILTONIAN CIRCUIT ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) and HAMILTONIAN PATH ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) be the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with  $m \geq (1 - \epsilon)\binom{n}{2}$  edges.

We prove that these six restricted problems remain NP-complete. Finally, we consider sufficient conditions for a graph to have a Hamiltonian circuit. These conditions are based on degree sums and neighborhood unions of independent vertices, respectively. Lowering the required bounds the problem HAMILTONIAN CIRCUIT jumps from 'easy' to 'NP-complete'.

**Keywords:** Computational Complexity, NP-Completeness, Hamiltonian Circuit, Hamiltonian Path, Independent Set

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### 1. MOTIVATION AND NOTATION

One of the most well-known problems in the theory of NP-completeness is the  $k$ -Satisfiability problem:

**$k$ -SATISFIABILITY**

INSTANCE: A set  $V$  of Boolean variables and a formula  $F$  of  $r$  different clauses in conjunctive normal form where each clause contains  $k$  literals in disjunctive normal form.

QUESTION: Is there a satisfying truth assignment for  $F$ ?

Recently it has been shown that  $k$ -SATISFIABILITY remains NP-complete when restricted to sparse as well as to dense formulas (cf. [7],[9]).

Let  $(k, s)$ -SATISFIABILITY be the  $k$ -SATISFIABILITY problem restricted to formulas  $F$  where each variable occurs at most  $s$  times. In [7] Kratochvíl, Savický and Tuza proved the following result for sparse formulas.

**Theorem 1.1.** *For each integer  $k \geq 3$  there exists an integer  $f(k)$  such that*

1. *every instance of  $(k, s)$ -SATISFIABILITY is satisfiable for  $s \leq f(k)$ ;*
2.  *$(k, s)$ -SATISFIABILITY is NP-complete for  $s \geq f(k) + 1$ .*

Note that  $(k, s)$ -SATISFIABILITY is solvable in polynomial time for each  $s \leq k$  (cf. [7]).

Let  $k$ -SATISFIABILITY  $(r > r_0)$  be the  $k$ -SATISFIABILITY problem restricted to formulas  $F$  with  $r > r_0$  clauses. In [9] we proved the following result for dense formulas.

**Theorem 1.2.** *For each  $k \geq 3$  and each  $l \geq 4$  with  $n \geq lk^2$   $k$ -SATISFIABILITY  $(r > \binom{n}{k}(2^k - 1 - 4/k))$  is NP-complete.*

Note that each formula  $F$  of  $k$ -SATISFIABILITY  $(r > \binom{n}{k}(2^k - 1))$  is unsatisfiable since there always exists a set of  $k$  variables such that all  $2^k$  possible clauses over the corresponding literals belong to  $F$ .

In section 2 we prove that the problems INDEPENDENT SET, HAMILTONIAN CIRCUIT and HAMILTONIAN PATH remain NP-complete when restricted to sparse or dense graphs. In section 3 we state several sufficient conditions in terms of degree-sums and neighborhood unions of vertices for a graph to have a Hamiltonian circuit, which can be checked in polynomial time. We show that HAMILTONIAN CIRCUIT becomes NP-complete when the corresponding bounds required for the degree-sums and neighborhood unions are lowered.

In this paper we only consider undirected and simple graphs (i.e., graphs without loops and multiple edges). Let  $G$  be a graph. By  $V(G)$  we denote the *vertex-set* of  $G$ , and by  $E(G)$  the *edge-set* of  $G$ . The cardinalities of  $V(G)$  and  $E(G)$  will be denoted by  $n$  and  $m$ , respectively. For a vertex  $v \in V(G)$  the *neighborhood*  $N(v)$  of  $v$  is the set of all vertices adjacent to  $v$ . The *degree* of a vertex  $v$  is denoted by  $d_G(v) = |N(v)|$ , or shortly,  $d(v)$ . The number of vertices in a maximum independent set of  $G$  is denoted by  $\alpha(G)$ . For  $k \leq \alpha(G)$  we define

$$\sigma_k(G) = \min\left\{\sum_{v \in S} d(v) \mid S \text{ is an independent set of } k \text{ vertices}\right\}$$

and

$$NC_k(G) = \min\left\{\left|\bigcup_{v \in S} N(v)\right| \mid S \text{ is an independent set of } k \text{ vertices}\right\}.$$

In these definitions we follow the convention that the minimum over an empty set is  $+\infty$ .

For further terminology and notations not defined here we refer to [6] (concerning complexity) and to [2] (concerning graph theory), respectively.

## 2. THREE PROBLEMS IN GRAPH THEORY

We now show that the problems INDEPENDENT SET, HAMILTONIAN CIRCUIT and HAMILTONIAN PATH remain NP-complete when restricted to sparse or dense graphs. Our technique of proof will be standard as described in [6]. In each proof we choose a known NP-complete problem  $\Pi_2$  and transform it to one of the considered problems  $\Pi_1$ . Clearly, in all cases our problem  $\Pi_1$  belongs to NP under restriction.

For each fixed pair  $\alpha, c > 0$  let INDEPENDENT SET ( $m \leq cn^\alpha$ ) and INDEPENDENT SET ( $m \geq \binom{n}{2} - cn^\alpha$ ) be the problem INDEPENDENT SET restricted to graphs with  $m \leq cn^\alpha$  or  $m \geq \binom{n}{2} - cn^\alpha$  edges, respectively. Analogously, for each fixed pair  $\alpha, c > 0$ , HAMILTONIAN CIRCUIT ( $m \leq n + cn^\alpha$ ) and HAMILTONIAN PATH ( $m \leq n + cn^\alpha$ ) are the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with  $m \leq n + cn^\alpha$  edges. For each  $\epsilon > 0$  let HAMILTONIAN CIRCUIT ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) and HAMILTONIAN PATH ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) be the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with  $m \geq (1 - \epsilon)\binom{n}{2}$  edges.

**Theorem 2.1.** *INDEPENDENT SET ( $m \leq cn^\alpha$ ) is NP-complete.*

**Proof.** We transform INDEPENDENT SET to INDEPENDENT SET ( $m \leq cn^\alpha$ ). Let  $G_1 = (V_1, E_1)$  be a graph on  $n_1$  vertices and  $m_1$  edges making up an arbitrary instance of INDEPENDENT SET. We now construct a graph  $G_2 = (V_2, E_2)$  on  $n_2$  vertices and  $m_2 = m_1$  edges by adding  $n_2 - n_1$  isolated vertices such that  $n_2 \geq \max\{n_1, \lceil (\frac{1}{c} \binom{n_1}{2})^{1/\alpha} \rceil\}$ . Then  $\alpha(G) = \alpha(G_1) + (n_2 - n_1)$  and  $G_2$  has  $m_2 = m_1 \leq \binom{n_1}{2} \leq cn_2^\alpha$  edges. For each fixed pair  $\alpha, c > 0$  we have  $m_2 = O(n_2^\alpha)$  which is bounded above by a polynomial function of  $n_1$ , since  $m_1 = O(n_1^2)$ .

For each positive  $K_1 \leq n_1$  let  $K_2 = K_1 + (n_2 - n_1)$ . Then  $0 < K_2 \leq n_2$  and  $G_2$  has an independent set of cardinality  $K_2$  or more if and only if  $G_1$  has an independent set of cardinality  $K_1$  or more. ■

Note that INDEPENDENT SET ( $m \leq k$ ) can be solved in time  $O(n^k)$ .

**Theorem 2.2.** *INDEPENDENT SET ( $m \geq \binom{n}{2} - cn^\alpha$ ) is NP-complete.*

**Proof.** We transform INDEPENDENT SET to INDEPENDENT SET ( $m \geq \binom{n}{2} - cn^\alpha$ ) and proceed as in the proof of theorem 2.1. This time we add a complete graph  $K_p$  on  $p = n_2 - n_1$  vertices  $v_1, v_2, \dots, v_p$  and join them to all vertices of  $G_1$ . Thus  $G_2$  has  $m_2 = \binom{n_2}{2} - \binom{n_1}{2} + m_1 \geq \binom{n_2}{2} - \binom{n_1}{2} \geq \binom{n_2}{2} - cn_2^\alpha$  edges. Furthermore,  $\alpha(G_1) = \alpha(G_2)$  by this construction. Now let  $K \leq n_1$  be positive. Then  $G_2$  has an independent set of cardinality  $K$  or more if and only if  $G_1$  has an independent set of cardinality  $K$  or more. ■

Note that INDEPENDENT SET ( $m \geq \binom{n}{2} - k$ ) can be solved in time  $O(n^k)$ .

**Remark.** Considering the problem CLIQUE in the complement of  $G$ , theorems 2.1 and 2.2 show that for each fixed pair  $\alpha, c > 0$  CLIQUE remains NP-complete for graphs on  $n$  vertices and  $m \leq cn^\alpha$  or  $m \geq \binom{n}{2} - cn^\alpha$  edges, respectively.

**Theorem 2.3.** *HAMILTONIAN CIRCUIT ( $m \leq n + cn^\alpha$ ) is NP-complete.*

**Proof.** We transform HAMILTONIAN PATH to HAMILTONIAN CIRCUIT ( $m \leq n + cn^\alpha$ ). Let  $G_1 = (V_1, E_1)$  be a graph on  $n_1$  vertices and  $m_1$  edges making up an arbitrary instance of HAMILTONIAN PATH. We now

construct a graph  $G_2 = (V_2, E_2)$  on  $n_2$  vertices and  $m_2$  edges by adding  $p \geq \max\{2, \lceil n_1^{2/\alpha} c^{-1/\alpha} \rceil - n_1\}$  vertices  $v_1, v_2, \dots, v_p$  inducing the path  $v_1 v_2 \dots v_p$  and joining  $v_1$  and  $v_p$  to all vertices of  $G_1$ . Thus  $n_2 = n_1 + p$  and  $m_2 = m_1 + p - 1 + 2n_1 = m_1 + n_2 + n_1 - 1 \leq n_2 + (n_1 - 1) + \binom{n_1}{2} < n_2 + n_1^2$ . Now  $n_2 \geq n_1^{2/\alpha} c^{-1/\alpha} \Leftrightarrow n_1^2 \leq c n_2^\alpha$  and thus  $m_2 \leq n_2 + c n_2^\alpha$ . Furthermore, for each fixed pair  $\alpha, c > 0$  we have  $m_2 = O(n_2^2)$  which is bounded above by a polynomial function of  $n_1$ .

If  $G_2$  has a Hamiltonian circuit  $C$ , then  $C$  contains the path  $u_1 v_1 v_2 \dots v_p u_2$  for two vertices  $u_1, u_2 \in V_1$ . Hence  $C$  contains also a path from  $u_1$  to  $u_2$  in  $G_1$  containing all vertices of  $G_1$ . Thus  $G_1$  has a Hamiltonian path. Conversely, if  $G_1$  has a Hamiltonian path, say  $u_1 u_2 \dots u_{n_1}$ , then  $u_1 u_2 \dots u_{n_1} v_1 v_2 \dots v_p u_1$  is a Hamiltonian circuit in  $G_2$ . Thus  $G_1$  has a Hamiltonian path if and only if  $G_2$  has a Hamiltonian circuit. ■

Note that HAMILTONIAN CIRCUIT ( $m \leq n + k$ ) can be solved in time  $O(n^k)$ .

**Theorem 2.4.** *HAMILTONIAN CIRCUIT ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) is NP-complete.*

**Proof.** We transform HAMILTONIAN PATH to HAMILTONIAN CIRCUIT ( $m \geq (1 - \epsilon)\binom{n}{2}$ ). Let  $G_1 = (V_1, E_1)$  be a graph making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph  $G_2 = (V_2, E_2)$  by adding a complete graph  $K_p$  on  $p \geq 3$  vertices  $v_1, v_2, \dots, v_p$  and joining  $v_1$  and  $v_2$  to all vertices of  $G_1$ , where  $p = n_2 - n_1$  and  $n_2 = \lceil \frac{2n_1}{\epsilon} \rceil$ . Thus  $n_2 = n_1 + p$  and  $G_2$  has

$$\begin{aligned} m_2 &= \binom{n_2 - n_1}{2} + 2n_1 + m_1 = \binom{n_2}{2} - n_1(n_2 - \frac{5}{2}) + \frac{n_1^2}{2} + m_1 \\ &> \binom{n_2}{2} - n_1(n_2 - 1) \geq \binom{n_2}{2} - \frac{\epsilon}{2} n_2(n_2 - 1) = (1 - \epsilon) \binom{n_2}{2} \end{aligned}$$

edges. For each fixed  $\epsilon > 0$  the graph  $G_2$  has size  $O(n_2^2)$  which is bounded above by a polynomial function of  $n_1$ .

If  $G_2$  has a Hamiltonian circuit then  $G_1$  has a Hamiltonian path since  $G[G_2 - \{v_1, v_2\}]$  consists of two components  $G[\{v_3, v_4, \dots, v_p\}]$  and  $G_1$ . If  $G_1$  has a Hamiltonian path, say  $u_1 u_2 \dots u_{n_1}$ , then  $u_1 u_2 \dots u_{n_1} v_1 v_3 v_4 \dots v_p v_2 u_1$  is a Hamiltonian circuit in  $G_2$ . Thus  $G_1$  has a Hamiltonian path if and only if  $G_2$  has a Hamiltonian circuit. ■

**Theorem 2.5** *HAMILTONIAN PATH* ( $m \leq n + cn^\alpha$ ) is NP-complete.

**Proof.** We transform HAMILTONIAN PATH to HAMILTONIAN PATH ( $m \leq n + cn^\alpha$ ). Let  $G_1 = (V_1, E_1)$  be a graph making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph  $G_2 = (V_2, E_2)$  by adding  $p \geq \max\{1, \lceil n_1^{2/\alpha} c^{-1/\alpha} \rceil - n_1\}$  vertices  $v_1, v_2, \dots, v_p$  inducing the path  $v_1 v_2 \dots v_p$  and joining  $v_1$  to all vertices of  $G_1$ . Thus  $n_2 = n_1 + p$  and  $G_2$  has  $m_2 = m_1 + p - 1 + n_1 = m_1 + n_2 - 1 \leq n_2 - 1 + \binom{n_1}{2} < n_2 + n_1^2$  edges. As in the proof of theorem 2.3 we obtain  $m_2 \leq n_2 + cn_2^\alpha$  and  $m_2 = O(n_1^{4/\alpha})$  for each fixed pair  $\alpha, c > 0$ .

If  $G_2$  has a Hamiltonian path  $P$  then  $P$  contains the path  $v_p \dots v_1 u$  for a vertex  $u \in V_1$ . Hence  $P$  contains also a path in  $G_1$  starting at  $u$  and containing all vertices of  $V_1$ . Thus  $G_1$  has a Hamiltonian path. Conversely, if  $G_1$  has a Hamiltonian path, say  $u_1 u_2 \dots u_{n_1}$ , then  $u_1 u_2 \dots u_{n_1} v_1 v_2 \dots v_p$  is a Hamiltonian path in  $G_2$ . Thus  $G_1$  has a Hamiltonian path if and only if  $G_2$  has a Hamiltonian path. ■

Note that HAMILTONIAN PATH ( $m \leq n + k$ ) can be solved in time  $O(n^k)$ .

**Theorem 2.6.** *HAMILTONIAN PATH* ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) is NP-complete.

**Proof.** We transform HAMILTONIAN PATH to HAMILTONIAN PATH ( $m \geq (1 - \epsilon)\binom{n}{2}$ ) and proceed as in the proof of theorem 2.4. This time, only one vertex ( $v_1$ ) of the complete graph  $K_p$  is joined to all vertices of  $G_1$ . Thus  $n_2 = n_1 + p$  and  $G_2$  has  $\binom{n_2 - n_1}{2} + n_1 + m_1$  edges.

If  $G_2$  has a Hamiltonian path then  $G_1$  has a Hamiltonian path, too, since  $G[G_2 - v_1]$  consists of two components  $G[\{v_2, v_3, \dots, v_p\}]$  and  $G_1$ . If  $G_1$  has a Hamiltonian path, say  $u_1 u_2 \dots u_{n_1}$ , then  $u_1 u_2 \dots u_{n_1} v_1 v_2 \dots v_p$  is a Hamiltonian path in  $G_2$ . Thus  $G_1$  has a Hamiltonian path if and only if  $G_2$  has a Hamiltonian path.

Everything else mentioned in the proof of theorem 2.4. remains valid. Thus the proof is complete. ■

### 3. SUFFICIENT CONDITIONS FOR HAMILTONIAN CIRCUITS

There are quite a lot of sufficient conditions for a graph to have a Hamiltonian circuit. The following result is due to Bondy [1] and generalizes the well-known theorems of Dirac ( $k = 0$ , [4]) and Ore ( $k = 1$ , [8]).

**Theorem 3.1.** *Let  $G$  be a  $k$ -connected graph of order  $n \geq 3$ . If  $\sigma_{k+1} \geq \frac{1}{2}((k+1)(n-1)+1)$  then  $G$  has a Hamiltonian circuit.*

A major improvement for the case  $k = 2$  has been established in [5] by Flandrin, Jung and Li.

**Theorem 3.2.** *Let  $G$  be a 2-connected graph of order  $n$  such that*  

$$d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$$
  
*for any independent set  $\{u, v, w\}$  of vertices. Then  $G$  has a Hamiltonian circuit.*

Note that the conditions required in theorem 3.1. and 3.2. can be checked in polynomial time of  $O(n^{k+1})$  and  $O(n^3)$ , respectively.

The best known sufficient condition using neighborhood unions of two independent vertices is due to Broersma, van den Heuvel and Veldman [3].

**Theorem 3.3.** *Let  $G$  be a graph of order  $n$  and  $NC_2 \geq \frac{1}{2}n$ . Then either  $G$  has a Hamiltonian circuit, or  $G$  is the Petersen graph, or  $G$  belongs to one of three families of exceptional graphs.*

**Remark.** It can be decided in polynomial time whether a given graph  $G$  is isomorphic to the Petersen graph (a graph on ten vertices) or belongs to one of the three families of exceptional graphs.

As in the previous section we now restrict HAMILTONIAN CIRCUIT to graphs with  $\sigma_{k+1} \geq (\frac{1}{2} - \epsilon)(k+1)n$ ,  $d(u) + d(v) + d(w) \geq (1 - \epsilon)n + |N(u) \cap N(v) \cap N(w)|$  for any three independent vertices or  $NC_2 \geq (\frac{1}{2} - \epsilon)n$ , respectively. The corresponding problems will be HAMILTONIAN CIRCUIT ( $\sigma_{k+1} \geq (\frac{1}{2} - \epsilon)(k+1)n$ ), HAMILTONIAN CIRCUIT ( $d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$ ) or HAMILTONIAN CIRCUIT ( $NC_2 \geq (\frac{1}{2} - \epsilon)n$ ), respectively.

**Theorem 3.4.** *HAMILTONIAN CIRCUIT ( $\sigma_{k+1} \geq (\frac{1}{2} - \epsilon)(k+1)n$ ) is NP-complete.*

**Proof.** We transform HAMILTONIAN PATH to HAMILTONIAN CIRCUIT ( $\sigma_{k+1} \geq (\frac{1}{2} - \epsilon)(k+1)n$ ). Let  $G_1 = (V_1, E_1)$  be a graph making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph  $G_2 = (V_2, E_2)$  by adding a complete graph  $K_p$  on  $p$  vertices  $v_1, v_2, \dots, v_p$  and  $p-1$  vertices  $v_{p+1}, v_{p+2}, \dots, v_{2p-1}$ , where  $p = \lceil (\frac{1}{2} - \epsilon)n_2 \rceil$  and  $n_2 = \lceil \frac{n_1}{2\epsilon} \rceil$ . Then  $n_2 = n_1 + 2p - 1$ . Let  $E_2 = E_1 \cup \{v_i v_j | 1 \leq i < j \leq p\} \cup \{v_i v_j | 1 \leq i \leq p, p+1 \leq j \leq 2p-1\} \cup \{v_i v | v \in V_1, 1 \leq i \leq p\}$ .

Then  $\sigma_{k+1} \geq (k+1)p \geq (\frac{1}{2} - \epsilon)(k+1)n_2$  for  $0 \leq k+1 \leq p$ , since  $\sigma_1 \geq (\frac{1}{2} - \epsilon)n_2$  and  $\alpha(G_2) = \alpha(G_1) + (p-1) \geq 1 + (p-1) = p$  by the construction. Thus there always exists an independent set  $I$  of size  $k+1$ , e.g.,  $I = \{v_{p+1}, v_{p+2}, \dots, v_{p+k}\} \cup \{v\}$  for a vertex  $v \in V_1$ . For each fixed  $\epsilon > 0$  we have  $m_2 = O(n_2^2)$  which is bounded above by a polynomial function of  $n_1$ .

If  $G_2$  has a Hamiltonian circuit then  $G_1$  has a Hamiltonian path since  $G[G_2 - \{v_1, v_2, \dots, v_p\}]$  consists of  $p$  components  $\{v_{p+1}\}, \{v_{p+2}\}, \dots, \{v_{2p-1}\}$  and  $G_1$ . If  $G_1$  has a Hamiltonian path, say  $u_1 u_2 \dots u_{n_1}$ , then  $u_1 u_2 \dots u_{n_1} v_1 v_{p+1} v_2 v_{p+2} \dots v_{2p-1} v_p u_1$  is a Hamiltonian circuit in  $G_2$ . Thus  $G_1$  has a Hamiltonian path if and only if  $G_2$  has a Hamiltonian circuit. ■

**Remark.** For  $n_1 = 2$  let  $G_1$  consist of two isolated vertices  $\{u, v\}$ . Then the graph  $G_2$  shows that the required conditions in theorems 3.1, 3.2 and 3.3 are sharp, since  $G[G_2 - \{v_1, v_2, \dots, v_p\}]$  consists of  $p+1$  components  $\{v_{p+1}\}, \{v_{p+2}\}, \dots, \{v_{2p-1}\}, \{u\}$  and  $\{v\}$ . Thus  $G_2$  has no Hamiltonian circuit.

**Theorem 3.5.** *HAMILTONIAN CIRCUIT* ( $d(u) + d(v) + d(w) \geq (1-\epsilon)n + |N(u) \cap N(v) \cap N(w)|$ ) is NP-complete.

**Proof.** We follow the proof of theorem 3.4 and let  $\epsilon_1 = \frac{\epsilon}{2}, p = \lceil (\frac{1}{2} - \epsilon_1)n_2 \rceil$ . Set  $S = \{v_1, v_2, \dots, v_p\}$ ,  $T = V_2 - S$ . If  $I = \{u, v, w\}$  is an independent set of three vertices, then  $I \subseteq T$  by the construction of  $G_2$ . For a set  $X \subseteq V(G)$  and a vertex  $v \in V(G)$  let  $N_X(v) := N(v) \cap X$ . By  $d_X(v)$  we denote the degree of  $v$  in  $X$ . Then

$$\begin{aligned} d(u) + d(v) + d(w) &= d_S(u) + d_S(v) + d_S(w) + d_T(u) + d_T(v) + d_T(w) \\ &\geq 3p + |N_T(u) \cap N_T(v) \cap N_T(w)| \\ &\geq 2(\frac{1}{2} - \epsilon_1)n_2 + |N_S(u) \cap N_S(v) \cap N_S(w)| \\ &\quad + |N_T(u) \cap N_T(v) \cap N_T(w)| \\ &= (1 - \epsilon)n_2 + |N(u) \cap N(v) \cap N(w)|. \end{aligned}$$

■

**Theorem 3.6.** *HAMILTONIAN CIRCUIT* ( $NC_2 \geq (\frac{1}{2} - \epsilon)n$ ) is NP-complete.

**Proof.** We follow the proof of theorem 3.4. With  $\alpha(G_2) \geq p+1 \geq 2$  and  $\sigma_1(G_2) \geq (\frac{1}{2} - \epsilon)n_2$  we have  $NC_2(G_2) \geq (\frac{1}{2} - \epsilon)n_2$ . ■



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