ON CHROMATICITY OF GRAPHS

Ewa Łazuka

Department of Applied Mathematics, Technical University of Lublin
Bernardyńska 13, 20–950 Lublin, Poland

Abstract

In this paper we obtain the explicit formulas for chromatic polynomials of cacti. From the results relating to cacti we deduce the analogous formulas for the chromatic polynomials of n–gon–trees. Besides, we characterize unicyclic graphs by their chromatic polynomials. We also show that the so–called clique–forest–like graphs are chromatically equivalent.

Keywords: chromatic polynomial, chromatically equivalent graphs, chromatic characterization

1991 Mathematics Subject Classification: 05C15

1. Introduction

The graphs considered here are finite, undirected, simple and loopless. Let $V(G)$ be the vertex set of a graph G and $E(G)$ be the edge set of G. Let $P(G,\lambda)$ denote the *chromatic polynomial* of the graph G. Two graphs are said to be *chromatically equivalent* if their chromatic polynomials are equal. A graph G is *chromatically unique* if $P(G,\lambda) = P(H,\lambda)$ implies that H is isomorphic to G. A class of graphs is *chromatically characterized* by their chromatic polynomials if for each graph G from this class we have $P(G,\lambda) = P(H,\lambda)$ if and only if H belongs to this class.

A *cactus* is a connected graph where any two cycles (assumed as graphs but not as sequences) have no edge in common. Chao and Whitehead [2] proved that the cactus graphs with the same number of vertices, the same number of edges and the same number of cycles of each length are chromatically equivalent. Here, in part 2, we shall give the explicit formula for the chromatic polynomial of these cacti.

Let $n \geq 3$ be an integer. The graphs called *n–gon–trees* are defined by recursion. The smallest n–gon–tree is the n–cycle, i.e. a cycle with n

vertices. A n–gon–tree with $k+1$ n–gons is obtained from a n–gon–tree with k n–gons by adding a new n–gon which has exactly one edge in common with some n–gon of a n–gon–tree with k n–gons. In [1], Chao and Li studied chromatic polynomials of these "trees of polygons" and obtained the chromatic characterization of n–gon–trees. Besides, Wakelin and Woodall gave another proof of this characterization in [8]. None of these papers provided an explicit formula of the chromatic polynomial of graphs mentioned above. In part 2, using the preceding results of this paper, we shall find the chromatic polynomial of n–gon–trees.

In part 3 we shall give the chromatic characterization of *unicyclic graphs*, i.e. connected graphs which contain exactly one cycle.

In part 4 we prove that graphs of a large class of nonisomorphic connected planar graphs, the so–called *clique–forest–like graphs*, are chromatically equivalent. This class includes *forest–like graphs* defined by Chao and Whitehead in [2]. In part 4 we generalize their result.

In this paper we make use of some results of graphs and chromatic polynomials due to Harary [5], Read [6] and Whitney [9].

2. CHROMATICITY OF CACTI AND n–GON–TREES

In this part the cycle with two vertices is meant as the complete graph of order two (no multiple edges are allowed).

We shall denote by $O(n_1,\ldots,n_m)$ a cactus with m cycles with n_i vertices for $i=1,2,\ldots,m$, respectively, We assume for the cactus $O(n_1,\ldots,n_m)$ that $n_i \geq 2$, for $i=1,2,\ldots,m$.

We shall use the following notations in the next part of this paper:

$$I_0^m = \{\emptyset\},$$

$$I_k^m = \{\ \{i_1,i_2,\ldots,i_k\} :\ i_1<i_2<\ldots<i_k;\ i_1,i_2,\ldots,i_k \in \{1,2,\ldots,m\}\ \},$$

$$S(A)=\sum_{i\in A} n_i,\ \text{for } A\in I_k^m.$$

Thus, $S(\emptyset)=0$ and $S(I_1^m)=\sum_{i=1}^m n_i$.
Although the proof of the below theorem is very long, we shall present it in full because Theorem 1 will be used to derive further corollaries.

Theorem 1. *A cactus* $O(n_1,\ldots,n_m)$*, where* $m \geq 1$ *and* $n_i \geq 2$ *for* $i = 1,2,\ldots,m$*, has the following chromatic polynomial*

$$P\left(O(n_1,\ldots,n_m),\lambda\right) = \frac{1}{\lambda^{m-1}}\left\{\sum_{k=0}^{m}\sum_{A\in I_k^m}(-1)^{S(A)}\cdot(\lambda-1)^{S(I_1^m)-S(A)+k}\right\}. \quad (1)$$

Proof. We use induction on m, i.e. on the number of the cycles of the cactus $O(n_1,\ldots,n_m)$.
1. Let $m = 1$. According to [6, Theorem 6], we can write as follows

$$P(O(n_1),\lambda) = P(C_{n_1},\lambda) = (-1)^{n_1}(\lambda-1) + (\lambda-1)^{n_1} =$$
$$\frac{1}{\lambda^0}[\ (\lambda-1)^{n_1} + (-1)^{n_1}(\lambda-1)^1\ ] =$$
$$\frac{1}{\lambda^0}\left\{\sum_{k=0}^{1}\sum_{A\in I_k^1}(-1)^{S(A)}\cdot(\lambda-1)^{S(I_1^1)-S(A)+k}\right\},$$

so for $m = 1$ Formula (1) is true.
2. Assume that Formula (1) holds for $m = t$; we will prove it for $m = t+1$. By [6, Theorem 3], the chromatic polynomial of the cactus $O(n_1,\ldots,n_{t+1})$, which came into being from the cactus $O(n_1,\ldots,n_t)$ by adding a new cycle $C_{n_{t+1}}$ with n_{t+1} vertices, is of the following form

$$P(O(n_1,\ldots,n_{t+1}),\lambda) = \frac{1}{\lambda}P(O(n_1,\ldots,n_t),\lambda)\cdot P(C_{n_{t+1}},\lambda).$$

Now, using the chromatic characterization of cycle [6] and by the induction hypothesis we obtain

$$P(O(n_1,\ldots,n_{t+1}),\lambda) =$$
$$\frac{1}{\lambda^t}\left\{\sum_{k=0}^{t}\sum_{A\in I_k^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+k}\right\}\{(-1)^{n_{t+1}}(\lambda-1)+(\lambda-1)^{n_{t+1}}\} =$$

$$\frac{1}{\lambda^t}\left\{\sum_{A\in I_0^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)} + \sum_{A\in I_1^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+1} + \right.$$
$$\cdots\cdots\cdots$$
$$\left.\sum_{A\in I_k^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+k} + \cdots + \sum_{A\in I_t^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+t}\right\}\times$$

$$
\begin{aligned}
&\{(-1)^{n_{t+1}}(\lambda-1)+(\lambda-1)^{n_{t+1}}\} = \\
&\frac{1}{\lambda^t}\left\{\sum_{A\in I_0^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+1}+\right. \\
&\sum_{A\in I_1^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+2}+\cdots+ \\
&\sum_{A\in I_k^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+k+1}+\cdots+ \\
&\sum_{A\in I_t^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+t+1}+ \\
&\sum_{A\in I_0^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}}+ \\
&\sum_{A\in I_1^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+1}+\cdots+ \\
&\sum_{A\in I_k^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+k}+\cdots+ \\
&\left.\sum_{A\in I_t^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+t}\right\} =
\end{aligned}
$$

(now, we shall write out some additional components and transpose some components in the sum above for the clarity of the proof)

$$
\begin{aligned}
&\frac{1}{\lambda^t}\left\{\sum_{A\in I_0^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}}+\right. \\
&\left[\sum_{A\in I_0^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+1}+\right. \\
&\left.\sum_{A\in I_1^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+1}\right]+ \\
&\left[\sum_{A\in I_1^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+2}+\right. \\
&\left.\sum_{A\in I_2^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+2}\right]+\cdots+
\end{aligned}
$$

$$\left[\sum_{A\in I_{k-1}^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+k}+\right.$$

$$\left.\sum_{A\in I_k^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+k}\right]+$$

$$\left[\sum_{A\in I_k^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+k+1}+\right.$$

$$\left.\sum_{A\in I_{k+1}^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+k+1}\right]+\cdots+$$

$$\left[\sum_{A\in I_{t-1}^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+t}+\right.$$

$$\left.\sum_{A\in I_t^t}(-1)^{S(A)}(\lambda-1)^{S(I_1^t)-S(A)+n_{t+1}+t}\right]+$$

$$\left.\sum_{A\in I_t^t}(-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+t+1}\right\}=$$

(now, we shall reduce the sums in the square brackets)

$$\frac{1}{\lambda^t}\left\{\sum_{A\in I_0^{t+1}}(-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)}+\right.$$

$$\sum_{A\in I_1^{t+1}}(-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+1}+$$

$$\sum_{A\in I_2^{t+1}}(-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+2}+\cdots+$$

$$\sum_{A\in I_k^{t+1}}(-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+k}+$$

$$\sum_{A\in I_{k+1}^{t+1}}(-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+k+1}+\cdots+$$

$$\sum_{A\in I_t^{t+1}} (-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+t} +$$

$$\left.\sum_{A\in I_{t+1}^{t+1}} (-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+t+1}\right\} =$$

$$\frac{1}{\lambda^t}\left\{\sum_{k=0}^{t+1}\sum_{A\in I_k^{t+1}} (-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+k}\right\}.$$

This proves Formula (1) for $m = t+1$.

The last component in the sum could be written in the above form because

$$\sum_{A\in I_t^t} (-1)^{S(A)+n_{t+1}}(\lambda-1)^{S(I_1^t)-S(A)+t+1} =$$

$$(-1)^{S(I_1^t)+n_{t+1}}(\lambda-1)^{t+1} = (-1)^{S(I_1^{t+1})}(\lambda-1)^{t+1} =$$

$$\sum_{A\in I_{t+1}^{t+1}} (-1)^{S(A)}(\lambda-1)^{S(I_1^{t+1})-S(A)+t+1}.$$

Hence, Formula (1) holds for each integer $m \geq 1$. ■

Corollary 1. *A cactus O_n^m with m n-cycles, $m \geq 1, n \geq 2$, has the following chromatic polynomial*

$$P(O_n^m,\lambda) = \frac{1}{\lambda^{m-1}}\sum_{k=0}^{m}\left\{\binom{m}{k}(-1)^{kn}(\lambda-1)^{(m-k)n+k}\right\}, \tag{2}$$

where $\binom{m}{k}$ denotes the binomial coefficient.

Proof. By Theorem 1 for $n_i = n$ where $i = 1, 2, \ldots, m$, we have

$$P(O_n^m, \lambda) = \frac{1}{\lambda^{m-1}} \left\{ \sum_{k=0}^{m} \sum_{A \in I_k^m} (-1)^{S(A)} (\lambda - 1)^{S(I_1^m) - S(A) + k} \right\} =$$

$$\frac{1}{\lambda^{m-1}} \left\{ \sum_{k=0}^{m} \sum_{A \in I_k^m} (-1)^{kn} (\lambda - 1)^{(m-k)n + k} \right\} =$$

$$\frac{1}{\lambda^{m-1}} \sum_{k=0}^{m} \left\{ a_k (-1)^{kn} (\lambda - 1)^{(m-k)n + k} \right\}.$$

The coefficients a_k are given by

$$a_0 = \sum_{A \in I_0^m} 1 = 1 \qquad \text{and} \qquad a_k = \sum_{A \in I_k^m} 1\ , \quad \text{for} \quad k = 1, 2, \ldots, m.$$

It is easy to see that $a_k = \binom{m}{k}$ and $a_0 = \binom{m}{0} = 1$. ■

For $n = 2$ Corollary 1 gives the chromatic polynomial of trees with $m + 1$ vertices.

Using Corollary 1 we can find at once the chromatic polynomial of an n–gon–tree. Let $m \geq 1$ and $n \geq 3$ be integers. Let C_n^m denote n–gon–tree with m n–gons.

Corollary 2. *A n–gon–tree C_n^m has the chromatic polynomial of the following form*

$$P(C_n^m, \lambda) = \frac{1}{(\lambda - 1)^{m-1} \lambda^{m-1}} \sum_{k=0}^{m} \left\{ \binom{m}{k} (-1)^{kn} (\lambda - 1)^{(m-k)n + k} \right\}. \tag{3}$$

Proof. By well–known [6, Theorem 3] we have

$$P(C_n^m, \lambda) = \frac{(P(C_n, \lambda))^m}{(P(K_2, \lambda))^{m-1}}\ ,$$

where C_n is a n–cycle and K_2 is a complete graph with two vertices.

Thus,

$$P(C_n^m,\lambda)=\frac{(P(C_n,\lambda))^m}{(\lambda-1)^{m-1}\lambda^{m-1}}=\frac{1}{(\lambda-1)^{m-1}}\frac{(P(C_n,\lambda))^m}{\lambda^{m-1}}\ .$$

By comparison the forms of O_n^m and C_n^m we can write

$$P(C_n^m,\lambda)=\frac{1}{(\lambda-1)^{m-1}}P(O_n^m,\lambda).$$

Now, according to Formula (2), we obtain

$$P(C_n^m,\lambda)=\frac{1}{(\lambda-1)^{m-1}\lambda^{m-1}}\sum_{k=0}^{m}\left\{\binom{m}{k}(-1)^{kn}(\lambda-1)^{(m-k)n+k}\right\},$$

which proves the corollary. ■

The class of n–gon–trees is chromatically characterized [1,8]. The class of cacti is not chromatically characterized as shown by Chao and Whitehead in [2]. We find the characterization of the cacti with exactly one cycle, i.e. the unicyclic graphs.

3. Chromaticity of unicyclic graphs

Eisenberg [4] gave the necessary condition for a graph to be an unicyclic graph. We shall prove the necessity and sufficiency of this condition.

Theorem 2. *A graph G with $n \geq 3$ vertices is a unicyclic graph with a p–cycle, $3 \leq p \leq n$, if and only if*

$$P(G,\lambda)=(\lambda-1)^n+(-1)^p(\lambda-1)^{n-p+1}. \tag{4}$$

Proof. *Necessity.* Let G be a unicyclic graph with n vertices whose cycle has p vertices.
Without loss of generality we assume that G consists of the cycle and a tree with exactly one vertex in common. Then the tree has $n-p+1$ vertices.

According to [6] we have $P(C_n,\lambda) = (\lambda-1)^n+(-1)^n(\lambda-1)$ and $P(T_n,\lambda)=\lambda(\lambda-1)^{n-1}$, where C_n denotes a n–cycle and T_n denotes a tree with n vertices. Thus, see [6], G has the following chromatic polynomial

$$P(G,\lambda)=\frac{P(C_p,\lambda)P(T_{n-p+1},\lambda)}{P(K_1,\lambda)}=$$

$$\frac{[(\lambda-1)^p+(-1)^p(\lambda-1)]\,\lambda(\lambda-1)^{n-p}}{\lambda}=(\lambda-1)^n+(-1)^p(\lambda-1)^{n-p+1}.$$

This chromatic polynomial can also be obtained from Theorem (1) for $m=n-p+1$, where one of the cycles has p vertices and the others have two vertices. However, the above method of proof is far easier and quicker.

Sufficiency. Let G be a graph with n vertices whose chromatic polynomial is equal to

$$P(G,\lambda)=(\lambda-1)^n+(-1)^p(\lambda-1)^{n-p+1}.$$

This gives,

$$P(G,\lambda)=\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}(-1)^k \ + \ (-1)^p\sum_{k=0}^{n-p+1}\binom{n-p+1}{k}\lambda^{n-p+1-k}(-1)^k.$$

Let $P(G,\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n$. The form of the chromatic polynomial implies that for $p\le 1$ G would not have n vertices.

For $p=2$ we have $a_1=\binom{n}{1}(-1)^1+(-1)^2\binom{n-1}{0}(-1)^0=-(n-1)$ and $a_{n-1}=\binom{n}{n-1}(-1)^{n-1}+(-1)^2\binom{n-1}{n-2}(-1)^{n-2}=(-1)^{n-1}\neq 0$, so G would have to be a connected graph with n vertices and $n-1$ edges, i. e., G would have to be a tree.

For $p=n+1$ we have $P(G,\lambda)=(\lambda-1)^n+(-1)^{n+1}$. If n is even, then $P(G,1)=-1$ which is contrary to the definition of the chromatic polynomial. If n is odd, then $P(G,1)=1$. Moreover, for $p=n+1$ we have $a_0=1$ and $a_{n-1}=\binom{n}{n-1}(-1)^{n-1}+(-1)^{n+1}=(-1)^{n-1}(n+1)\neq 0$. According to the well-known properties of the chromatic polynomials [6], G would have to be a connected graph with n vertices. Since $n\ge 3$, G could not be properly coloured with only one colour.

For $p>n+1$, $P(G,\lambda)$ would not be the chromatic polynomial of any graph.

Now, for p fulfilling the condition $3\le p\le n$, we have

$$\begin{aligned}
&a_0=1,\\
&a_1=\binom{n}{1}(-1)^1=-n,\\
&a_{n-1}=\binom{n}{n-1}(-1)^{n-1}+(-1)^p\binom{n-p+1}{n-p}(-1)^{n-p}=\\
&\qquad n(-1)^{n-1}+(n-p+1)(-1)^n=(-1)^{n-1}(p-1)\neq 0\ .
\end{aligned}$$

Following [6], G is a connected graph with n vertices and n edges, so it must contain exactly one cycle.

Now, let G be a unicyclic graph with n vertices containing a k–cycle, where $k \neq p$. Then, according to "*Necessity*", we have $P(G, \lambda) = (\lambda - 1)^n + (-1)^k(\lambda - 1)^{n-k+1}$, which contradicts the above formula for $P(G, \lambda)$ and G must contain a p–cycle. This completes the proof of the theorem. ■

The above theorem implies the following conclusion.

Corollary 3. [3] *A unicyclic graph G with a $(n-1)$–cycle and $n > 3$ vertices is chromatically unique.*

4. Chromatically equivalent graphs

A cycle is said to be a *mini–cycle* if no pair of its vertices is joined by a chord. This definition is equivalent to the one given by Chao and Whitehead in [2]. Figure 1 gives examples of mini-cycles C' and C'' and a nonmini-cycle C in the graph G.

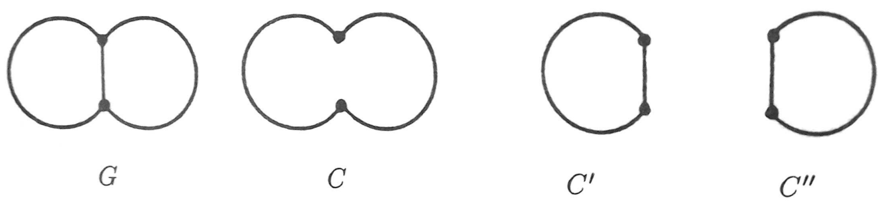

Figure 1

Chao and Whitehead [2] defined a *forest–like graph* G as graph in which every pair of cycles has at most one edge in common and its dual graph G^d is a forest. G^d is obtained from G by replacing each mini–cycle by a vertex, and by joining two vertices in G^d by one edge if and only if the corresponding mini–cycles in G have one edge in common.

Now, we define a new class of graphs, the so–called *clique–forest–like graphs*, as follows:

(i) Each forest–like graph is a clique–forest–like graph.

(ii) If H is a clique–forest–like graph then G is a clique–forest–like graph if and only if G can be obtained from H by adding a tree which has one vertex in common with H, or by adding a cycle which has one vertex or one edge in common with H.

It is easy to see that if G is a clique–forest–like graph, then its dual graph G^d consists of cliques having at most one vertex in common (Figure 2 gives an example). We shall say that G^d is a forest of clique–trees.

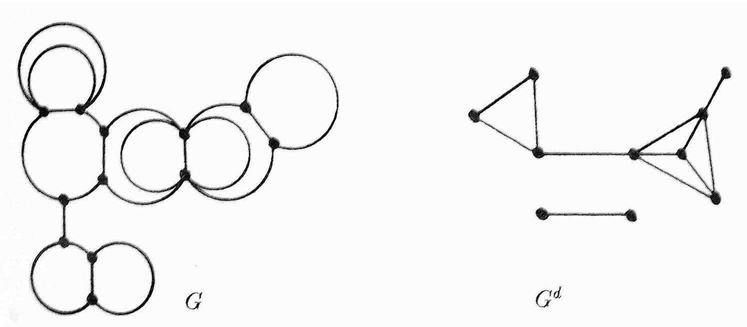

Figure 2

Theorem 3. *All clique–forest–like graphs with n vertices, e edges and the same number of mini–cycles of each length are chromatically equivalent.*

In order to prove this theorem we need the following lemma, the proof of which is a direct consequence of the construction of G.

Lemma 1. *Let G be a clique–forest–like graph. Then G contains at least one mini–cycle in which there exist edges belonging to none of other mini–cycles.*

***Proof of Theorem* 3.** Let G and H be such graphs. We show that they have the same chromatic polynomial, i.e., $P(G,\lambda) = P(H,\lambda)$.

According to Lemma 1 we know that G contains a mini–cycle C in which there exist edges belonging only to this cycle. We consider this clique–tree in G^d which contains a vertex corresponding with the mini–cycle C in G. Let i denote a number of vertices of the clique–tree mentioned above. Now, we label all vertices in this clique–tree and their corresponding mini–cycles in G by $1, 2, \ldots, i$ traversing the clique–tree according to the *depth* *first search* [7]. But we can label a vertex v in G^d (and its corresponding mini–cycles in G too) if and only if all vertices in each clique–subtree with the root v have been already labelled. Then, we repeat this process to each of the remaining clique–trees starting with the number $i+1$ at the next clique–tree. In this way all mini–cycles in G are labelled and none of them have the same label.

Now, we shall use Whitney's *broken cycle theorem* [9]. We break the mini–cycles $1, 2, 3, \ldots$ successively in the following way: If there exist edges contained only in this cycle, we remove one of them. If the cycle j consists only of the edges belonging also to other mini–cycles, we remove any edge contained in the cycle with the label smaller than j, i.e. in the cycle which has been already broken. We do not need to consider the nonmini–cycles in G because in this process we break all nonmini–cycles by breaking all mini–cycles.

Now, we apply the same process to H. The graphs G and H have the same number of vertices, the same number of edges and the same number of broken mini–cycles of each length. According to Whitney's broken cycle theorem we obtain $P(G, \lambda) = P(H, \lambda)$, so G and H are chromatically equivalent. That completes the proof of this theorem. ■

Acknowledgment

I would like to thank Doctor H. Bielak and Professor M. Borowiecki for their constructive comments which resulted in improvements to this paper.

References

[1] C. Y. Chao and N. Z. Li, *On trees of polygons*, Archiv Math. **45** (1985) 180–185.

[2] C. Y. Chao and E. G. Whitehead Jr., *On chromatic equivalence of graphs*, in: Y. Alavi and D. R. Lick, ed., Theory and Applications of Graphs, Lecture Notes in Math. 642 (Springer, Berlin, 1978) 121–131.

[3] G. L. Chia, *A note on chromatic uniqueness of graphs*, J. Graph Theory **10** (1986) 541–543.

[4] B. Eisenberg, *Generalized lower bounds for the chromatic polynomials*, in: A. Dold and B. Eckmann, eds., Recent Trends in Graph Theory, Lecture Notes in Math. 186 (Springer, Berlin, 1971) 85–94.

[5] F. Harary, Graph Theory (Addison–Wesley, Reading, MA, 1969).

[6] R. C. Read, *An introduction to chromatic polynomials*, J. Combin. Theory. **4** (1968) 52–71.

[7] R. E. Tarjan, *Depth first search and linear graph algorithms*, SIAM J. Comput. **1** (1972) 146–160.

[8] C. D. Wakelin and D. R. Woodall, *Chromatic polynomials, polygon trees, and outerplanar graphs*, J. Graph Theory **16** (1992) 459–466.

[9] H. Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. **38** (1932) 572–579.

Received 17 February 1994
Revised 9 July 1994