REDUCIBLE PROPERTIES OF GRAPHS

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Abstract

Let $\mathbb{L}$ be the set of all hereditary and additive properties of graphs. For $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{L}$, the reducible property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is defined as follows: $G \in \mathcal{R}$ if and only if there is a partition $V(G) = V_1 \cup V_2$ of the vertex set of G such that $\langle V_1 \rangle_G \in \mathcal{P}_1$ and $\langle V_2 \rangle_G \in \mathcal{P}_2$. The aim of this paper is to investigate the structure of the reducible properties of graphs with emphasis on the uniqueness of the decomposition of a reducible property into irreducible ones.

Keywords: hereditary property of graphs, additivity, reducibility

1991 Mathematics Subject Classification: 05C15, 05C75

1. Introduction

We consider finite undirected graphs without loops and multiple edges. In general, we follow the notation and terminology of [3]. For the sake of brevity, we simply say that "the graph G contains a subgraph H" instead of "the graph G contains a subgraph isomorphic to H".

Let $\mathcal{I}$ be the set of all mutually non–isomorphic graphs. If $\mathcal{P}$ is nonempty subset of $\mathcal{I}$, then $\mathcal{P}$ also denotes the property that a graph G is a member of $\mathcal{P}$. A property $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$ and *additive* if for each graph G all of whose components have property $\mathcal{P}$ it follows $G \in \mathcal{P}$, too (see [1], [2], [8], [9]). The set $\mathbb{L}$ of all hereditary and additive properties of graphs, partially ordered by set inclusion forms a complete distributive lattice (see [2], [4]). For any hereditary property $\mathcal{P} \neq \mathcal{I}$ there is a number $c(\mathcal{P})$ called *completeness* of $\mathcal{P}$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$. A hereditary property $\mathcal{P}$ can be uniquely determined by the set of *minimal forbidden graphs* which can be defined in the following way:

$$\boldsymbol{F}(\mathcal{P}) = \{F \in \mathcal{I} | F \notin \mathcal{P} \text{ but each proper subgraph of } F \text{ belongs to } \mathcal{P}\}.$$

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be any properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)-$*partition* of G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $\langle V_i \rangle_G$ has the property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as a set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)-$partition.

A property $\mathcal{P} \in \mathbb{L}$ is said to be *reducible* if there exist $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{L}$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, otherwise $\mathcal{P}$ is called *irreducible* (cf. [4], [6]).

A subset W of vertices of a graph G is called $\mathcal{P}-$*independent* if and only if the induced subgraph $\langle W \rangle_G$ belongs to $\mathcal{P}$. A subset $W \subseteq V(G)$ is said to be *maximal* $\mathcal{P}-$*independent* if it is $\mathcal{P}-$independent and there exists no subset of vertices of G which is $\mathcal{P}-$independent and properly contains W. The maximum cardinality of $\mathcal{P}-$independent set of a graph G is denoted by $\alpha_{\mathcal{P}}(G)$.

We start with an easy observation.

Lemma 1. *Let* $\mathcal{P}, \mathcal{P}_1$ and $\mathcal{P}_2$ *be any hereditary properties of graphs. If* $\mathcal{P}_1 = \mathcal{P}_2$, *then* $\mathcal{P} \circ \mathcal{P}_1 = \mathcal{P} \circ \mathcal{P}_2$.

Proof. If $G \in \mathcal{P} \circ \mathcal{P}_1$, then there exists a $(V_1, V_2)-$partition of $V(G)$ such that $\langle V_1 \rangle_G \in \mathcal{P}$ and $\langle V_2 \rangle_G \in \mathcal{P}_1$. As $\mathcal{P}_1 = \mathcal{P}_2$, it implies that $\langle V_2 \rangle_G \in \mathcal{P}_2$ and therefore $G \in \mathcal{P} \circ \mathcal{P}_2$.

The second inclusion can be proved analogously. ■

According to the previous lemma, one can ask whether it is possible to simplify the equation $\mathcal{P} \circ \mathcal{P}_1 = \mathcal{P} \circ \mathcal{P}_2$ by cancellation of $\mathcal{P}$. In what follows we shall give a particular answer.

In the beginning we prove three useful lemmas.

Lemma 2. *Let* $\mathcal{P}_1, \mathcal{P}_2$ *be hereditary properties of graphs. If* $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$, *then there exists a graph* $G \in \mathcal{P}_2$ *such that* $G \in \boldsymbol{F}(\mathcal{P}_1)$.

Proof. We notice that $\mathcal{P}_2 \setminus \mathcal{P}_1$ is nonempty, because of $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$. If G belongs to $\mathcal{P}_2 \setminus \mathcal{P}_1$, then either G is a member of $\boldsymbol{F}(\mathcal{P}_1)$ or G possesses $H \in \boldsymbol{F}(\mathcal{P}_1)$ as a subgraph. Since $\mathcal{P}_2$ is hereditary, it follows that $H \in \mathcal{P}_2$ and the proof is complete. ■

Lemma 3. *Let $\mathcal{P}, \mathcal{P}_1$ and $\mathcal{P}_2$ be any hereditary properties of graphs. Then*

(1) $\mathcal{P} \circ (\mathcal{P}_1 \cap \mathcal{P}_2) \subseteq \mathcal{P} \circ \mathcal{P}_1 \cap \mathcal{P} \circ \mathcal{P}_2$

(2) $\mathcal{P} \circ (\mathcal{P}_1 \cup \mathcal{P}_2) = \mathcal{P} \circ \mathcal{P}_1 \cup \mathcal{P} \circ \mathcal{P}_2$.

Proof. (1) Let G be an arbitrary graph belonging to $\mathcal{P} \circ (\mathcal{P}_1 \cap \mathcal{P}_2)$. Then the graph G must have a $(V_1, V_2)-$partition of $V(G)$ such that $\langle V_1 \rangle_G \in \mathcal{P}$ and $\langle V_2 \rangle_G \in \mathcal{P}_1 \cap \mathcal{P}_2$. It means that $\langle V_2 \rangle_G \in \mathcal{P}_1$ and $\langle V_2 \rangle_G \in \mathcal{P}_2$. Thus, $G \in \mathcal{P} \circ \mathcal{P}_1$ and simultaneously $G \in \mathcal{P} \circ \mathcal{P}_2$.

The proof of the statement (2) goes in a similar manner. ■

Lemma 4. *Let l be a non-negative integer. If $\mathcal{P}$ is a hereditary property of graphs with $c(\mathcal{P}) \geq l$, then for each graph G of order at least $l+1$, $\alpha_{\mathcal{P}}(G) \geq l+1$ holds.*

Proof. Let F be an arbitrary forbidden subgraph of $\mathcal{P}$. As

$$c(\mathcal{P}) = \min\{|V(F)| - 2 | F \in \boldsymbol{F}(\mathcal{P})\},$$

we have $l+2 \leq c(\mathcal{P})+2 \leq |V(F)|$. It implies that each subgraph of G with at most $l+1$ vertices contains no $F \in \boldsymbol{F}(\mathcal{P})$. Therefore, $\alpha_{\mathcal{P}}(G) \geq l+1$. ■

2. CANCELLATION BY DEGENERATE HEREDITARY PROPERTIES

If $\mathcal{P}$ is a hereditary property, then by $\chi(\mathcal{P})$ we understand the graph theoretic invariant defined as follows:

$$\chi(\mathcal{P}) = \min\{\chi(F) | F \in \boldsymbol{F}(\mathcal{P})\}.$$

A hereditary property $\mathcal{P}$ is called *degenerate* if and only if $\chi(\mathcal{P}) = 2$, otherwise it is said to be *non-degenerate* (see e.g. [7]).

Now, we can prove our main results. We recall that we want to answer the question whether it is possible to simplify the equation $\mathcal{P} \circ \mathcal{P}_1 = \mathcal{P} \circ \mathcal{P}_2$ by cancellation of $\mathcal{P}$. The following theorem provides an affirmative answer in the case when $\mathcal{P}$ has some bipartite graph forbidden.

Theorem 5. *Let $\mathcal{P}$ be an additive degenerate hereditary property. Let $\mathcal{P}_1, \mathcal{P}_2$ be any additive hereditary properties. If $\mathcal{P} \circ \mathcal{P}_1 = \mathcal{P} \circ \mathcal{P}_2$, then $\mathcal{P}_1 = \mathcal{P}_2$.*

Proof. We shall prove the assertion of theorem indirectly.

Since $\mathcal{P}$ is degenerate, $\boldsymbol{F}(\mathcal{P})$ must contain a graph $F \in \boldsymbol{F}(\mathcal{P})$ with $\chi(F) = 2$. It follows that there exists a $(U_1, U_2)-$partition of $V(F)$ such that $\langle U_1 \rangle_F \in \mathcal{O}$ and $\langle U_2 \rangle_F \in \mathcal{O}$, where $\mathcal{O}$ stands for the set of all edgeless graphs. Moreover, as $\mathcal{P}$ is additive, F must be connected (for details see [2]). Let us denote by q the integer $\max\{|U_1|, |U_2|\}$. By an easy observation we get that F is a subgraph of the complete bipartite graph $K_{q,q}$. Without loss of generality, we can suppose $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$. Then, according to Lemma 2, it is possible to choose a graph $G^* \in \mathcal{P}_2$ which does not belong to $\mathcal{P}_1$. Further, consider the set

$$M = \{V \subseteq V(G^*) | V \text{ is maximal } \mathcal{P}-\text{independent}\}.$$

It is easy to see that M is not void. So, we can define the graphs H, G_1 and G as follows:

$$H = \bigcup_{V \in M} \bigcup_{i=1}^{q} \langle V \rangle_{G^*}, G_1 = \bigcup_{i=1}^{q} G^*, G = H + G_1.$$

As $\mathcal{P}$ and $\mathcal{P}_2$ are additive properties, it is easy to check that $H \in \mathcal{P}$ and $G_1 \in \mathcal{P}_2$. Then clearly $G \in \mathcal{P} \circ \mathcal{P}_2$. We claim that $G \notin \mathcal{P} \circ \mathcal{P}_1$.

Suppose, to the contrary, $G \in \mathcal{P} \circ \mathcal{P}_1$. Then there exists a $(W_1, W_2)-$partition of $V(G)$ such that $\langle W_1 \rangle_G \in \mathcal{P}$ and $\langle W_2 \rangle_G \in \mathcal{P}_1$. Using the notations V_1 and V_2 for the sets $V(H)$ and $V(G_1)$ respectively, we shall distinguish two cases.

Case 1. Assume $|W_1 \cap V_2| \leq q - 1$.
Then it is not difficult to see that G^* is a subgraph of $\langle W_2 \rangle_G$, contradicting the fact $\langle W_2 \rangle_G \in \mathcal{P}_1$.

Case 2. Assume $|W_1 \cap V_2| \geq q$.
Obviously, for an arbitrary fixed copy of G^*, $W_1 \cap V(G^*)$ is a $\mathcal{P}-$independent subset of G^*. Thus, according to the definition of M, there exists a set $V^* \in M$ such that $W_1 \cap V(G^*) \subseteq V^*$. Provided that there is a copy of $\langle V^* \rangle_G \subseteq H$ with $V^* \cap W_1 = \emptyset$ we obtain

$$G^* \subseteq \langle V^* \rangle_{G^*} + \langle W_2 \cap V(G^*) \rangle_{G^*} \subseteq \langle W_2 \rangle_G,$$

which contradicts our assumption $\langle W_2 \rangle_G \in \mathcal{P}_1$. Therefore, at least one vertex of each copy of $\langle V^* \rangle_{G^*} \subseteq H$ must belong to W_1.
Observe then, that $K_{q,q}$ is a subgraph of the graph

$$G_2 + \langle W_1 \cap V_2 \rangle_G \subseteq \langle W_1 \rangle_G,$$

where G_2 stands for the graph

$$G_2 = \bigcup_{i=1}^{q} \langle V^* \cap W_1 \rangle_{G^*}.$$

But as stated above, $F \subseteq K_{q,q}$, which is a contradiction to our assumption $\langle W_1 \rangle_G \in \mathcal{P}$. So we are done in the second case.

Since G has no vertex $(\mathcal{P}, \mathcal{P}_1)-$partition, $\mathcal{P} \circ \mathcal{P}_1 \neq \mathcal{P} \circ \mathcal{P}_2$ holds. ■

3. Other results

The next theorem provides an entire solution of the cancellation problem when the completeness of $\mathcal{P}$ is equal to one.

Theorem 6. *Let $\mathcal{P}$ be an additive hereditary property of graphs, $c(\mathcal{P}) = 1$. Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties and $\mathcal{P}_1 \neq \mathcal{P}_2$. Then $\mathcal{P} \circ \mathcal{P}_1 \neq \mathcal{P} \circ \mathcal{P}_2$.*

Proof. Without loss of generality, we can suppose that $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$. Then, by Lemma 2, there exists a graph $G^* \in \mathcal{P}_2$ which does not belong to $\mathcal{P}_1$.

Let l denote the $\mathcal{P}-$independence number of G^*. Lemma 4 yields that $l = \alpha_{\mathcal{P}}(G^*) \geq 2$. Let us define the set M in the following way:

$$M = \{V \subseteq V(G^*) | V \text{ is maximal } \mathcal{P} - \text{independent}\}.$$

Then it is easy to verify that the graph

$$H = \bigcup_{V \in M} \bigcup_{i=1}^{l} \langle V \rangle_{G^*}$$

has property $\mathcal{P}$ and the graph $G = H + G^*$ belongs to $\mathcal{P} \circ \mathcal{P}_2$. We shall show that $G \notin \mathcal{P} \circ \mathcal{P}_1$.

Suppose indirectly that $G \in \mathcal{P} \circ \mathcal{P}_1$. Then there exists some $(\mathcal{P}, \mathcal{P}_1)-$partition of the vertex set $V(G)$. Let (W_1, W_2) be the vertex partition mentioned above. Further, let V_1 stands for the set $V(H)$ and V_2 denotes the set $V(G^*)$. The following cases may occur.

Case 1. $|W_1 \cap V_1| = 0$.
This means that there is a set $V^* \in M$ such that $W_1 \cap V_2 \subseteq V^*$. Therefore,

$$G^* \subseteq \langle V^* \rangle_H + \langle W_2 \cap V_2 \rangle_{G^*} \subseteq \langle W_2 \rangle_G,$$

which is a contradiction.

Case 2. $|W_1 \cap V_1| \geq 1$ and $\langle W_1 \cap V_1 \rangle_G$ is edgeless graph.
Then $W_1 \cap V_2$ is empty or independent set (otherwise $\langle W_1 \rangle_G$ contains a triangle, which contradicts the fact $c(\mathcal{P}) = 1$). It implies that

$$|W_1 \cap V_2| \leq \alpha_O(G^*) \leq \alpha_{\mathcal{P}}(G^*) = l$$

(we recall that $\mathcal{O}$ denotes the set of all graphs without edges). On the other hand, $c(\mathcal{P})$ is equal to one, which implies that for each $V \in M$ the induced graph $\langle V \rangle_{G^*} \subseteq H$ contains at least one edge and that is why $|W_2 \cap V_1| \geq l$. Thus

$$G^* \subseteq \langle W_2 \cap V_1 \rangle_G + \langle W_2 \cap V_2 \rangle_G = \langle W_2 \rangle_G$$

and we get again a contradiction.

Case 3. $|W_1 \cap V_1| \geq 2$ and $\langle W_1 \cap V_1 \rangle_G$ has an edge.
In this case, either $W_1 \cap V_2$ is nonempty and $\langle W_1 \rangle_{G^*}$ contains a triangle, or $G^* \subseteq \langle W_2 \rangle_G$. In the former situation $\langle W_1 \rangle_G$ cannot have the property $\mathcal{P}$, and in the latter one, $\langle W_2 \rangle_G$ does not belong to $\mathcal{P}_1$.

Thus, we have $G \notin \mathcal{P} \circ \mathcal{P}_1$. ∎

It turns out that Theorem 5 can be extended to all hereditary properties provided $\boldsymbol{F}(\mathcal{P})$ contains a tree which is not too large with respect to the completeness of $\mathcal{P}$.

Theorem 7. *Let $\mathcal{P}$ be an additive hereditary property, $T \in \boldsymbol{F}(\mathcal{P})$ is a tree, $|V(T)| \leq c(\mathcal{P}) + 3$. If $\mathcal{P}_1, \mathcal{P}_2$ are hereditary properties of graphs, $\mathcal{P}_1 \neq \mathcal{P}_2$, then $\mathcal{P} \circ \mathcal{P}_1 \neq \mathcal{P} \circ \mathcal{P}_2$.*

Proof. Since $|V(T)| \leq c(\mathcal{P}) + 3$ we have

$$\Delta(T) \leq |V(T)| - 1 \leq c(\mathcal{P}) + 2.$$

Without loss of generality, we can suppose $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$, which implies that there is a graph $G^* \in \mathcal{P}_2 \setminus \mathcal{P}_1$. We define the graphs

$$H_1 = \bigcup_{V \in M} \bigcup_{i=1}^{c(\mathcal{P})+2} \langle V \rangle_{G^*},$$

$$H_2 = \bigcup_{i=1}^{c(\mathcal{P})+2} K_{c(\mathcal{P})+1},$$

$$H = H_1 \cup H_2 \text{ and } G = H + G^*,$$

where M denotes the set of all maximal $\mathcal{P}-$independent subsets of $V(G^*)$. We assert that $G \in \mathcal{P} \circ \mathcal{P}_2 \setminus \mathcal{P} \circ \mathcal{P}_1$.

Indeed, it is easy to check that $G \in \mathcal{P} \circ \mathcal{P}_2$. In order to obtain a contradiction, assume $G \in \mathcal{P} \circ \mathcal{P}_1$. Then there exists a $(W_1, W_2)-$partition of $V(G)$ such that $\langle W_1 \rangle_G \in \mathcal{P}$ and simultaneously $\langle W_2 \rangle_G \in \mathcal{P}_1$. We introduce the symbols V_1 and V_2 for the vertex sets $V(H)$ and $V(G^*)$ respectively, in order to simplify notation.

Case 1. $|W_1 \cap V_2| = 0$.
It follows that $V_2 \subseteq W_2$ and $G^* \subseteq \langle W_2 \rangle_G$. As $G^* \in \mathcal{P}_2 \setminus \mathcal{P}_1$ we get a contradiction.

Case 2. $0 < |W_1 \cap V_2| \leq c(\mathcal{P}) + 1$.
To avoid G^* as a subgraph of $\langle W_2 \rangle_G$, the inequality

$$|W_2 \cap V(K_{c(\mathcal{P})+1})| < |W_1 \cap V_2| \leq c(\mathcal{P}) + 1$$

must be satisfied for all copies of $K_{c(\mathcal{P})+1} \subseteq H$. Hence,

$$|W_1 \cap V(K_{c(\mathcal{P})+1})| \geq c(\mathcal{P}) + 1 - |W_1 \cap V_2| + 1 = c(\mathcal{P}) - |W_1 \cap V_2| + 2.$$

It makes each vertex $u \in W_1 \cap V(K_{c(\mathcal{P})+1})$ have a degree at least $c(\mathcal{P}) - |W_1 \cap V_2| + 1 + |W_1 \cap V_2| = c(\mathcal{P}) + 1$. As at least one vertex of each copy of $K_{c(\mathcal{P})+1}$ belongs to W_1, any vertex $w \in W_1 \cap V_2$ has a degree at least $c(\mathcal{P}) + 1$. Therefore, $\langle W_1 \rangle_G$ contains a subgraph with minimum degree at least $c(\mathcal{P}) + 1$. Then, by Lemma 3 of [5], an arbitrary tree on $c(\mathcal{P}) + 3$ vertices (occasionally excluding a star on $c(\mathcal{P})+3$ vertices, but this case can be solved by a small modification of this proof and therefore it is omitted) is contained in $\langle W_1 \rangle_G$, which contradicts the fact $\langle W_1 \rangle_G \in \mathcal{P}$.

Case 3. $|W_1 \cap V_2| \geq c(\mathcal{P}) + 2$.
In similar manner as in the proof of Theorem 5, we obtain that either $G^* \subseteq \langle W_2 \rangle_G$ or $\langle W_1 \rangle_G$ possesses a complete bipartite graph $K_{c(\mathcal{P})+2, c(\mathcal{P})+2}$. Since T is also bipartite and $|V(T)| \leq c(\mathcal{P}) + 3$, we have $T \subseteq K_{c(\mathcal{P})+2, c(\mathcal{P})+2} \subseteq \langle W_1 \rangle_G$. In both cases $G \notin \mathcal{P} \circ \mathcal{P}_1$.

Hence, G is a graph which belongs to $\mathcal{P} \circ \mathcal{P}_2$ but does not have the property $\mathcal{P} \circ \mathcal{P}_1$, i.e. $\mathcal{P} \circ \mathcal{P}_1 \neq \mathcal{P} \circ \mathcal{P}_2$. ■

Acknowledgement

We would like to thank Roman Vasky for his stimulating ideas and first steps in proving our main result.

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Received 20 January 1994
Revised 15 November 1994