THE CROSSING NUMBERS OF CERTAIN CARTESIAN PRODUCTS

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Abstract

In this article we determine the crossing numbers of the Cartesian products of given three graphs on five vertices with paths.

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Preliminaries

Let G be a simple graph with the vertex set V and the edge set E. A *drawing* is a mapping of a graph into a surface. The vertices go into distinct points, *nodes*. An edge and its incident vertices map into a homeomorphic image of the closed interval [0,1] with the relevant nodes as endpoints and the interior, an *arc*, containing no node. A *good drawing* is one in which no two arcs incident to a common node have a common point; and no two arcs have more than one point in common. A common point of two arcs is a *crossing*. The *crossing number* $\nu(G)$ of a graph G is the minimum number of crossings in any good drawing of G in the plane.

The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has a vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \times G_2) = \{\{(u_i, v_j), (u_h, v_k)\} : \quad u_i = u_h \quad \text{and} \quad \{v_j, v_k\} \in E(G_2) \\ \text{or} \quad \{u_i, u_h\} \in E(G_1) \quad \text{and} \quad v_j = v_k\}.$$

Let C_n be the *cycle*, P_n the *path* of length n and S_n the *star* $K_{1,n}$. The crossing numbers of the Cartesian products of all 4–vertex graphs with cycles are determined in [1] and [2] and with paths and stars in [3] and [4]. In this paper we determine the precise values of the crossing numbers of some products $G \times P_n$ where G is 5–vertex graph.

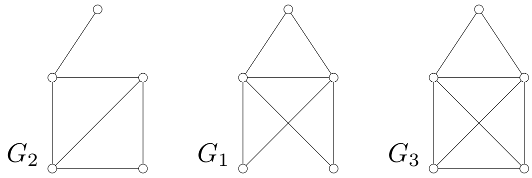

Figure 1

RESULTS

Three graphs of order five are shown in Figure 1. We assume $n \geq 1$ and find it convenient to consider the graph $G_k \times P_n$, $k \in \{1,2,3\}$, in the following way. It has $5(n+1)$ vertices and edges that are the edges in the $n+1$ copies G_k^i, $i = 0,1,...,n$, and five paths of length n. Furthermore, we call the former edges red and the latter ones blue.

Let $H^{i,j}$ be a subgraph of $G_k \times P_n$, $k \in \{1,2,3\}$, induced by the vertices of $G_k^i, G_k^{i+1}, ..., G_k^j$ for $0 \leq i < j \leq n$. The subgraph $H^{i,j} - G_k^i$ is obtained by the removal of all edges of G_k^i from the graph $H^{i,j}$.

Lemma 1. *If D is a good drawing of $G_1 \times P_n$, $n \geq 2$, in which every G_1^i, $i = 0,1,...,n$, has at most one crossing, then D has at least $2(n-1)$ crossings.*

Proof. We show that in every drawing $D^{0,i}$ of $H^{0,i}$, $i = 2,3,...,n$, induced by D there are at least two crossings more than the number of crossings in the drawing $D^{0,i-1}$ induced by $D^{0,i}$.

Consider the drawing $D^{0,i}$ of $H^{0,i}$ induced by D. By the assumption of Lemma 1 in the drawing $D^{0,i-1}$ induced by $D^{0,i}$ there is no region with 5 vertices and at most one region with 4 vertices of G_1^{i-1} on its boundary. (The crossings are considered to be vertices of the map.) Suppose that in $D^{0,i-1}$ there is one region with four vertices of G_1^{i-1} on its boundary. In this case G_1^{i-1} has one crossing with a blue edge joining G_1^{i-2} to G_1^{i-1}

and in $D^{0,i}$ all vertices of G_1^i must lie outside this regoin. Therefore, in the drawing $D^{0,i}$ there are at least two crossings between the edges of $H^{0,i-1}$ and the edges of $H^{i-1,i} - G_1^{i-1}$. Otherwise, $D^{0,i-1}$ induces the map with at most three vertices of G_1^{i-1} on the boundary of every region and the edges of $H^{i-1,i} - G_1^{i-1}$ have at least two crossings in $D^{0,i}$.

Since i runs through $2, 3, ..., n$, the drawing D has at least $2(n-1)$ crossings. ■

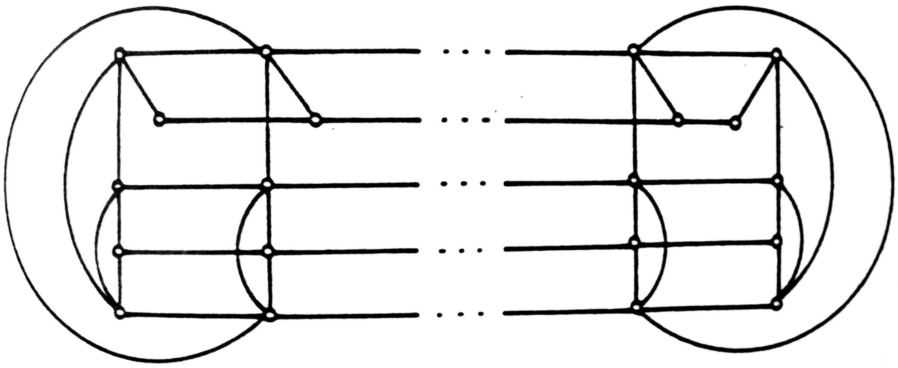

Figure 2

Theorem 1. $\nu(G_1 \times P_n) = 2(n-1)$ *for* $n \geq 1$.

Proof. The drawing in Figure 2 shows that $\nu(G_1 \times P_n) \leq 2(n-1)$ for $n \geq 1$. We prove the reverse inequality by induction on n. The case $n = 1$ is trivial.

Assume that the result is true for $n = k$, $k \geq 1$, and suppose that there is a good drawing of $G_1 \times P_{k+1}$ with fewer than $2k$ crossings. By Lemma 1, some of G_1^i must then be crossed at least twice. By the removal of all edges of this G_1^i we obtain a graph, which is homeomorphic to $G_1 \times P_k$ or which contains the subgraph $G_1 \times P_k$, and has a drawing with fewer than $2(k-1)$ crossings. This contradicts the induction hypothesis. ■

If we join all vertices of the graph G_2 (Figure 1) with a vertex x different from the vertices of G_2, we obtain the graph which cannot be drawn without having a G_2-edge crossed because it contains a subgraph $K_{3,3}$. If we join all vertices of the graph G_2 with vertices of a connected graph G, we again obtain the graph which cannot be drawn without having a G_2-edge crossed.

Lemma 2. *If* D *is a good drawing of* $G_2 \times P_n$, $n \geq 1$, *in which every* G_2^i, $i = 0, 1, ..., n$, *has at most two crossings, then* D *has at least* $3n - 1$ *crossings.*

Proof. By the assumption of Lemma 2 the red edges of two different G_2^i and G_2^j cannot cross. Otherwise, G_2^i (G_2^j) has at least three crossings (at least two crossings with the red edges of G_2^j (G_2^i) and at least one crossing with the blue edges joining G_2^i to G_2^{i-1} or G_2^{i+1} (G_2^j to G_2^{j-1} or G_2^{j+1})).

Consider the drawing $D^{i,i+1}$ of $H^{i,i+1}$, $i \in \{0, 1, ..., n-2\}$, induced by D.

Case 1. Let no edges of G_2^{i+1} cross each other in $D^{i,i+1}$. Then the drawing D^{i+1} of G_2^{i+1} induced by $D^{i,i+1}$ induces the map with two quadrangular regions and two triangular regions. By the assumption of Lemma 2 in the drawing D all copies $G_2^0, G_2^1, ..., G_2^i, G_2^{i+2}, ..., G_2^n$ must lie in the quadrangular region of D^{i+1}. In $D^{i,i+1}$ there is exactly one crossing between the red edges of G_2^{i+1} and the blue edges of $H^{i,i+1}$ (Figure 3).

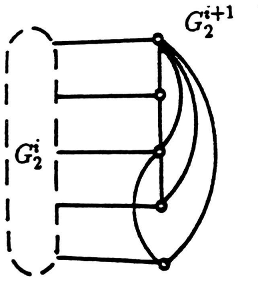

Figure 3

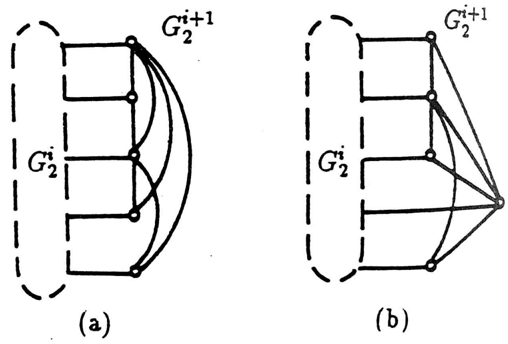

Figure 4

The drawing $D^{i,i+1}$ divides the quadrangular region of D^{i+1} into new regions with at most two vertices of G_2^{i+1} on the boundary of every region. (The crossings are again considered to be vertices of the map.) Consider now the drawing $D^{i,i+2}$ of $H^{i,i+2}$, $i \in \{0, 1, ..., n-2\}$, induced by D. In the drawing $D^{i,i+2}$ there are at least three crossings between the edges of $H^{i,i+1}$ and the edges of $H^{i+1,i+2} - G_2^{i+1}$.

Case 2. Let in the drawing D^{i+1} of G_2^{i+1} induced by $D^{i,i+1}$ there be a region with all vertices of G_2^{i+1} on its boundary (G_2^{i+1} has an internal crossing). Then the drawing $D^{i,i+1}$ divides this region of D^{i+1} into new regions with at most two vertices (Figure 4(a)) or with at most three vertices

(Figure 4(b)) on the boundary of every region. Consider now the drawing $D^{i,i+2}$ of $H^{i,i+2}$, $i \in \{0,1,...,n-2\}$, induced by D. In the drawing $D^{i,i+2}$ there are at least three crossings between the edges of $H^{i,i+1}$ and the edges of $H^{i+1,i+2} - G_2^{i+1}$.

Since $H^{0,1}$ has at least two crossings and i runs through $0,1,...,n-2$, the drawing D has at least $3(n-1)+2$ crossings. ■

Theorem 2. *$\nu(G_2 \times P_n) = 3n-1$ for $n \geq 1$.*

Proof. The drawing in Figure 5 shows that $\nu(G_2 \times P_n) \leq 3n-1$ for $n \geq 1$. The proof of the reverse inequality proceeds by induction on n in the same way as in Theorem 1 using Lemma 2. ■

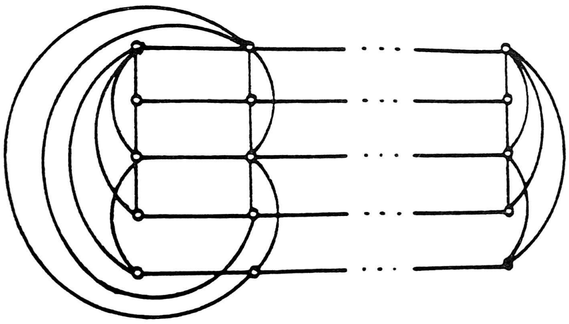

Figure 5

Theorem 3. *$\nu(G_3 \times P_n) = 3n-1$ for $n \geq 1$.*

Proof. Into drawing of $G_2 \times P_n$ in Figure 5 we can draw edges so that we obtain a good drawing of $G_3 \times P_n$ with at most $3n-1$ crossings. As $G_2 \times P_n$ is a subgraph of $G_3 \times P_n$ and $\nu(G_2 \times P_n) = 3n-1$, then $\nu(G_3 \times P_n) \geq 3n-1$. ■

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