ON EDGE $H$-IRREGULARITY STRENGTHS OF SOME GRAPHS

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Abstract

For a graph $G$ an edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. In this case we say that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. An $H$-covering of graph $G$ is an $(H_1, H_2, \ldots, H_t)$-edge covering in which every subgraph $H_i$ is isomorphic to a given graph $H$.

Let $G$ be a graph admitting $H$-covering. An edge $k$-labeling $\alpha : E(G) \to \{1, 2, \ldots, k\}$ is called an $H$-irregular edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ their weights are different.
wt_α(H') and wt_α(H'') are distinct. The weight of a subgraph H under an edge k-labeling α is the sum of labels of edges belonging to H. The edge H-irregularity strength of a graph G, denoted by ehs(G, H), is the smallest integer k such that G has an H-irregular edge k-labeling.

In this paper we determine the exact values of ehs(G, H) for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs. Moreover the subgraph H is isomorphic to only $C_4$, $C_3$, and $K_4$.

**Keywords:** H-irregular edge labeling, edge H-irregularity strength, prism, antiprism, triangular ladder, diagonal ladder, wheel, gear graph.

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1. Introduction

Consider a simple and finite graph $G = (V, E)$ of order at least 2. An edge k-labeling is a function $\alpha : E(G) \to \{1, 2, \ldots, k\}$, where k is a positive integer. Then the associated weight of a vertex $x \in V(G)$ is $w_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)$, where the sum is taken over all edges incident to x. Such a labeling $\alpha$ is called irregular if the obtained weights of all vertices are different. The smallest positive integer $k$ for which there exists an irregular labeling of $G$ is called the irregularity strength of $G$ and is denoted by $s(G)$. If it does not exist, then we write $s(G) = \infty$.

One can easily see that $s(G) < \infty$ if and only if $G$ contains no isolated edges and has at most one isolated vertex.

The notion of the irregularity strength was firstly introduced by Chartrand et al. in [7]. Some results on the irregularity strength can be found in [2, 3, 5, 6, 8, 9, 11–14].

A vertex k-labeling $\beta : V(G) \to \{1, 2, \ldots, k\}$ is called an edge irregular k-labeling of the graph $G$ if the weights $w_\beta(xy) \neq w_\beta(x'y')$ for every two distinct edges $xy$ and $x'y'$, where the weight of an edge $xy \in E(G)$ is $w_\beta(xy) = \beta(x) + \beta(y)$. The minimum $k$ for which a graph $G$ admits an edge irregular k-labeling is called the edge irregularity strength of $G$, denoted by $es(G)$. The notion of the edge irregularity strength was defined by Ahmad et al. in [1].

A family of subgraphs $H_1, H_2, \ldots, H_t$ is said to be an edge-covering of $G$ if each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. In this case we say that $G$ admits an $(H_1, H_2, \ldots, H_t)$-(edge) covering. If every subgraph $H_i$, $i = 1, 2, \ldots, t$, is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering.

Motivated by the irregularity strength and the edge irregularity strength of a graph $G$ Ashraf et al. in [4] introduced a new parameter, edge $H$-irregularity strength, as a natural extension of the parameters $s(G)$ and $es(G)$. Let $G$ be a graph admitting $H$-covering. An edge $k$-labeling $\alpha$ is called an $H$-irregular
edge \(k\)-labeling of the graph \(G\) if for every two different subgraphs \(H'\) and \(H''\) isomorphic to \(H\) we have

\[
wt_\alpha(H') = \sum_{e \in E(H')} \alpha(e) \neq \sum_{e \in E(H'')} \alpha(e) = wt_\alpha(H'').
\]

The edge \(H\)-irregularity strength of a graph \(G\), denoted by \(ehs(G, H)\), is the smallest integer \(k\) for which \(G\) has an \(H\)-irregular edge \(k\)-labeling.

Next theorem proved in [4] gives the lower bound of the edge \(H\)-irregularity strength of a graph \(G\).

**Theorem 1** [4]. Let \(G\) be a graph admitting an \(H\)-covering and \(t\) is the number of all the subgraphs isomorphic with \(H\). Then

\[
ehs(G, H) \geq \left\lceil 1 + \frac{t-1}{|E(H)|} \right\rceil.
\]

Note that the parameter \(t\) is the number of all subgraphs of \(G\) isomorphic to \(H\). In this paper we determine exact values of the edge \(H\)-irregularity strength for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs for some \(H\). Moreover the subgraph \(H\) is isomorphic to only \(C_4\), \(C_3\) and \(K_4\).

2. Prism and Antiprism

The prism \(D_n\) can be defined as the Cartesian product \(C_n \square P_2\) of a cycle on \(n\) vertices with a path on 2 vertices. Let \(V(C_n \square P_2) = \{x_i, y_i : 1 \leq i \leq n\}\) be the vertex set and \(E(C_n \square P_2) = \{x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n\} \cup \{x_iy_i : 1 \leq i \leq n\}\) be the edge set, where the indices are taken modulo \(n\). Hence, the graph \(D_n\) has \(2n\) vertices and \(3n\) edges.

**Theorem 2.** Let \(D_n = C_n \square P_2\), \(n \geq 3, n \neq 4\), be a prism. Then

\[
ehs(D_n, C_4) = \left\lceil \frac{n + 3}{4} \right\rceil.
\]

**Proof.** The prism \(D_n, n \geq 3, n \neq 4\), admits a \(C_4\)-covering with exactly \(n\) cycles \(C_4\). We denote these 4-cycles by the symbols \(C_4^i, i = 1, 2, \ldots, n\), such that the vertex set of \(C_4^i\) is \(V(C_4^i) = \{x_i, x_{i+1}, y_i, y_{i+1}\}\) and the edge set is \(E(C_4^i) = \{x_ix_{i+1}, y_iy_{i+1}, x_iy_i, x_{i+1}y_{i+1}\}\).

From Theorem 1 it follows that \(ehs(D_n, C_4) \geq \left\lceil \frac{n+3}{4} \right\rceil\). To show that \(\left\lceil \frac{n+3}{4} \right\rceil\) is an upper bound for the edge \(C_4\)-irregularity strength of \(D_n\) we define a \(C_4\)-irregular edge labeling \(\alpha_1 : E(D_n) \to \{1, 2, \ldots, \left\lceil \frac{n+3}{4} \right\rceil\}\), in the following way. We distinguish to cases according to the parity of \(n\).
Case 1. When \( n \) is odd, then

\[
\alpha_1(x_i x_{i+1}) = \begin{cases} 
\left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+1-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(y_i y_{i+1}) = \begin{cases} 
\left\lfloor \frac{i+1}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+2-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(x_i y_i) = \begin{cases} 
\left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+3-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n.
\end{cases}
\]

Case 2. When \( n \) is even, then

\[
\alpha_1(x_i x_{i+1}) = \begin{cases} 
\left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lceil \frac{n+1-i}{2} \right\rceil & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(y_i y_{i+1}) = \begin{cases} 
\left\lfloor \frac{i+1}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lceil \frac{n+3}{2} \right\rceil & \text{for } i = \frac{n}{2} + 1,
\end{cases}
\]

\[
\alpha_1(x_i y_i) = \begin{cases} 
\left\lfloor \frac{n}{4} + 1 \right\rfloor & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 0 \pmod{4}, \\
\frac{n+2}{4} & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 2 \pmod{4},
\end{cases}
\]

\[
\left\lceil \frac{n+3-i}{2} \right\rceil & \text{for } \frac{n}{2} + 2 \leq i \leq n.
\end{cases}
\]

It is easy to see that under the labeling \( \alpha_1 \) all edge labels are at most \( \left\lfloor \frac{n+3}{4} \right\rfloor \).

The \( C_4 \)-weights of the cycles \( C_n^4 \), \( i = 1, 2, \ldots, n \), under the edge labeling \( \alpha_1 \), are given by

\[
wt_{\alpha_1}(C_n^4) = \sum_{e \in E(C_n^4)} \alpha_1(e) = \alpha_1(x_i x_{i+1}) + \alpha_1(y_i y_{i+1}) + \alpha_1(x_i y_i) + \alpha_1(x_i+1 y_{i+1}).
\]

Case 1. When \( n \) is odd, then

\[
wt_{\alpha_1}(C_n^4) = \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{i+1}{2} \right\rfloor = 2i + 2 \quad \text{for } 1 \leq i \leq \frac{n-1}{2},
\]

\[
wt_{\alpha_1}\left(\frac{n+1}{4} \right) = \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+3}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+3}{4} \right\rfloor = n + 3,
\]

\[
wt_{\alpha_1}(C_n^4) = \left\lfloor \frac{n+1-i}{2} \right\rfloor + \left\lfloor \frac{n+2-i}{2} \right\rfloor + \left\lfloor \frac{n+3-i}{2} \right\rfloor + \left\lfloor \frac{n+2-i}{2} \right\rfloor = 2n + 5 - 2i \quad \text{for } \frac{n+1}{2} + 1 \leq i \leq n - 1,
\]

\[
wt_{\alpha_1}(C_n^4) = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = 5.
\]
Case 2. When \( n \) is even, then

\[
wt_{\alpha_1}(C_{4}^i) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2i + 2 \quad \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1,
\]

\[
wt_{\alpha_1}\left(C_{4}^{\frac{n}{2}}\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil = n + 2 \quad \text{for } n \equiv 0 \pmod{4},
\]

\[
wt_{\alpha_1}\left(C_{4}^{\frac{n}{2}+1}\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+3}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \frac{n+2}{4} = n + 3 \quad \text{for } n \equiv 2 \pmod{4},
\]

\[
wt_{\alpha_1}(C_{4}^{n}) = \left\lceil \frac{n+1-i}{2} \right\rceil + \left\lceil \frac{n+1-i}{2} \right\rceil + \left\lceil \frac{n+3-i}{2} \right\rceil + \left\lceil \frac{n+3-i}{2} \right\rceil = 2n + 5 - 2i \quad \text{for } \frac{n}{4} + 2 \leq i \leq n-1,
\]

\[
wt_{\alpha_1}(C_{4}^{n-1}) = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil = 5.
\]

Combining the previous we get that

\[
wt_{\alpha_1}(C_{4}^i) = \begin{cases} 2(1+i) & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ 1 + 2(n + 2 - i) & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}
\]

One can see that the weights of cycles \( C_{4}^i \), for \( i = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil \), are even and in increasing order, therefore \( wt_{\alpha_1}(C_{4}^{i+1}) > wt_{\alpha_1}(C_{4}^{i}) \).

On the other hand, the weights of cycles \( C_{4}^i \), for \( i = \left\lceil \frac{n}{2} \right\rceil + 1, \ldots, n \), are odd and in decreasing order, therefore \( wt_{\alpha_1}(C_{4}^{i+1}) < wt_{\alpha_1}(C_{4}^{i}) \).

Thus the edge weights are distinct numbers from the set \{4, 5, \ldots, n + 3\}. This shows that \( \text{ehs}(D_{n}, C_{4}) \leq \left\lceil \frac{n+3}{4} \right\rceil \). Hence the proof is concluded. \( \blacksquare \)

The antiprism \( A_{n} \) \([10]\), \( n \geq 3 \), is a 4-regular graph (Archimedean convex polytope), consisting of \( 2n \) vertices and \( 4n \) edges. The vertex set and edge set of \( A_{n} \) are defined as: \( V(A_{n}) = \{x_i, y_i : 1 \leq i \leq n\} \), \( E(A_{n}) = \{x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_{i+1}y_i : 1 \leq i \leq n\} \), with indices taken modulo \( n \).

**Theorem 3.** Let \( A_{n}, n \geq 4 \), be an antiprism. Then

\[
\text{ehs}(A_{n}, C_{3}) = \left\lceil \frac{2n + 2}{3} \right\rceil.
\]

**Proof.** The antiprism \( A_{n}, n \geq 4 \), admits a \( C_{3} \)-covering with exactly \( 2n \) cycles \( C_{3} \).

The first type of the cycle \( C_{3} \) has the vertex set \( V(C_{3}^1) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n\} \) and the edge set \( E(C_{3}^1) = \{x_ix_{i+1}, x_{i+1}y_i, y_ix_i : 1 \leq i \leq n\} \). The second type
of the cycle $C_3$ has the vertex set $V(C_3) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n\}$ and the edge set $E(C_3) = \{y_iy_{i+1}, y_ix_{i+1}, y_{i+1}x_{i+1} : 1 \leq i \leq n\}$. Note that the indices are taken modulo $n$.

From Theorem 1 it follows that $\text{ehs}(A_n, C_3) \geq \lceil \frac{2n+2}{3} \rceil$. To show that $\lceil \frac{2n+2}{3} \rceil$ is an upper bound for the edge $C_3$-irregularity strength of $A_n$ we define a $C_3$-
irregular edge labeling $\alpha_2 : E(A_n) \to \{1, 2, \ldots, \lceil \frac{2n+2}{3} \rceil\}$, in the following way. We distinguish two cases.

**Case 1.** When $n \equiv 0, 4, 5 \pmod{6}$, then

$$\alpha_2(x_i, x_{i+1}) = \begin{cases} 
 i & \text{for } i = 1, 2, \\
 i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq t - 1, \\
 \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = t, \\
 n - i + 2 + \left\lfloor \frac{n-i}{3} \right\rfloor & \text{for } t + 1 \leq i \leq n - 2, \\
 n - i + 2 & \text{for } i = n - 1, n,
\end{cases}$$

where $t = \begin{cases} 
 \frac{n+1}{2} & \text{if } n \equiv 5 \pmod{6}, \\
 \frac{n}{2} + 1 & \text{if } n \equiv 0, 4 \pmod{6}.
\end{cases}$

$$\alpha_2(y_iy_{i+1}) = \begin{cases} 
 i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
 \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 5 \pmod{6}, \\
 n - i + 2 + \left\lfloor \frac{n-i}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n - 1, \\
 1 & \text{for } i = n,
\end{cases}$$

$$\alpha_2(x_iy_i) = \begin{cases} 
 i + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
 \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 0, 5 \pmod{6}, \\
 \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 4 \pmod{6}, \\
 2 & \text{for } i = n,
\end{cases}$$

$$\alpha_2(y_ix_{i+1}) = \begin{cases} 
 1 & \text{for } i = 1, \\
 i + 1 + \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
 \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 0, 4 \pmod{6}, \\
 \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 5 \pmod{6}, \\
 n - i + 2 + \left\lfloor \frac{n-i}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}$$
Case 2. When \( n \equiv 1, 2, 3 \pmod{6} \), then

\[
\alpha_2(x_ix_{i+1}) = \begin{cases} 
  i & \text{for } i = 1, 2, \\
  i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq t - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = t \text{ and } n \equiv 2 \pmod{6}, \\
  n - i + 3 + \left\lceil \frac{n-i-2}{3} \right\rceil & \text{for } t + 1 \leq i \leq n - 2, \\
  n - i + 2 & \text{for } i = n - 1, n,
\end{cases}
\]

where \( t = \begin{cases} 
  \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{6}, \\
  \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{6}.
\end{cases} \)

\[
\alpha_2(y_iy_{i+1}) = \begin{cases} 
  i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\
  n - i + 2 + \left\lceil \frac{n-i-1}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n - 1, \\
  1 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(x_iy_i) = \begin{cases} 
  i + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 2 \pmod{6}, \\
  n - i + 3 + \left\lceil \frac{n-i-1}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
  2 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(y_ix_{i+1}) = \begin{cases} 
  1 & \text{for } i = 1, \\
  i + 1 + \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 2 \pmod{6}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 3 \pmod{6}, \\
  n - i + 2 + \left\lceil \frac{n-i}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
\]

Now we compute the \( C_3 \)-weights under the edge labeling \( \alpha_2 \) as follows. For the weights of 3-cycles of the first type we get

\[
\text{wt}_{\alpha_2}(C_3^1) = \sum_{e \in E(C_3^1)} \alpha_2(e) = \alpha_2(x_ix_{i+1}) + \alpha_2(x_iy_i) + \alpha_2(y_ix_{i+1})
\]

\[
= \begin{cases} 
  4i - 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
  4n - 4i + 6 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\]
and for the weights of 3-cycles of the second type we have

\[ wt_{\alpha_2}(C^3_i) = \sum_{e \in E(C^3_i)} \alpha_2(e) = \alpha_2(y_iy_{i+1}) + \alpha_2(y_ix_{i+1}) + \alpha_2(y_{i+1}x_{i+1}) \]

\[
= \begin{cases} 
4i + 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1, \\
4n - 4i + 4 & \text{for } \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n.
\end{cases}
\]

Combining these two cases one can see that the weights of the cycles \( C^3_i \) are different to the weights of the cycles \( C^3_j \). This shows that \( \alpha_2 \) is an edge \( C_3 \)-irregular labeling of \( A_n \). Therefore, \( \text{ehs}(A_n, C_4) \leq \left\lceil \frac{2n+2}{3} \right\rceil \) and we arrive at the desired result.

\[ \square \]

3. Triangular Ladder and Diagonal Ladder

Let \( L_n \cong P_n \square P_2, n \geq 2 \), be a ladder with the vertex set \( V(L_n) = \{x_i, y_i : i = 1, 2, \ldots, n\} \) and the edge set \( E(L_n) = \{x_ix_{i+1}, y_iy_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_iy_i : i = 1, 2, \ldots, n\} \). The triangular ladder \( TL_n, n \geq 2 \), is obtained from a ladder \( L_n \) by adding the edges \( y_ix_{i+1} \) for \( i = 1, 2, \ldots, n-1 \).

**Theorem 4.** Let \( TL_n, n \geq 2 \), be a triangular ladder. Then

\[ \text{ehs}(TL_n, C_3) = \left\lfloor \frac{2n}{3} \right\rfloor. \]

**Proof.** The triangular ladder \( TL_n, n \geq 2 \), admits a \( C_3 \)-covering with exactly \( 2(n-1) \) cycles \( C_3 \). There are two types of cycles \( C_3 \) that cover \( TL_n \). The first type of cycles \( C_3 \) has the vertex set \( V(C^3_3) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n-1\} \) and the edge set \( E(C^3_3) = \{x_ix_{i+1}, y_iy_{i+1}, y_ix_{i+1} : 1 \leq i \leq n-1\} \). The second type of cycles \( C_3 \) has the vertex set \( V(C^3_3) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n-1\} \) and the edge set \( E(C^3_3) = \{y_iy_{i+1}, y_ix_{i+1}, y_{i+1}x_{i+1} : 1 \leq i \leq n-1\} \).

According to Theorem 1 it follows that \( \text{ehs}(TL_n, C_3) \geq \left\lfloor \frac{2n}{3} \right\rfloor \). To show that \( \left\lceil \frac{2n}{3} \right\rceil \) is an upper bound for the edge \( C_3 \)-irregularity strength of \( TL_n \) we define a \( C_3 \)-irregular edge labeling \( \alpha_3 : E(TL_n) \rightarrow \{1, 2, \ldots, \left\lceil \frac{2n}{3} \right\rceil \} \) as follows. Let us consider three cases.

**Case 1.** When \( i \equiv 0 \pmod{3} \), then

\[
\alpha_3(x_ix_{i+1}) = \alpha_3(y_iy_{i+1}) = \frac{2i}{3} \quad \text{for } i = 3, 6, \ldots, n-1, \\
\alpha_3(y_ix_{i+1}) = \frac{2i+3}{3} \quad \text{for } i = 3, 6, \ldots, n-1, \\
\alpha_3(x iy_i) = \frac{2i}{3} \quad \text{for } i = 3, 6, \ldots, n.
\]
Case 2. When \( i \equiv 1 \pmod{3} \), then

\[
\alpha_3(x_ix_{i+1}) = \alpha_3(y_iy_{i+1}) = \alpha_3(y_ix_{i+1}) = \frac{2i+1}{3}
\]

for \( i = 1, 4, \ldots, n-1 \),
\[
\alpha_3(x_iy_i) = \frac{2i+1}{3}
\]

for \( i = 1, 4, \ldots, n \).

Case 3. When \( i \equiv 2 \pmod{3} \), then

\[
\alpha_3(x_ix_{i+1}) = \frac{2i-1}{3}
\]

for \( i = 2, 5, \ldots, n-1 \),
\[
\alpha_3(y_iy_{i+1}) = \alpha_3(y_ix_{i+1}) = \frac{2i+2}{3}
\]

for \( i = 2, 5, \ldots, n-1 \),
\[
\alpha_3(x_iy_i) = \frac{2i+2}{3}
\]

for \( i = 2, 5, \ldots, n \).

It is a routine matter to verify that under the labeling \( \alpha_3 \) all edge labels are at most \( \left\lfloor \frac{2n}{3} \right\rfloor \). It is not difficult to see that under the edge labeling \( \alpha_3 \) the weights of the cycles \( C_3^i \), \( 1 \leq i \leq n-1 \), are of the form

\[
wt_{\alpha_3}(C_3^i) = \alpha_3(x_ix_{i+1}) + \alpha_3(x_iy_i) + \alpha_3(y_ix_{i+1}) = 2i + 1.
\]

The weights of the cycles \( C_3^i \), \( 1 \leq i \leq n-1 \), are of the form

\[
wt_{\alpha_3}(C_3^i) = \alpha_3(y_iy_{i+1}) + \alpha_3(x_iy_{i+1}) + \alpha_3(y_ix_{i+1}) = 2(i+1).
\]

Combining these two cases we obtained that the weights are different for any two distinct cycles \( C_3 \). Thus \( \text{ehs}(\mathcal{L}_n, C_3) \leq \left\lfloor \frac{2n}{3} \right\rfloor \). This completes the proof. \( \blacksquare \)

The diagonal ladder \( DL_n \) is obtained from a ladder \( L_n \) by adding the edges \( \{x_iy_{i+1}, x_{i+1}y_i : 1 \leq i \leq n-1 \} \). So the diagonal ladder \( DL_n \) contains \( 2n \) vertices and \( 5n - 4 \) edges.

**Theorem 5.** Let \( DL_n \), \( n \geq 2 \), be a diagonal ladder. Then

\[
\text{ehs}(DL_n, K_4) = \left\lfloor \frac{n+4}{6} \right\rfloor.
\]

**Proof.** The diagonal ladder \( DL_n, n \geq 2 \), admits a \( K_4 \)-covering with exactly \( (n-1) \) complete graphs \( K_4 \). The \( K_4^i \) has the vertex set \( V(K_4^i) = \{x_i, y_i, x_{i+1}, y_{i+1} : 1 \leq i \leq n-1 \} \) and the edge set \( E(K_4^i) = \{x_ix_{i+1}, x_{i+1}y_{i+1}, y_{i+1}y_i, x_iy_i, x_{i+1}y_i, y_{i+1}x_i, x_iy_{i+1}, x_{i+1}y_{i+1} : 1 \leq i \leq n-1 \} \).

With respect to Theorem 1 it follows that \( \text{ehs}(DL_n, K_4) \geq \left\lfloor \frac{n+4}{6} \right\rfloor \). To show that \( \text{ehs}(DL_n, K_4) \leq \left\lfloor \frac{n+4}{6} \right\rfloor \) we define a \( K_4 \)-irregular edge labeling \( \alpha_4 : \)
\[ E(DL_n) \rightarrow \{1, 2, \ldots, \left\lceil \frac{n+4}{6} \right\rceil \}, \text{ in the following way.} \]

\[
\begin{align*}
\alpha_4(x_iy_{i+1}) &= \left\lceil \frac{i}{6} \right\rceil \quad &\text{for } 1 \leq i \leq n - 1, \\
\alpha_4(x_iy_i) &= \begin{cases} 
\left\lceil \frac{i}{6} \right\rceil & \text{for } 1 \leq i \leq 5, \\
\left\lceil \frac{i+1}{6} \right\rceil & \text{for } 6 \leq i \leq n - 1, 
\end{cases} \\
\alpha_4(x_iy_i) &= \begin{cases} 
\left\lceil \frac{i}{6} \right\rceil & \text{for } 1 \leq i \leq 4, \\
\left\lceil \frac{i+2}{6} \right\rceil & \text{for } 5 \leq i \leq n,
\end{cases} \\
\alpha_4(x_iy_i) &= \begin{cases} 
1 & \text{for } i = 1, \\
\left\lceil \frac{i+5}{6} \right\rceil & \text{for } 2 \leq i \leq n - 1, 
\end{cases} \\
\alpha_4(x_{i+1}y_i) &= \begin{cases} 
1 & \text{for } i = 1, 2, \\
\left\lceil \frac{i+4}{6} \right\rceil & \text{for } 3 \leq i \leq n - 1.
\end{cases}
\end{align*}
\]

One can verify that under the labeling \( \alpha_4 \) all edge labels are at least 1 and at most \( \left\lceil \frac{n+4}{6} \right\rceil \). To show that \( \alpha_4 \) is edge \( K_4 \)-irregular labeling it will be enough to show that \( wt_{\alpha_4}(K^i_4) < wt_{\alpha_4}(K^{i+1}_4) \). It is a simple mathematical exercise that the weights of the subgraphs \( K^i_4 \), \( i = 1, 2, 3, 4, 5 \) are \( wt_{\alpha_4}(K^i_4) = 5 + i \).

For \( i = 6, 7, \ldots, n - 1 \) we get

\[
wt_{\alpha_4}(K^i_4) = \sum_{e \in E(K^i_4)} \alpha_4(e) = \alpha_4(x_iy_{i+1}) + \alpha_4(y_iy_{i+1}) + \alpha_4(x_iy_i) + \alpha_4(x_{i+1}y_i)
\]

\[
+ \alpha_4(x_iy_i) + \alpha_4(x_{i+1}y_i) = \left\lceil \frac{i+1}{6} \right\rceil + \left\lceil \frac{i}{6} \right\rceil + \left\lceil \frac{i+2}{6} \right\rceil + \left\lceil \frac{i+3}{6} \right\rceil + \left\lceil \frac{i+4}{6} \right\rceil
\]

\[
\left\lceil \frac{i+4}{6} \right\rceil = 5 + i.
\]

This proves that \( wt_{\alpha_4}(K^{i+1}_4) = wt_{\alpha_4}(K^i_4) + 1 \) for \( i = 1, 2, \ldots, n - 1 \). Therefore, \( \alpha_4 \) is an edge \( K_4 \)-irregular labeling of \( DL_n \). Thus \( ehs(DL_n, K_4) \leq \left\lceil \frac{n+4}{6} \right\rceil \). This concludes the proof.

\section{Wheel and Gear Graph}

A wheel \( W_n \), \( n \geq 3 \), is a graph obtained by joining all vertices of cycle \( C_n \) to a further vertex \( c \), called the center. Thus \( W_n \) contains \( n + 1 \) vertices, say, \( c, x_1, x_2, \ldots, x_n \) and \( 2n \) edges, say, \( cx_i, x_ix_{i+1}, 1 \leq i \leq n \), where the indices are taken modulo \( n \).

\begin{theorem}
Let \( W_n, n \geq 4 \), be a wheel. Then

\[
ehs(W_n, C_3) = \left\lceil \frac{n+2}{3} \right\rceil.
\]
\end{theorem}
Proof. The wheel $W_n$, $n \geq 4$, admits a $C_3$-covering with exactly $n$ cycles $C_3$. Every cycle $C_3$ in $W_n$ is of the form $C_3 = cx_ix_{i+1}$, where $i = 1, 2, \ldots, n$ with indices taken modulo $n$.

According to Theorem 1 we have that $\text{ehs}(W_n, C_3) \geq \lceil \frac{n+2}{3} \rceil$. To show that

$$\text{ehs}(G_n, C_4) \geq \lceil \frac{n+3}{4} \rceil$$

we define a $C_4$-irregular edge labeling $\alpha_5 : E(W_n) \to \{1, 2, \ldots, \lceil \frac{n+2}{3} \rceil \}$ as follows.

$$\alpha_5(x_i, x_{i+1}) = \begin{cases} 
  i - \lfloor \frac{i}{3} \rfloor & \text{for } i = 1, 2, \\
  \lfloor \frac{n+2}{3} \rfloor & \text{for } i = 3 \leq i \leq \lfloor \frac{n+1}{2} \rfloor - 1, \\
  \lfloor \frac{n+2}{3} \rfloor & \text{for } i = \lfloor \frac{n+1}{2} \rfloor, \\
  n - i + 1 - \lfloor \frac{n-i}{3} \rfloor & \text{for } 1 \leq i \leq n,
\end{cases}
$$

$$\alpha_5(cx_i) = \begin{cases} 
  1 & \text{for } i = 1, \\
  1 - \lfloor \frac{i-2}{3} \rfloor & \text{for } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\
  \lfloor \frac{n+2}{3} \rfloor - 1 & \text{for } i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \equiv 0, 1 \pmod{3}, \\
  \lfloor \frac{n+2}{3} \rfloor - 1 & \text{for } i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \equiv 2 \pmod{3}, \\
  n - i + 1 - \lfloor \frac{n-i-1}{3} \rfloor & \text{for } \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1, \\
  2 & \text{for } i = n.
\end{cases}
$$

It is a matter of routine checking that under the labeling $\alpha_5$ all edge labels are at most $\lfloor \frac{n+2}{3} \rfloor$. For the $C_3$-weight of the cycle $C_3$ we get

$$\text{wt}_{\alpha_5}(C_3^3) = \alpha_5(cx_i) + \alpha_5(cx_{i+1}) + \alpha_5(x_ix_{i+1}) = \begin{cases} 
  2i + 1 & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\
  2(n+2-i) & \text{for } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n.
\end{cases}$$

Clearly, the weights of $C_3^3$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ are odd and increasing. On the other hand the weights of $C_3^3$ for $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n$ are even and decreasing. So, it concludes that all the weights of $C_3^3$ are different. Thus $\alpha_5$ is an edge $C_3$-irregular labeling of $W_n$ and $\text{ehs}(W_n, C_3) \leq \lfloor \frac{n+2}{3} \rfloor$. This completes the proof.

A gear graph $G_n$ is obtained from $W_n$ by inserting a vertex to each edge on the cycle $C_n$. Then the vertex set of $G_n$ is $V(G_n) = \{c, x_i, y_i : 1 \leq i \leq n\}$ and the edge set is $E(G_n) = \{x_iy_i, y_ix_{i+1}, cx_i : 1 \leq i \leq n\}$ with indices taken modulo $n$.

Theorem 7. Let $G_n, n \geq 3$, be a gear graph. Then

$$\text{ehs}(G_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. The gear $G_n, n \geq 3$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. According to Theorem 1 we obtain that $\text{ehs}(G_n, C_4) \geq \left\lceil \frac{n+3}{4} \right\rceil$. To show that
\[ \left\lceil \frac{n+3}{4} \right\rceil \text{ is an upper bound for the edge } C_4 \text{-irregularity strength of } G_n \text{ we define a } \]
\[ C_4 \text{-irregular edge labeling } \alpha_6 : E(G_n) \rightarrow \{1, 2, \ldots, \left\lceil \frac{n+3}{4} \right\rceil \}, \text{ in the following way.} \]
\[ \alpha_6(x_iy_i) = \begin{cases} 
\left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
\left\lfloor \frac{n}{4} \right\rfloor & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \not\equiv 2 \pmod{4}, \\
\left\lfloor \frac{n-i+2}{2} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n, \\
\end{cases} \]
\[ \alpha_6(y_ix_{i+1}) = \begin{cases} 
\left\lfloor \frac{i+1}{2} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\left\lfloor \frac{n-i+1}{2} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n, \\
\end{cases} \]
\[ \alpha_6(cx_i) = \begin{cases} 
\left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\left\lfloor \frac{n-i+3}{2} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \\
\end{cases} \]

It is easy to verify that under the labeling \( \alpha_6 \) all edge labels are at most \( \left\lceil \frac{n+3}{4} \right\rceil \). For the \( C_4 \)-weight of the cycle \( C_4 \), \( i = 1, 2, \ldots, n \), under the edge labeling \( \alpha_6 \), we get
\[ wt_{\alpha_6}(C_4^i) = \sum_{e \in E(C_4^i)} \alpha_6(e) = \alpha_6(cx_i) + \alpha_6(cx_{i+1}) + \alpha_6(x_iy_i) + \alpha_6(y_ix_{i+1}) \]
\[ = \begin{cases} 
2i + 2 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
2n + 5 - 2i & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \\
\end{cases} \]

Clearly, the weights of \( C_4^i \) for \( 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \) are even and increasing. On the other hand the weights of \( C_4^i \) for \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n \) are odd and decreasing. So, it concluded that all the weights of \( C_4^i \) are different. Thus \( \alpha_6 \) is an edge \( C_4 \)-irregular labeling of \( G_n \). Hence \( \text{ehs}(G_n, C_4) \leq \left\lceil \frac{n+3}{4} \right\rceil \). This completes the proof of theorem.

5. Conclusion

In this paper we have investigated the edge \( H \)-irregularity strength of some graphs. We have found the exact values of this parameter for several families of graphs namely, prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs.

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References

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