ON EDGE $H$-IRREGULARITY STRENGTHS OF SOME GRAPHS

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Abstract

For a graph $G$ an edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. In this case we say that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. An $H$-covering of graph $G$ is an $(H_1, H_2, \ldots, H_t)$-edge covering in which every subgraph $H_i$ is isomorphic to a given graph $H$.

Let $G$ be a graph admitting $H$-covering. An edge $k$-labeling $\alpha : E(G) \to \{1, 2, \ldots, k\}$ is called an $H$-irregular edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ their weights

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wt_α(H') and wt_α(H'') are distinct. The weight of a subgraph H under an edge k-labeling α is the sum of labels of edges belonging to H. The edge H-irregularity strength of a graph G, denoted by ehs(G, H), is the smallest integer k such that G has an H-irregular edge k-labeling.

In this paper we determine the exact values of ehs(G, H) for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs. Moreover the subgraph H is isomorphic to only C_4, C_3 and K_4.

**Keywords:** H-irregular edge labeling, edge H-irregularity strength, prism, antiprism, triangular ladder, diagonal ladder, wheel, gear graph.

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## 1. Introduction

Consider a simple and finite graph G = (V, E) of order at least 2. An edge k-labeling is a function α: E(G) → {1, 2, ..., k}, where k is a positive integer. Then the associated weight of a vertex x ∈ V(G) is w_α(x) = Σ_{xy ∈ E(G)} α(xy), where the sum is taken over all edges incident to x. Such a labeling α is called irregular if the obtained weights of all vertices are different. The smallest positive integer k for which there exists an irregular labeling of G is called the irregularity strength of G and is denoted by s(G). If it does not exist, then we write s(G) = ∞. One can easily see that s(G) < ∞ if and only if G contains no isolated edges and has at most one isolated vertex.

The notion of the irregularity strength was firstly introduced by Chartrand et al. in [7]. Some results on the irregularity strength can be found in [2, 3, 5, 6, 8, 9, 11–14].

A vertex k-labeling β: V(G) → {1, 2, ..., k} is called an edge irregular k-labeling of the graph G if the weights w_β(xy) ≠ w_β(x'y') for every two distinct edges xy and x'y', where the weight of an edge xy ∈ E(G) is w_β(xy) = β(x)+β(y). The minimum k for which a graph G admits an edge irregular k-labeling is called the edge irregularity strength of G, denoted by es(G). The notion of the edge irregularity strength was defined by Ahmad et al. in [1].

A family of subgraphs H_1, H_2, ..., H_t is said to be an edge-covering of G if each edge of E(G) belongs to at least one of the subgraphs H_i, i = 1, 2, ..., t. In this case we say that G admits an (H_1,H_2,...,H_t)-(edge) covering. If every subgraph H_i, i = 1, 2, ..., t, is isomorphic to a given graph H, then the graph G admits an H-covering.

Motivated by the irregularity strength and the edge irregularity strength of a graph G Ashraf et al. in [4] introduced a new parameter, edge H-irregularity strength, as a natural extension of the parameters s(G) and es(G). Let G be a graph admitting H-covering. An edge k-labeling α is called an H-irregular
edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ we have

$$wt_{\alpha}(H') = \sum_{e \in E(H')} \alpha(e) \neq \sum_{e \in E(H'')} \alpha(e) = wt_{\alpha}(H'').$$

The edge $H$-irregularity strength of a graph $G$, denoted by $ehs(G, H)$, is the smallest integer $k$ for which $G$ has an $H$-irregular edge $k$-labeling.

Next theorem proved in [4] gives the lower bound of the edge $H$-irregularity strength of a graph $G$.

**Theorem 1** [4]. Let $G$ be a graph admitting an $H$-covering and $t$ is the number of all the subgraphs isomorphic with $H$. Then

$$ehs(G, H) \geq \left\lceil 1 + \frac{t - 1}{|E(H)|} \right\rceil.$$

Note that the parameter $t$ is the number of all subgraphs of $G$ isomorphic to $H$. In this paper we determine exact values of the edge $H$-irregularity strength for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs for some $H$. Moreover the subgraph $H$ is isomorphic to only $C_4$, $C_3$ and $K_4$.

2. Prism and Antiprism

The prism $D_n$ can be defined as the Cartesian product $C_n \square P_2$ of a cycle on $n$ vertices with a path on 2 vertices. Let $V(C_n \square P_2) = \{x_i, y_i : 1 \leq i \leq n\}$ be the vertex set and $E(C_n \square P_2) = \{x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n\}$ be the edge set, where the indices are taken modulo $n$. Hence, the graph $D_n$ has $2n$ vertices and $3n$ edges.

**Theorem 2.** Let $D_n = C_n \square P_2$, $n \geq 3$, $n \neq 4$, be a prism. Then

$$ehs(D_n, C_4) = \left\lceil \frac{n + 3}{4} \right\rceil.$$

**Proof.** The prism $D_n$, $n \geq 3$, $n \neq 4$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. We denote these 4-cycles by the symbols $C_4^i$, $i = 1, 2, \ldots, n$, such that the vertex set of $C_4^i$ is $V(C_4^i) = \{x_i, x_{i+1}, y_i, y_{i+1}\}$ and the edge set is $E(C_4^i) = \{x_ix_{i+1}, y_iy_{i+1}, x_iy_i, x_{i+1}y_{i+1}\}$.

From Theorem 1 it follows that $ehs(D_n, C_4) \geq \left\lceil \frac{n+3}{4} \right\rceil$. To show that $\left\lceil \frac{n+3}{4} \right\rceil$ is an upper bound for the edge $C_4$-irregularity strength of $D_n$ we define a $C_4$-irregular edge labeling $\alpha_1 : E(D_n) \to \{1, 2, \ldots, \left\lceil \frac{n+3}{4} \right\rceil\}$, in the following way. We distinguish to cases according to the parity of $n$. 
Case 1. When $n$ is odd, then

\[
\alpha_1(x_i x_{i+1}) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+1-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(y_i y_{i+1}) = \begin{cases} 
\left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+2-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(x_i y_i) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+3-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n.
\end{cases}
\]

Case 2. When $n$ is even, then

\[
\alpha_1(x_i x_{i+1}) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lceil \frac{n+1-i}{2} \right\rceil & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(y_i y_{i+1}) = \begin{cases} 
\left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lceil \frac{n+3}{2} \right\rceil & \text{for } i = \frac{n}{2} + 1,
\end{cases}
\]

\[
\frac{n+2}{2} - i \right\rceil & \text{for } \frac{n}{2} + 2 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(x_i y_i) = \begin{cases} 
\frac{n}{4} + 1 & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 0 \pmod{4}, \\
\frac{n+2}{4} & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 2 \pmod{4}, \\
\left\lceil \frac{n+3-i}{2} \right\rceil & \text{for } \frac{n}{2} + 2 \leq i \leq n.
\end{cases}
\]

It is easy to see that under the labeling $\alpha_1$ all edge labels are at most $\left\lceil \frac{n+3}{4} \right\rceil$. The $C_4$-weights of the cycles $C_4^i$, $i = 1, 2, \ldots, n$, under the edge labeling $\alpha_1$, are given by

\[
wt_{\alpha_1}(C_4^i) = \sum_{e \in E(C_4^i)} \alpha_1(e) = \alpha_1(x_i x_{i+1}) + \alpha_1(y_i y_{i+1}) + \alpha_1(x_i y_i) + \alpha_1(x_{i+1} y_{i+1}).
\]

Case 1. When $n$ is odd, then

\[
wt_{\alpha_1}(C_4^i) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2i + 2 \quad \text{for } 1 \leq i \leq \frac{n-1}{2},
\]

\[
wt_{\alpha_1}(C_4^\frac{n+1}{2}) = \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n+3}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n+3}{2} \right\rceil = n + 3,
\]

\[
wt_{\alpha_1}(C_4^\frac{n+1-i}{2}) = \left\lceil \frac{n+1-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil + \left\lceil \frac{n+3-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil = 2n + 5 - 2i \quad \text{for } \frac{n+1}{2} + 1 \leq i \leq n - 1,
\]

\[
wt_{\alpha_1}(C_4^n) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{3}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil = 5.
\]
Case 2. When \( n \) is even, then
\[
\text{wt}_{\alpha_1}(C_{4}^i) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2i + 2 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} - 1,
\]
\[
\text{wt}_{\alpha_1}\left(C_{4}^{\frac{n}{4}}\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + 1 = n + 2 \quad \text{for} \quad n \equiv 0 \pmod{4},
\]
\[
\text{wt}_{\alpha_1}\left(C_{4}^{\frac{n+1}{4}}\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+3}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil = n + 3 \quad \text{for} \quad n \equiv 2 \pmod{4},
\]
\[
\text{wt}_{\alpha_1}\left(C_{4}^{n}\right) = \left\lceil \frac{n+1-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil + \left\lceil \frac{n+3-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil = 2n - 2i \\
\text{for} \quad \frac{n}{2} + 2 \leq i \leq n - 1,
\]
\[
\text{wt}_{\alpha_1}\left(C_{4}^{n-3}\right) = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = 5.
\]

Combining the previous we get that
\[
\text{wt}_{\alpha_1}(C_{4}^i) = \begin{cases} 
2(1+i) & \text{for} \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
1 + 2(n+2-i) & \text{for} \quad \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n.
\end{cases}
\]

One can see that the weights of cycles \( C_{4}^i \), for \( i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \), are even and in increasing order, therefore \( \text{wt}_{\alpha_1}(C_{4}^{i+1}) > \text{wt}_{\alpha_1}(C_{4}^{i}) \).

On the other hand, the weights of cycles \( C_{4}^i \), for \( i = \left\lceil \frac{n}{2} \right\rceil + 1, \ldots, n \), are odd and in decreasing order, therefore \( \text{wt}_{\alpha_1}(C_{4}^{i+1}) < \text{wt}_{\alpha_1}(C_{4}^{i}) \).

Thus the edge weights are distinct numbers from the set \( \{4, 5, \ldots, n+3\} \).
This shows that \( \text{ehs}(D_n, C_4) \leq \left\lfloor \frac{n+3}{4} \right\rfloor \). Hence the proof is concluded. \( \blacksquare \)

The antiprism \( A_n \) \([10]\), \( n \geq 3 \), is a 4-regular graph (Archimedean convex polytope), consisting of \( 2n \) vertices and \( 4n \) edges. The vertex set and edge set of \( A_n \) are defined as: \( V(A_n) = \{x_i, y_i : 1 \leq i \leq n\} \), \( E(A_n) = \{x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_{i+1}y_i : 1 \leq i \leq n\} \), with indices taken modulo \( n \).

**Theorem 3.** Let \( A_n, n \geq 4 \), be an antiprism. Then
\[
\text{ehs}(A_n, C_3) = \left\lfloor \frac{2n + 2}{3} \right\rfloor.
\]

**Proof.** The antiprism \( A_n, n \geq 4 \), admits a \( C_3 \)-covering with exactly \( 2n \) cycles \( C_3 \).
The first type of the cycle \( C_3 \) has the vertex set \( V(C_3) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n\} \)
and the edge set \( E(C_3) = \{x_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_{i+1}y_i : 1 \leq i \leq n\} \). The second type
of the cycle $C_3$ has the vertex set $V(C_3^i) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n\}$ and the edge set $E(C_3^i) = \{y_iy_{i+1}, y_ix_{i+1}, y_{i+1}x_{i+1} : 1 \leq i \leq n\}$. Note that the indices are taken modulo $n$.

From Theorem 1 it follows that $\text{ehs}(A_n, C_3) \geq \left\lceil \frac{2n+2}{3} \right\rceil$. To show that $\left\lceil \frac{2n+2}{3} \right\rceil$ is an upper bound for the edge $C_3$-irregularity strength of $A_n$ we define a $C_3$-irregular edge labeling $\alpha_2 : E(A_n) \rightarrow \{1, 2, \ldots, \left\lceil \frac{2n+2}{3} \right\rceil\}$, in the following way. We distinguish two cases.

**Case 1.** When $n \equiv 0, 4, 5 \pmod{6}$, then

$$\alpha_2(x_ix_{i+1}) = \begin{cases} i & \text{for } i = 1, 2, \\ i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq t - 1, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = t, \\ n - i + 2 + \left\lfloor \frac{n-i-2}{3} \right\rfloor & \text{for } t + 1 \leq i \leq n - 2, \\ n - i + 2 & \text{for } i = n - 1, n, \end{cases}$$

where $t = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 5 \pmod{6}, \\ \frac{n}{2} + 1 & \text{if } n \equiv 0, 4 \pmod{6}. \end{cases}$

$$\alpha_2(y_iy_{i+1}) = \begin{cases} i + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 5 \pmod{6}, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 0, 4 \pmod{6}, \\ n - i + 2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n - 1, \\ 1 & \text{for } i = n, \end{cases}$$

$$\alpha_2(x_ix_i) = \begin{cases} i + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \equiv 0, 5 \pmod{6}, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \equiv 4 \pmod{6}, \\ n - i + 3 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n - 1, \\ 2 & \text{for } i = n, \end{cases}$$

$$\alpha_2(y_ix_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ i + 1 + \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 0, 4 \pmod{6}, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 5 \pmod{6}, \\ n - i + 2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$
Case 2. When \( n \equiv 1, 2, 3 \pmod{6} \), then

\[
\alpha_2(x_ix_{i+1}) = \begin{cases} 
    i & \text{for } i = 1, 2, \\
    i + \lceil \frac{i}{3} \rceil & \text{for } 3 \leq i \leq t - 1, \\
    \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = t \text{ and } n \equiv 2 \pmod{6}, \\
    n - i + 3 + \left\lceil \frac{n-1}{3} \right\rceil & \text{for } t + 1 \leq i \leq n - 2, \\
    n - i + 2 & \text{for } i = n - 1, n,
\end{cases}
\]

where \( t = \begin{cases} 
    \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{6}, \\
    \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{6}.
\end{cases} \)

\[
\alpha_2(y_iy_{i+1}) = \begin{cases} 
    i + 1 + \left\lceil \frac{i-1}{3} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\
    \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\
    n - i + 2 + \left\lceil \frac{n-1}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n - 1, \\
    1 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(x_iy_i) = \begin{cases} 
    i + \left\lceil \frac{i-1}{3} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
    \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 2 \pmod{6}, \\
    n - i + 3 + \left\lceil \frac{n-1}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
    2 & \text{for } i = n,
\end{cases}
\]

Now we compute the \( C_3 \)-weights under the edge labeling \( \alpha_2 \) as follows. For the weights of 3-cycles of the first type we get

\[
wt_{\alpha_2}(C^i_3) = \sum_{e \in E(C^i_3)} \alpha_2(e) = \alpha_2(x_ix_{i+1}) + \alpha_2(x_iy_i) + \alpha_2(y_ix_{i+1})
\]

\[
= \begin{cases} 
    4i - 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
    4n - 4i + 6 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\]
and for the weights of 3-cycles of the second type we have

\[
wt_{\alpha_2}(C_3^i) = \sum_{e \in E(C_3^i)} \alpha_2(e) = \alpha_2(y_iy_{i+1}) + \alpha_2(y_ix_{i+1}) + \alpha_2(y_{i+1}x_{i+1})
\]

\[
= \begin{cases} 
4i + 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1, \\
4n - 4i + 4 & \text{for } \left\lfloor \frac{n+1}{2} \right\rfloor \leq i \leq n.
\end{cases}
\]

Combining these two cases one can see that the weights of the cycles \(C_3^i\) are different to the weights of the cycles \(C_3^j\). This shows that \(\alpha_2\) is an edge \(C_3\)-irregular labeling of \(A_6\). Therefore, \(ehs(A_6, C_3) \leq \left\lceil \frac{2n+2}{3} \right\rceil\) and we arrive at the desired result. ~

\[\Box\]

3. Triangular Ladder and Diagonal Ladder

Let \(L_n \cong P_n \sqcup P_2, n \geq 2\), be a ladder with the vertex set \(V(L_n) = \{x_i, y_i : i = 1, 2, \ldots, n\}\) and the edge set \(E(L_n) = \{x_ix_{i+1}, y_iy_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_iy_i : i = 1, 2, \ldots, n\}\). The triangular ladder \(TL_n, n \geq 2\), is obtained from a ladder \(L_n\) by adding the edges \(y_ix_{i+1}\) for \(i = 1, 2, \ldots, n-1\).

Theorem 4. Let \(TL_n, n \geq 2\), be a triangular ladder. Then

\[ehs(TL_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil.\]

Proof. The triangular ladder \(TL_n, n \geq 2\), admits a \(C_3\)-covering with exactly \(2(n-1)\) cycles \(C_3\). There are two types of cycles \(C_3\) that cover \(TL_n\). The first type of cycles \(C_3\) has the vertex set \(V(C_3^i) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n-1\}\) and the edge set \(E(C_3^i) = \{x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\}\). The second type of cycles \(C_3\) has the vertex set \(V(C_3^j) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n-1\}\) and the edge set \(E(C_3^j) = \{y_iy_{i+1}, y_{i+1}x_{i+1} : 1 \leq i \leq n-1\}\).

According to Theorem 1 it follows that \(ehs(TL_n, C_3) \geq \left\lceil \frac{2n}{3} \right\rceil\). To show that \(\left\lceil \frac{2n}{3} \right\rceil\) is an upper bound for the edge \(C_3\)-irregularity strength of \(TL_n\) we define a \(C_3\)-irregular edge labeling \(\alpha_3 : E(TL_n) \to \{1, 2, \ldots, \left\lceil \frac{2n}{3} \right\rceil\}\) as follows. Let us consider three cases.

Case 1. When \(i \equiv 0 \pmod{3}\), then

\[
\alpha_3(x_ix_{i+1}) = \begin{cases} 
\frac{2i}{3} & \text{for } i = 3, 6, \ldots, n-1, \\
\frac{2i+3}{3} & \text{for } i = 3, 6, \ldots, n-1, \\
\frac{2i}{3} & \text{for } i = 3, 6, \ldots, n.
\end{cases}
\]

\[\Box\]
Case 2. When $i \equiv 1 \pmod{3}$, then

$$\alpha_3(x_ix_{i+1}) = \alpha_3(y_iy_{i+1}) = \alpha_3(y_ix_i) = \frac{2i+1}{3}$$

for $i = 1, 4, \ldots, n-1$,

$$\alpha_3(x_iy_i) = \frac{2i+1}{3}$$

for $i = 1, 4, \ldots, n$.

Case 3. When $i \equiv 2 \pmod{3}$, then

$$\alpha_3(x_ix_{i+1}) = \frac{2i-1}{3}$$

for $i = 2, 5, \ldots, n-1$,

$$\alpha_3(y_iy_{i+1}) = \alpha_3(y_ix_i) = \frac{2i+2}{3}$$

for $i = 2, 5, \ldots, n-1$,

$$\alpha_3(x_iy_i) = \frac{2i+2}{3}$$

for $i = 2, 5, \ldots, n$.

It is a routine matter to verify that under the labeling $\alpha_3$ all edge labels are at most $\left\lceil \frac{2n}{3} \right\rceil$. It is not difficult to see that under the edge labeling $\alpha_3$ the weights of the cycles $C^i_3$, $1 \leq i \leq n-1$, are of the form

$$wt_{\alpha_3}(C^i_3) = \alpha_3(x_ix_{i+1}) + \alpha_3(x_iy_i) + \alpha_3(y_ix_{i+1}) = 2i + 1.$$

The weights of the cycles $C^i_3$, $1 \leq i \leq n-1$, are of the form

$$wt_{\alpha_3}(C^i_3) = \alpha_3(y_iy_{i+1}) + \alpha_3(x_{i+1}y_{i+1}) + \alpha_3(y_ix_{i+1}) = 2(i+1).$$

Combining these two cases we obtained that the weights are different for any two distinct cycles $C_3$. Thus $ehs(TL_n, C_3) \leq \left\lceil \frac{2n}{3} \right\rceil$. This completes the proof. 

The diagonal ladder $DL_n$ is obtained from a ladder $L_n$ by adding the edges $\{x_iy_{i+1}, x_{i+1}y_i : 1 \leq i \leq n-1\}$. So the diagonal ladder $DL_n$ contains $2n$ vertices and $5n-4$ edges.

**Theorem 5.** Let $DL_n$, $n \geq 2$, be a diagonal ladder. Then

$$ehs(DL_n, K_4) = \left\lceil \frac{n+4}{6} \right\rceil.$$

**Proof.** The diagonal ladder $DL_n$, $n \geq 2$, admits a $K_4$-covering with exactly $(n-1)$ complete graphs $K_4$. The $K_4^i$ has the vertex set $V(K_4^i) = \{x_i, y_i, x_{i+1}, y_{i+1} : 1 \leq i \leq n-1\}$ and the edge set $E(K_4^i) = \{x_ix_{i+1}, x_iy_i, x_{i+1}y_{i+1}, y_{i+1}y_{i+1}, x_{i+1}y_{i} : 1 \leq i \leq n-1\}$.

With respect to Theorem 1 it follows that $ehs(DL_n, K_4) \geq \left\lceil \frac{n+4}{6} \right\rceil$. To show that $ehs(DL_n, K_4) \leq \left\lceil \frac{n+4}{6} \right\rceil$ we define a $K_4$-irregular edge labeling $\alpha_4$:
\[ E(DL_n) \rightarrow \{1, 2, \ldots, \lceil \frac{n+4}{6} \rceil \}, \text{ in the following way.} \]

\[
\alpha_4(y_iy_{i+1}) = \left\lfloor \frac{i}{6} \right\rfloor \quad \text{for } 1 \leq i \leq n-1, \\
\alpha_4(x_ix_{i+1}) = \begin{cases} \left\lfloor \frac{i}{6} \right\rfloor & \text{for } 1 \leq i \leq 5, \\ \left\lfloor \frac{i+1}{6} \right\rfloor & \text{for } 6 \leq i \leq n-1, \\ \end{cases} \\
\alpha_4(x_iy_i) = \begin{cases} \left\lfloor \frac{i}{6} \right\rfloor & \text{for } 1 \leq i \leq 4, \\ \left\lfloor \frac{i+2}{6} \right\rfloor & \text{for } 5 \leq i \leq n, \\ \end{cases} \\
\alpha_4(x_iy_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ \left\lfloor \frac{i+5}{6} \right\rfloor & \text{for } 2 \leq i \leq n-1, \\ \end{cases} \\
\alpha_4(x_{i+1}y_i) = \begin{cases} 1 & \text{for } i = 1, 2, \\ \left\lfloor \frac{i+4}{6} \right\rfloor & \text{for } 3 \leq i \leq n-1. \\ \end{cases}
\]

One can verify that under the labeling \( \alpha_4 \) all edge labels are at least 1 and at most \( \left\lceil \frac{n+4}{6} \right\rceil \). To show that \( \alpha_4 \) is edge \( K_4 \)-irregular labeling it will be enough to show that \( wt_{\alpha_4}(K_4^i) < wt_{\alpha_4}(K_4^{i+1}) \). It is a simple mathematical exercise that the weights of the subgraphs \( K_4^i, i = 1, 2, 3, 4, 5 \) are \( wt_{\alpha_4}(K_4^i) = 5 + i \).

For \( i = 6, 7, \ldots, n-1 \) we get

\[
wt_{\alpha_4}(K_4^i) = \sum_{e \in E(K_4^i)} \alpha_4(e) = \alpha_4(x_ix_{i+1}) + \alpha_4(y_iy_{i+1}) + \alpha_4(x_iy_i) + \alpha_4(x_{i+1}y_{i+1}) \\
+ \alpha_4(x_iy_{i+1}) + \alpha_4(x_{i+1}y_i) = \left\lceil \frac{i+1}{6} \right\rceil + \left\lceil \frac{i}{6} \right\rceil + \left\lceil \frac{i+2}{6} \right\rceil + \left\lceil \frac{i+3}{6} \right\rceil + \left\lceil \frac{i+5}{6} \right\rceil \\
+ \left\lceil \frac{i+4}{6} \right\rceil = 5 + i.
\]

This proves that \( wt_{\alpha_4}(K_4^{i+1}) = wt_{\alpha_4}(K_4^i) + 1 \) for \( i = 1, 2, \ldots, n-1 \). Therefore, \( \alpha_4 \) is an edge \( K_4 \)-irregular labeling of \( DL_n \). Thus \( \text{ehs}(DL_n, K_4) \leq \left\lceil \frac{n+4}{6} \right\rceil \). This concludes the proof.

\[ \text{Theorem 6.} \] Let \( W_n, n \geq 4, \) be a wheel. Then

\[ \text{ehs}(W_n, C_3) = \left\lceil \frac{n+2}{3} \right\rceil. \]
Proof. The wheel $W_n$, $n \geq 4$, admits a $C_3$-covering with exactly $n$ cycles $C_3$. Every cycle $C_3$ in $W_n$ is of the form $C_3 = c_{x_i, x_{i+1}}$, where $i = 1, 2, \ldots, n$ with indices taken modulo $n$.

According to Theorem 1 we have that $\text{ehs}(W_n, C_3) \geq \lceil \frac{n + 3}{4} \rceil$. To show that

$$\alpha_5(x_{i,x_{i+1}}) = \begin{cases} 
  i & \text{for } i = 1, 2, \\
  i - \lfloor \frac{i}{3} \rfloor & \text{for } 3 \leq i \leq \lceil \frac{n + 1}{2} \rceil - 1, \\
  \lceil \frac{n + 2}{3} \rceil & \text{for } i = \lceil \frac{n + 1}{2} \rceil, \\
  n - i + 1 - \lfloor \frac{n - i}{3} \rfloor & \text{for } \lceil \frac{n + 1}{2} \rceil + 1 \leq i \leq n,
\end{cases}$$

$$\alpha_5(c_{x_i}) = \begin{cases} 
  1 & \text{for } i = 1, \\
  i - 1 - \lfloor \frac{i - 2}{3} \rfloor & \text{for } 2 \leq i \leq \lceil \frac{n}{2} \rceil, \\
  \lceil \frac{n + 2}{3} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \equiv 0, 1 \mod 3, \\
  \lceil \frac{n + 2}{3} \rceil - 1 & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \equiv 2 \mod 3, \\
  n - i + 1 - \lfloor \frac{n - i - 1}{3} \rfloor & \text{for } \lceil \frac{n}{2} \rceil + 2 \leq i \leq n - 1, \\
  2 & \text{for } i = n.
\end{cases}$$

It is a matter of routine checking that under the labeling $\alpha_5$ all edge labels are at most $\lceil \frac{n + 2}{3} \rceil$. For the $C_3$-weight of the cycle $C_3$ we get

$$\text{wt}_{\alpha_5}(C_3) = \alpha_5(c_{x_i}) + \alpha_5(c_{x_{i+1}}) + \alpha_5(x_{i,x_{i+1}}) = \begin{cases} 
  2i + 1 & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\
  2(n + 2 - i) & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n.
\end{cases}$$

Clearly, the weights of $C_3^i$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ are odd and increasing. On the other hand the weights of $C_3^i$ for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$ are even and decreasing. So, it concludes that all the weights of $C_3^i$ are different. Thus $\alpha_5$ is an edge $C_3$-irregular labeling of $W_n$ and $\text{ehs}(W_n, C_3) \leq \lceil \frac{n + 2}{3} \rceil$. This completes the proof. \hfill \blacksquare

A gear graph $G_n$ is obtained from $W_n$ by inserting a vertex to each edge on the cycle $C_n$. Then the vertex set of $G_n$ is $V(G_n) = \{c, x_i, y_i : 1 \leq i \leq n\}$ and the edge set is $E(W_n) = \{x_{i,y_i}, y_{i,x_{i+1}}, c_{x_i} : 1 \leq i \leq n\}$ with indices taken modulo $n$.

Theorem 7. Let $G_n$, $n \geq 3$, be a gear graph. Then

$$\text{ehs}(G_n, C_4) = \left\lceil \frac{n + 3}{4} \right\rceil.$$

Proof. The gear $G_n$, $n \geq 3$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. According to Theorem 1 we obtain that $\text{ehs}(G_n, C_4) \geq \left\lceil \frac{n + 3}{4} \right\rceil$. To show that
\[\left\lceil \frac{n+3}{4} \right\rceil\] is an upper bound for the edge \(C_4\)-irregularity strength of \(G_n\) we define a \(C_4\)-irregular edge labeling \(\alpha_6 : E(G_n) \to \{1, 2, \ldots, \left\lceil \frac{n+3}{4} \right\rceil\}\), in the following way.

\[
\alpha_6(x_iy_i) = \begin{cases} 
\left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\
\left\lfloor \frac{n}{4} \right\rfloor & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \not\equiv 2 \pmod{4}, \\
\frac{n-2}{4} & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 2 \pmod{4}, \\
\frac{n-i+2}{2} & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, 
\end{cases}
\]

\[
\alpha_6(y_ix_{i+1}) = \begin{cases} 
\left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\left\lceil \frac{n-i+1}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, 
\end{cases}
\]

\[
\alpha_6(cx_i) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\left\lceil \frac{n-i+3}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. 
\end{cases}
\]

It is easy to verify that under the labeling \(\alpha_6\) all edge labels are at most \(\left\lceil \frac{n+3}{4} \right\rceil\). For the \(C_4\)-weight of the cycle \(C_4^n, i = 1, 2, \ldots, n\), under the edge labeling \(\alpha_6\), we get

\[
wt_{\alpha_6}(C_4^i) = \sum_{e \in E(C_4^i)} \alpha_6(e) = \alpha_6(cx_i) + \alpha_6(cx_{i+1}) + \alpha_6(x_iy_i) + \alpha_6(y_ix_{i+1})
\]

\[
= \begin{cases} 
2i + 2 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
2n + 5 - 2i & \text{for } \left\lfloor \frac{n}{2} \right\rceil + 1 \leq i \leq n. 
\end{cases}
\]

Clearly, the weights of \(C_4^n\) for \(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\) are even and increasing. On the other hand the weights of \(C_4^n\) for \(\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n\) are odd and decreasing. So, it concluded that all the weights of \(C_4^n\) are different. Thus \(\alpha_6\) is an edge \(C_4\)-irregular labeling of \(G_n\). Hence \(ehs(G_n, C_4) \leq \left\lceil \frac{n+3}{2} \right\rceil\). This completes the proof of theorem. 

\[
5. \text{ Conclusion}
\]

In this paper we have investigated the edge \(H\)-irregularity strength of some graphs. We have found the exact values of this parameter for several families of graphs namely, prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs.

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\section*{References}


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