RELAXED DP-COLORING AND ANOTHER GENERALIZATION OF DP-COLORING ON PLANAR GRAPHS WITHOUT 4-CYCLES AND 7-CYCLES

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Abstract

DP-coloring is generalized via relaxed coloring and variable degeneracy in [P. Sittitrai and K. Nakprasit, Sufficient conditions on planar graphs to have a relaxed DP-3-coloring, Graphs Combin. 35 (2019) 837–845], [K.M. Nakprasit and K. Nakprasit, A generalization of some results on list coloring and DP-coloring, Graphs Combin. 36 (2020) 1189–1201] and [P. Sittitrai and K. Nakprasit, An analogue of DP-coloring for variable degeneracy and its applications, Discuss. Math. Graph Theory]. In this work, we introduce another concept that includes two previous generalizations. We demonstrate its application on planar graphs without 4-cycles and 7-cycles. One implication is that the vertex set of every planar graph without 4-cycles and 7-cycles can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set. Additionally, we show that every planar graph without 4-cycles and 7-cycles is DP-(1,1,1)-colorable. This generalizes a result of Lih et al. [A note on list improper coloring planar graphs, Appl. Math. Lett. 14 (2001) 269–273] that every planar graph without 4-cycles and 7-cycles is (3,1)*-choosable.

Keywords: relaxed DP-colorings, variable degeneracy, planar graphs, discharging.

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1. Introduction

All considered graphs are finite, simple, undirected, and embedded in the plane. For a graph $G$, let its vertex set, edge set, face set, and minimum degree be denoted by $V(G)$, $E(G)$, $F(G)$, and $\delta(G)$, respectively. Let $d(x)$ denote the degree of $x$ where $x \in V(G) \cup F(G)$. A $k$-vertex (or $k^+$-vertex) is a vertex of degree $k$ (or at least $k$). Similar notation is applied to a cycle and a face. A face $f$ is simple if its boundary forms a cycle. A face $f$ and a vertex $v$ are incident if $v$ is on the boundary of $f$. We simply say two faces share an edge (or a vertex) instead of the boundary of two faces share an edge (or a vertex). Two faces are adjacent if they share at least one edge. If $G$ is a graph and $U \subseteq V(G)$, then $G[U]$ denote the subgraph of $G$ induced by $U$. A linear forest is a forest in which each component is a path.

Vizing [11] in 1976, and independently Erdős, Rubin, and Taylor [5] in 1979, introduced list coloring and choosability. An assignment $L$ of a graph $G$ assigns a list $L(v)$ (a set of colors) to each vertex $v$. A $k$-assignment $L$ is an assignment such that $|L(v)| = k$ for each vertex $v$. If a graph $G$ admits a proper coloring $f$ where $f(v) \in L(v)$ for each vertex $v$, then we say $G$ is $L$-colorable. A graph $G$ is $k$-choosable if it is $L$-colorable for each $k$-assignment $L$.

In 1999, Škrekovski [10] and Eaton and Hull [4] independently introduced the concept of relaxed list coloring. A graph $G$ with an assignment $L$ is $(L, d)^*$ choosable if each vertex $v$ of $G$ can be colored with a color $f(v) \in L(v)$ such that at most $d$ neighbors of $v$ receive the color $f(v)$. A graph $G$ is $(k, d)^*$-choosable if $G$ is $(L, d)^*$-choosable for each $k$-assignment $L$.

Dvořák and Postle [3] introduced a generalization of list coloring which they called correspondence coloring. Following Bernshteyn, Kostochka, and Pron [1], we call it a DP-coloring. Let $L$ be an assignment of a graph $G$. We call $(H, L)$ (or simply $H$) a cover of $G$ if it satisfies the following conditions.

(i) The vertex set of $H$ is $\bigcup_{u \in V(G)} \{ \{u\} \times L(u) \} = \{(u, c) : u \in V(G), c \in L(u)\}$.

(ii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (the matching may be empty).

(iii) If $uv \notin E(G)$, then no edges of $H$ connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

A transversal of $(H, L)$ is a vertex set $T \subseteq V(H)$ such that $|T \cap (\{u\} \times L(u))| = 1$ for each vertex $u$ in $G$. A DP-coloring of $(H, L)$ is a transversal $T$ of $(H, L)$ such that $T$ is independent. The DP-chromatic number of $G$ is the least number $k$ such that every cover $(H, L)$ of $G$ with $k$-assignment $L$ has a DP-coloring.

Since names of colors for distinct vertices in DP-coloring are irrelevant, we always assume in this paper that a $k$-assignment of a graph $G$ has $L(v) = \{1, \ldots, k\}$ for each $v \in V(G)$. In [9], Sittitrai and Nakprasit combined DP-coloring and relaxed list coloring as follows. Let $(H, L)$ be a cover of a graph $G$ with a $k$-assignment $L$. A transversal $T$ of $(H, L)$ is a $(t_1, \ldots, t_k)$-coloring if
every \((v,i) \in T\) has degree at most \(t_i\) in \(H[T]\). If \(G\) with a \(k\)-assignment \(L\) has a \((t_1,\ldots,t_k)\)-coloring for every cover \((H,L)\), then we say \(G\) is \(DP-(t_1,\ldots,t_k)\)-colorable. One can show that the fact that \(G\) is \(DP-(t_1,\ldots,t_k)\)-colorable where \(t_i = d\ (i \in \{1,\ldots,k\})\) implies \(G\) is \((k,d)^*\)-choosable.

In this work, we obtain the following result.

**Theorem 1.** Every planar graph without 4-cycles or 7-cycles is \(DP-(1,1,1)\)-colorable.

Theorem 1 generalizes the following result by Lih et al. [6].

**Theorem 2.** Every planar graph without 4-cycles or 7-cycles is \((3,1)^*\)-choosable.

Remark that the proof of \((3,1)^*\)-choosability by Lih et al. cannot be applied to Theorem 1. For example, Lih et al. use the fact that a 3-cycle \(abca\) is \((L,1)^*\)-colorable if \(|L(a)| \geq 2\) and \(|L(b)|,|L(c)| \geq 1\). But this fact is not true for DP-coloring. Let \(L(a) = \{1,2\}, L(b) = \{1\}, L(c) = \{2\}\), and let \((a,1)(b,1), (a,2)(c,2), (b,1)(c,2)\) be edges of a cover \(H\). One can see that \((H,L)\) has no \(DP-(1,1,1)\)-colorings.

Additionally, we show that every planar graph is \(DP-(0,2,2)\)-colorable. In fact, we present this second main result in a stronger form by using a concept similar to “variable degeneracy” but broader. One immediate consequence of the second main result is that the vertex set of a planar graph without 4-cycles or 7-cycles can be partitioned into three sets such that one set is independent and each of the two remaining sets induces a linear forest.

Some definitions are required to understand the second main result. The concept of variable degeneracy was introduced by Borodin, Kostochka, and Toft [2] as follows. Let \(f\) be a function from \(V(G)\) to the set of positive integers. A graph \(G\) is strictly \(f\)-degenerate if every subgraph \(G'\) has a vertex \(v\) with \(d_G(v) < f(v)\). Let \(f_i\), where \(i \in \{1,\ldots,s\}\), be a function from \(V(G)\) to the set of nonnegative integers. An \((f_1,\ldots,f_s)\)-partition of a graph \(G\) is a partition of \(V(G)\) into \(V_1,\ldots,V_s\) such that the induced subgraph \(G[V_i]\) is strictly \(f_i\)-degenerate for each \(i \in \{1,\ldots,s\}\). Equivalently, the vertices of \(V_i\) can be ordered from left to right such that each vertex in \(V_i\) has less than \(f_i(v)\) neighbors in \(V_i\) on the left.

DP-coloring with variable degeneracy was introduced by Nakprasit and Nakprasit [7] and Sittitrai and Nakprasit [8] as follows. Let \(F = \{f_1,\ldots,f_s\}\) and \(f_i \in \mathbb{Z}^+ \cup \{0\}\), where \(1 \leq i \leq s\). A \(DP-F\)-coloring \(T\) of a cover \((H,L)\) of \(G\) is a transversal \(T\) of \((H,L)\) in which its vertices can be ordered from left to right so that each element \((v,i)\) in \(T\) has less than \(f_i(v)\) neighbors on the left. We say that \(G\) is \(DP-F\)-colorable if \((G,H)\) has a \(DP-F\)-coloring for every cover \(H\).

We observe that the restriction in the previous definition is about the number of neighbors on the left of each element in a transversal. We may employ other restrictions as needed to different applications. This observation inspires us to
define the following concept. Let \( B \) be a condition imposed on ordered vertices. A DP-\( B \)-coloring of \((G, H)\) is a transversal \( T \) with ordered vertices from left to right such that each \((v, c) \in T\) satisfies condition \( B \) imposed on each element of \( H \). In this work, we demonstrate the use of this definition by the condition \( B_A \) defined as follows. Let \( T \) be a transversal of a cover \((H, L)\) of \( G \). We say that \( T \) is a DP-\( B_A \)-coloring if vertices in \( T \) can be ordered from left to right such that:

1. For each \((v, 1) \in T, (v, 1)\) has no neighbor on the left.
2. For each \((v, c) \in T\) where \( c \neq 1\), \((v, c)\) has at most one neighbor on the left and that neighbor (if it exists) is adjacent to at most one vertex on the left of \((v, c)\).

We say that \( G \) is DP-\( B_A \)-\( k \)-colorable if every cover \((H, L)\) of a graph \( G \) with \( k \)-assignment \( L \) has a DP-\( B_A \)-coloring.

**Theorem 3.** Every planar graph without 4-cycles or 7-cycles is DP-\( B_A \)-3-colorable.

**Corollary 4.** If \( G \) is a planar graph without 4-cycles or 7-cycles, then

1. \( G \) is DP-(0, 2, 2)-colorable.
2. \( V(G) \) can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set.

**Proof.** Suppose Theorem 3 holds. Then the first part of the corollary follows immediately from definitions. To obtain the second part, we define edges on \( H \) to match exactly the same colors in \( L(u) \) and \( L(v) \) for each \( uv \in E(G) \). One can see that the set of vertices with color 1 is independent and the set of vertices with color \( i \) induces a linear forest when \( i = 2 \) or 3.

2. **Forbidden Configurations Due to Cycles**

**Lemma 5.** Let \( G \) be a graph without 4-cycles and 7-cycles. Then the following statements hold.

1. There are no adjacent 3-faces.
2. If a 3-face is adjacent to a 5-face, then they share exactly one edge and two vertices.
3. A 5-face is not adjacent to two 3-faces.
4. If \( \delta(G) \geq 3 \), then each 6-face is not adjacent to a 3-face.
5. If \( \delta(G) \geq 3 \), then a 3-vertex is not incident to a 3-face and two 5-faces simultaneously.

**Proof.** (1) If two 3-faces are adjacent, then \( G \) has a 4-cycle, a contradiction.
(2) If a 3-face and a 5-face share three vertices (so they share one or two edges), then $G$ has a 4-cycle, a contradiction.

(3) Suppose to the contrary that a 5-faces $C$ is adjacent to two 3-faces. If those two 3-faces share vertex outside $V(C)$, then $G$ has a 4-cycle, for otherwise $G$ has a 7-cycle, a contradiction. Thus a 5-face is not adjacent to two 3-faces.

(4) Suppose to the contrary that a 6-face $f_1$ is adjacent to a 3-face $f_2$. First we suppose $f_1$ is not a simple face. Then its boundary walk forms two 3-cycles with a common vertex. Thus $f_1$ adjacent to $f_2$ yields a 4-cycle, a contradiction. Now we suppose $f_1$ is a simple face. Since $\delta(G) \geq 3$, $f_1$ and $f_2$ share exactly one edge. If $f_1$ and $f_2$ share exactly two vertices, then $G$ has a 4-cycle or a 7-cycle, a contradiction. Altogether, $f_1$ is not adjacent to $f_2$.

(5) Suppose that $\delta(G) \geq 3$. Observe that if a 5-face is adjacent to a 3-face or another 5-face, then they share exactly one edge and two vertices to avoid a 4-cycle or a 7-cycle. It follows that a 3-vertex incident to a 3-face and two 5-faces yields a 7-cycle.

3. Proof of Theorem 1

3.1. Structure of a minimal counterexample

Lemma 6. Suppose $G$ is a non-DP-$(t_1, \ldots, t_k)$-colorable graph but all of its proper induced subgraphs are DP-$(t_1, \ldots, t_k)$-colorable. Then the following statements hold.

1. $\delta(G) \geq k$.
2. If $t_i = d \geq 1$ for each $i \in \{1, \ldots, k\}$, then every neighbor of a $k$-vertex has degree at least $k + d$.

Proof. (1) Suppose to the contrary that $G$ has a vertex $v$ of degree at most $k - 1$. Let $L$ be a $k$-assignment of $G$ and let $(H, L)$ be a cover of $G$ that does not have a DP-$(t_1, \ldots, t_k)$-coloring. By our assumption, $G' = G - v$ has a DP-$(t_1, \ldots, t_k)$-coloring $T'$. Since $d(v) \leq k - 1$, there exists $(v, i) \in V(H)$ that does not have a neighbor in $T'$. So, we add $(v, i)$ to $T'$ to obtain a desired coloring, a contradiction.

(2) Suppose to the contrary that $u$ and $v$ are adjacent vertices where $d(u) = k$ and $d(v) \leq k + d - 1$. Let $L$ be a $k$-assignment of $G$ and let $(H, L)$ be a cover of $G$ that does not have a DP-$(t_1, \ldots, t_k)$-coloring. By assumption, $G' = G - \{u, v\}$ has a DP-$(t_1, \ldots, t_k)$-coloring $T'$. Then there is $(u, b) \in V(H)$ that does not have a neighbor in $T'$. Suppose $(v, c)$ is adjacent to $(u, b)$ in $H$. If $(v, c)$ has at most $d - 1$ neighbors in $T'$, then we add $(u, b)$ and $(v, c)$ in $T'$ to obtain a desired coloring, a contradiction. Suppose $(v, c)$ has at least $d$ neighbors in $T'$. Then there exists $(v, i) \in V(H)$ that does not have a neighbor in $T'$. So, we add $(u, b)$ and $(v, i)$ to $T'$ to obtain a desired coloring, a contradiction. This completes the proof.
The next result immediately follows.

**Corollary 7.** Suppose $G$ is a non-$DP-(1, 1, 1)$-colorable graph but all of its proper induced subgraphs are $DP-(1, 1, 1)$-colorable. Then the following statements hold.

1. $\delta(G) \geq 3$.
2. There are no adjacent 3-vertices.

**Lemma 8.** Suppose $G$ is a counterexample to Theorem 1 but all of its proper induced subgraphs are $DP-(1, 1, 1)$-colorable. If $f$ is a face of $G$, then the number of its incident 3-vertices plus the number of its adjacent 3-faces is at most $d(f)$.

**Proof.** Let $f$ be a face with a boundary walk $v_1, v_2, \ldots, v_k$. Let $f_i$ be a face sharing an edge $v_iv_{i+1}$ with $f$ where subscripts are taken modulo $k$. We claim that if $d(f_i) = d(v_i) = 3$, then $d(f_{i-1}) \geq 4$ and $d(v_{i-1}) \geq 4$. Suppose that $d(f_i) = d(v_i) = 3$. It follows from Corollary 7(2) that $d(v_{i-1}) \geq 4$. If $d(f_{i-1}) = 3$, then there are adjacent 3-cycles, a contradiction. So, the claim holds. It follows from the claim that the average number of $v_i$ and $f_i$ with degree 3 for each $i$ is at most 1. This implies the lemma.

### 3.2. Discharging procedure

Suppose $G$ is a counterexample to Theorem 1 but all of its proper induced subgraphs are $DP-(1, 1, 1)$-colorable. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face $x$ and let $\mu^*(x)$ denote the final charge of $x$ after the discharging process. By the Euler’s formula, \[ \sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8. \] We define discharging rules as follows.

**Discharging Rules.**

1. **(R1)** Each $5^+$-face gives $\frac{1}{3}$ to each adjacent 3-face.
2. **(R2)** Each 5-face gives $\frac{1}{3}$ to each incident 3-vertex.
3. **(R3)** Each $6^+$-face gives $\frac{2}{3}$ to each incident 3-vertex.

We aim to show that the final charge $\mu^*(x)$ for each $x \in V(G) \cup F(G)$ is nonnegative. Since the total of charge is not changed by the rules, we obtain a contradiction and prove the main result.

**Proof.** By Corollary 7(1), every vertex $v$ is a $3^+$-vertex. If $v$ is a $4^+$-vertex, then it does not involve in a discharging process and thus $\mu^*(v) = \mu(v) \geq 0$.

Consider a 3-vertex $v$. If $v$ is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{3} = 0$ by (R2) and (R3). If $v$ is incident to a 3-face, then it is incident to two $5^+$-faces and one of which is a $6^+$-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + \frac{2}{3} + \frac{2}{3} = 0$ by (R2) and (R3).
Consider a 3-face $f$. It follows from Lemma 5 that every face adjacent to $f$ is a $5^+$-face. Thus $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by (R1).

If $f$ is a 4-face, then its charge is not affected by the discharging procedure and thus $\mu^*(f) = \mu(f) = 0$.

Consider a 5-face $f$. Then $f$ is incident to at most two 3-vertices by Corollary 7(2) and is adjacent to at most one 3-face by Lemma 5(3). Thus $\mu^*(f) \geq \mu(f) - 3 \times \frac{1}{3} = 0$ by (R1) and (R2).

Consider a 6-face $f$. Then $f$ is incident to at most three 3-vertices by Corollary 7(2) and is not adjacent to a 3-face by Lemma 5(4). Thus $\mu^*(f) \geq \mu(f) - 3 \times \frac{2}{3} = 0$ by (R3).

If a 7-face is a simple face, then $G$ has a 7-cycle, for otherwise $G$ has a 4-cycle. Thus $G$ does not contain a 7-face.

Consider a $k$-face $f$ where $k \geq 8$. Suppose that $f$ has $r$ incident 3-vertices and $s$ adjacent 3-faces. We have that $\mu^*(f) = \mu(f) - r \times \frac{3}{3} - s \times \frac{2}{3}$ by (R1) and (R3). Since $r + s \leq k$ by Lemma 8 and $r \leq k/2$ by Corollary 7(2), we have $r \times \frac{3}{3} + s \times \frac{2}{3} = (r + s) \times \frac{5}{3} + r \times \frac{1}{3} \leq k \times \frac{1}{3} + \frac{k}{2} = \frac{k}{2}$. Thus $\mu^*(f) \geq \mu(f) - \frac{k}{2} = \frac{k}{2} - 4 \geq 0$.

4. Proof of Theorem 3

4.1. Structure of a minimal counterexample

First, we introduce a concept used in the next two lemmas. Let $G$ be a graph with a vertex $v$ and a cover $H$. Let $T'$ be a DP-$B_A$-coloring of $G - v$ with an appropriate order $R$. Adding $(v, i)$ to the right of $T'$ is the process to have the transversal $T' \cup \{(v, i)\}$ of $G$ with an order such that vertices in $T'$ are ordered first according to the order $R$ and then we put $(v, i)$ at the farthest right. If $(v, i)$ according to such order satisfies the condition of DP-$B_A$-coloring, then $T' \cup \{(v, i)\}$ is a DP-$B_A$-coloring of $G$ since all remaining vertices in $T$ satisfy the condition by the order $R$ already.

Lemma 9. If $G$ is a non-DP-$B_A$-3-colorable graph but all of its proper induced subgraphs are DP-$B_A$-3-colorable, then $\delta(G) \geq 3$.

Proof. Suppose to the contrary that $G$ has a vertex $v$ with degree at most 2. Let $L$ be a 3-assignment of $G$ and let $(H, L)$ be a cover of $G$ that does not have a DP-$B_A$-coloring. By minimality, $G' = G - v$ has a DP-$B_A$-coloring $T'$. Since $d(v) \leq 2$, there exists $(v, i) \in V(H)$ that does not have a neighbor in $T'$. We add $(v, i)$ to the right of $T'$. Since $(v, i)$ does not have a neighbor in $T'$, we obtain a desired coloring. This contradiction completes the proof.

Lemma 10. Suppose $G$ is a non-DP-$B_A$-3-colorable graph but all of its proper induced subgraphs are DP-$B_A$-3-colorable. If a 3-vertex $u$ in $G$ is adjacent to a
3-vertex, then \( u \) has two 5+-neighbors. Moreover, if \( x \) is a 5-neighbor of \( u \), then \( x \) has a 4+-neighbor.

**Proof.** Let a 3-vertex \( u \) be adjacent to \( x, y \) and a 3-vertex \( v \). By minimality, \( G - \{ u, v \} \) has a DP-B\(_A\)-coloring \( T \). Choose \( (u, c_u) \in V(H) \) such that \( (u, c_u) \) is not adjacent to vertices in \( T \) and choose \( (v, c_v) \) similarly. If \( c_u \neq 1 \), or \( c_u = c_v = 1 \) and \( (u, 1) \) is not adjacent to \( (v, 1) \), then we add \( (v, c_v) \) and then \( (u, c_u) \) to the right of \( T \). Since \( (v, c_v) \) is not adjacent to any vertices in \( T \) and it is the only vertex that may adjacent to \( (u, c_u) \), it follows that \( T \cup \{(u, c_u), (v, c_v)\} \) is a desired coloring.

By symmetry, it remains to consider the case that \( c_u = c_v = 1 \) and \( (u, 1) \) and \( (v, 1) \) are adjacent, and we call this case unfavorable situation. Note that \( (u, 2) \) has exactly one neighbor, say \( (x, x_2) \) in \( T \), otherwise we can choose 2 or 3 to be \( c_u \) and we can avoid unfavorable situation. If \( (x, x_2) \) has at most one neighbor in \( T \), then we add \( (u, 2) \) and subsequently \( (v, 1) \) to the right of \( T \). By assumption, \( (u, 2) \) satisfies the condition of a DP-B\(_A\)-coloring. Moreover, \( (v, 1) \) has no neighbors in \( T \cup \{(u, 2), (v, 1)\} \), and thus we have a desired coloring. This contradiction yields that \( (x, x_2) \) has at least two neighbors in \( T \).

We aim to show that \( x \) is a 5+-vertex. If we can add \( (x, x_1) \) or \( (x, x_3) \) where \( \{x_1, x_2, x_3\} = \{1, 2, 3\} \) to the right of \( T - \{(x, x_2)\} \) to get a DP-B\(_A\)-coloring \( T' \) of \( G - \{u, v\} \), then \( (u, 2) \) has no neighbors in \( T' \). Consequently, we can avoid unfavorable situation by having \( c_u = 2 \) and then obtain a desired coloring which is a contradiction. Thus we cannot add \( (x, x_1) \) or \( (x, x_3) \) to the right of \( T - \{(x, x_2)\} \) to get a DP-B\(_A\)-coloring of \( G - \{u, v\} \). It follows that each of \( (x, x_1) \) and \( (x, x_3) \) have neighbors in \( T \). Recall that \( (x, x_2) \) has at least two neighbors in \( T \). Altogether, \( x \) in \( G \) has at least five neighbors including \( u \). By symmetry, \( y \) is also a 5+-vertex.

Next we show that a 5-vertex \( x \) has a 4+-neighbor. Suppose \( x \) is a 5-vertex. By the above argument, \( (x, x_2) \) has exactly two neighbors, \( (x, x_1) \) has exactly one neighbor, and \( (x, x_3) \) has exactly one neighbor in \( T \). By symmetry, assume \( x_3 \neq 1 \) and \( (x, x_3) \) is adjacent to only \( (z, c_z) \) in \( T \). If we can add \( (z, c'_z) \) to the right of \( T - \{(x, x_2), (z, c_z)\} \) where \( c'_z \neq c_z \) to obtain a DP-B\(_A\)-coloring \( T'' \) of \( G - \{x, u, v\} \), then we can add \( (x, x_3) \) that has no neighbors in \( T'' \) to the right of \( T'' \) to obtain a DP-B\(_A\)-coloring of \( G - \{u, v\} \). Recall that \( (u, 2) \) is adjacent to only \( (x, x_2) \) in \( T \). Consequently, \( (u, 2) \) has no neighbors in \( T'' \cup \{(x, x_3)\} \). It follows that we can avoid unfavorable situation by having \( T'' \cup \{(x, 3)\} \) as a DP-B\(_A\)-coloring of \( G - \{u, v\} \) and choosing \( c_u = 2 \). Thus we assume that we cannot add \( (z, c'_z) \) to the right of \( T - \{(x, x_2), (z, c_z)\} \) to obtain a DP-B\(_A\)-coloring of \( G - \{x, u, v\} \). One can use a similar argument for the vertex \( x \) to prove that \( z \) is a 4+-vertex. Thus \( z \) is a 5-vertex with a 4+-neighbor or a 6+-vertex, and so is \( y \) by symmetry. This completes the proof. ■
4.2. Discharging procedure

Suppose $G$ is a counterexample to Theorem 3 but all of its proper induced subgraphs are DP-$B_A$-3-colorable. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face $x$ and let $\mu^*(x)$ denote the final charge of $x$ after the discharging process. By the Euler’s formula, $\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8$. We call a 3-vertex $v$ a bad 3-vertex if $v$ is adjacent to another 3-vertex, otherwise we call it a good 3-vertex. We define discharging rules as follows.

Discharging Rules.

(R0) Each $5^+$-vertex gives $\frac{1}{4}$ to each adjacent bad 3-vertex.

(R1) Each 5-face gives $\frac{1}{2}$ to each incident bad 3-vertex.

(R2) Each 5-face gives $\frac{1}{3}$ to each adjacent bad 3-vertex and $\frac{1}{2}$ to each adjacent good 3-vertex.

(R3) Each $6^+$-face gives $\frac{1}{3}$ to each adjacent bad 3-vertex and $\frac{2}{3}$ to each adjacent good 3-vertex.

We aim to show that the final charge $\mu^*(x)$ for each $x \in V(G) \cup F(G)$ is nonnegative. Since the total of charge is not changed by the rules, we obtain a contradiction and prove the main result.

Proof. By Lemma 9, every vertex $v$ is a $3^+$-vertex.

Consider a good 3-vertex $v$. If $v$ is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{4} = 0$ by (R2) and (R3). If $v$ is incident to a 3-face, then it is incident to two $5^+$-faces and one of which is a $6^+$-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + \frac{1}{3} + \frac{2}{3} = 0$ by (R2) and (R3).

Consider a bad 3-vertex $v$. By Lemma 10, $v$ is adjacent to two $5^+$-vertices. If $v$ is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 2 \times \frac{1}{4} + 3 \times \frac{1}{6} = 0$ by (R0), (R2), and (R3). If $v$ is incident to a 3-face, then it is incident to two $5^+$-faces one of which is a $6^+$-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + 2 \times \frac{1}{4} + \frac{1}{6} + \frac{1}{3} = 0$ by (R0), (R2), and (R3).

If $v$ is a 4-vertex, then it does not involve in a discharging process and thus $\mu^*(v) = \mu(v) = 0$.

Consider a 5-vertex $v$. If $v$ is adjacent to a bad 3-vertex, say $u$, then $v$ has a $4^+$-neighbor by Lemma 10. Consequently, $v$ is adjacent to at most four bad 3-vertices. Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{4} = 0$ by (R0).

Consider a $k$-vertex $v$ where $k \geq 6$. Then $\mu^*(v) \geq \mu(v) - k \times \frac{1}{4} = (k - 4) - k \times \frac{1}{4} > 0$ by (R0).

Consider a 3-face $f$. It follows from Lemma 5(1) that every face adjacent to $f$ is a $5^+$-face. Thus $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by (R1).

If $f$ is a 4-face, then its charge is not affected by the discharging procedure and thus $\mu^*(f) = \mu(f) = 0$. 


Consider a 5-face $f$. From Lemma 5(3), $f$ is adjacent to at most one 3-face. If $f$ is incident to at most two 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{2}{3} = 0$ by (R1) and (R2). If $f$ is incident to at least three 3-vertices, then $f$ is incident to exactly three 3-vertices in which two of them are bad 3-vertices by Lemma 10. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{3} - \frac{1}{3} = 0$ by (R1) and (R2).

Consider a 6-face $f$. From Lemma 5(4), $f$ is not adjacent to a 3-face. If $f$ is incident to at most three 3-vertices, then $\mu^*(f) \geq \mu(f) - 3 \times \frac{2}{3} = 0$ by (R3). If $f$ is incident to at least four 3-vertices, then $f$ is incident to exactly four 3-vertices in which all of them are bad 3-vertices by Lemma 10. It follows that $\mu^*(f) = \mu(f) - 4 \times \frac{1}{3} > 0$ by (R3).

If a 7-face is a simple face, then $G$ has a 7-cycle, otherwise $G$ has a 4-cycle. Thus $G$ does not contain a 7-face.

Finally, consider a $k$-face $f$ where $k \geq 8$. Assume that all subscripts are taken modulo $k$. Let $v_1, v_2, \ldots, v_k$ be the vertices on the boundary of $f$, and let $f_i$ be a face sharing an edge $v_iv_{i+1}$ with $f$. We construct a new discharging rule for $f$ such that each of its incident 3-faces and adjacent 3-faces gains charge by the new rule not less than it gains by the original rules.

First, let $f$ send $\frac{1}{2}$ to each $v_i$. If $f_i$ is a 3-face, then let $\alpha(i) = 1$, otherwise $\alpha(i) = 0$. If $v_i$ is a 3-vertex, then let $\beta(i) = 1$, otherwise $\beta(i) = 0$. Let $v_i$ send charge $\frac{\alpha(i)}{6}$ to $f_i$ and $\beta(i+1)(\frac{1}{3} - \frac{\alpha(i)}{6})$ to $v_{i+1}$. Similarly, let $v_i$ send charge $\frac{\alpha(i-1)}{6}$ to $f_{i-1}$ and $\beta(i-1)(\frac{1}{3} - \frac{\alpha(i-1)}{6})$ to $v_{i-1}$. Then each 3-face $f_i$ gains $2 \times \frac{1}{3}$ from $v_i$ and $v_{i+1}$, and each 4'-vertex gains a nonnegative charge by the new rule.

Consider a good 3-vertex $v_i$. Note that at most one of $f_{i-1}$ and $f_i$ is a 3-face to avoid a 4-cycle. By symmetry, assume $f_{i-1}$ is not a 3-face. Then $v_i$ receives $\frac{1}{2}$ from $f$, receives $\frac{1}{3}$ from $v_{i-1}$, receives at least $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ from $v_{i+1}$, and sends at most $\frac{1}{6}$ to $f_i$. Thus $v_i$ gains charge at least $\frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$ by the new rule.

Consider bad 3-vertices $v_i$ and $v_{i+1}$. By Lemma 10, $v_{i-1}$ and $v_{i+2}$ are 5'-vertices. Since charge sent from $v_i$ to $v_{i+1}$ and charge sent from $v_{i+1}$ to $v_i$ are the same, we ignore this distribution in the calculation. Note that if $f_i$ is a 3-face, then none of $f_{i-1}$ and $f_{i+1}$ are 3-faces. Assume $f_i$ is a 3-face. Then $v_i$ receives $\frac{1}{2}$ from $f$, receives $\frac{1}{3}$ from $v_{i-1}$, and sends $\frac{1}{6}$ to $f_i$. Thus $v_i$ gains $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{7}{12} > \frac{1}{3}$ by the new rule. Assume $f_i$ is not a 3-face. Then $v_i$ receives $\frac{1}{2}$ from $f$, receives at least $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ from $v_{i-1}$, and sends at most $\frac{1}{6}$ to $f_{i-1}$. Thus $v_i$ gains at least $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{5}{12} > \frac{1}{3}$ by the new rule.

Altogether, let $f$ send charge at most $\frac{k}{2}$ with a distribution to its incident 3-faces and adjacent 3-faces that satisfies the original rules. Thus $\mu^*(f) \geq \mu(f) - \frac{k}{2} = \frac{k}{2} - 4 \geq 0$.

This completes the proof.

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