RELAXED DP-COLORING AND ANOTHER GENERALIZATION OF DP-COLORING ON PLANAR GRAPHS WITHOUT 4-CYCLES AND 7-CYCLES

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Abstract

DP-coloring is generalized via relaxed coloring and variable degeneracy in [P. Sittitrai and K. Nakprasit, Sufficient conditions on planar graphs to have a relaxed DP-3-coloring, Graphs Combin. 35 (2019) 837–845], [K.M. Nakprasit and K. Nakprasit, A generalization of some results on list coloring and DP-coloring, Graphs Combin. 36 (2020) 1189–1201] and [P. Sittitrai and K. Nakprasit, An analogue of DP-coloring for variable degeneracy and its applications, Discuss. Math. Graph Theory]. In this work, we introduce another concept that includes two previous generalizations. We demonstrate its application on planar graphs without 4-cycles and 7-cycles. One implication is that the vertex set of every planar graph without 4-cycles and 7-cycles can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set. Additionally, we show that every planar graph without 4-cycles and 7-cycles is DP-(1, 1, 1)-colorable. This generalizes a result of Lih et al. [A note on list improper coloring planar graphs, Appl. Math. Lett. 14 (2001) 269–273] that every planar graph without 4-cycles and 7-cycles is (3, 1)^*-choosable.

Keywords: relaxed DP-colorings, variable degeneracy, planar graphs, discharging.

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1. Introduction

All considered graphs are finite, simple, undirected, and embedded in the plane. For a graph $G$, let its vertex set, edge set, face set, and minimum degree be denoted by $V(G)$, $E(G)$, $F(G)$, and $\delta(G)$, respectively. Let $d(x)$ denote the degree of $x$ where $x \in V(G) \cup F(G)$. A $k$-vertex (or $k^+$-vertex) is a vertex of degree $k$ (or at least $k$). Similar notation is applied to a cycle and a face. A face $f$ is simple if its boundary forms a cycle. A face $f$ and a vertex $v$ are incident if $v$ is on the boundary of $f$. We simply say two faces share an edge (or a vertex) instead of the boundary of two faces share an edge (or a vertex). Two faces are adjacent if they share at least one edge. If $G$ is a graph and $U \subseteq V(G)$, then $G[U]$ denote the subgraph of $G$ induced by $U$. A linear forest is a forest in which each component is a path.

Vizing [11] in 1976, and independently Erdős, Rubin, and Taylor [5] in 1979, introduced list coloring and choosability. An assignment $L$ of a graph $G$ assigns a list $L(v)$ (a set of colors) to each vertex $v$. A $k$-assignment $L$ is an assignment such that $|L(v)| = k$ for each vertex $v$. If a graph $G$ admits a proper coloring $f$ where $f(v) \in L(v)$ for each vertex $v$, then we say $G$ is $L$-colorable. A graph $G$ is $k$-choosable if it is $L$-colorable for each $k$-assignment $L$.

In 1999, Škrekovski [10] and Eaton and Hull [4] independently introduced the concept of relaxed list coloring. A graph $G$ with an assignment $L$ is $(L, d)^*$ choosable if each vertex $v$ of $G$ can be colored with a color $f(v) \in L(v)$ such that at most $d$ neighbors of $v$ receive the color $f(v)$. A graph $G$ is $(k, d)^*$-choosable if $G$ is $(L, d)^*$-choosable for each $k$-assignment $L$.

Dvořák and Postle [3] introduced a generalization of list coloring which they called correspondence coloring. Following Bernshteyn, Kostochka, and Pron [1], we call it a DP-coloring. Let $L$ be an assignment of a graph $G$. We call $(H, L)$ (or simply $H$) a cover of $G$ if it satisfies the following conditions.

(i) The vertex set of $H$ is $\bigcup_{u \in V(G)} \{{u} \times L(u)\} = \{(u, c) : u \in V(G), c \in L(u)\}$.

(ii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (the matching may be empty).

(iii) If $uv \notin E(G)$, then no edges of $H$ connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

A transversal of $(H, L)$ is a vertex set $T \subseteq V(H)$ such that $|T \cap (\{u\} \times L(u))| = 1$ for each vertex $u$ in $G$. A DP-coloring of $(H, L)$ is a transversal $T$ of $(H, L)$ such that $T$ is independent. The DP-chromatic number of $G$ is the least number $k$ such that every cover $(H, L)$ of $G$ with $k$-assignment $L$ has a DP-coloring.

Since names of colors for distinct vertices in DP-coloring are irrelevant, we always assume in this paper that a $k$-assignment of a graph $G$ has $L(v) = \{1, \ldots, k\}$ for each $v \in V(G)$. In [9], Sittitrai and Nakprasit combined $DP$-coloring and relaxed list coloring as follows. Let $(H, L)$ be a cover of a graph $G$ with a $k$-assignment $L$. A transversal $T$ of $(H, L)$ is a $(t_1, \ldots, t_k)$-coloring if
every \((v,i) \in T\) has degree at most \(t_i\) in \(H[T]\). If \(G\) with a \(k\)-assignment \(L\) has a \((t_1,\ldots,t_k)\)-coloring for every cover \((H,L)\), then we say \(G\) is \(DP-(t_1,\ldots,t_k)\)-colorable. One can show that the fact that \(G\) is \(DP-(t_1,\ldots,t_k)\)-colorable where \(t_i = d\ (i \in \{1,\ldots,k\})\) implies \(G\) is \((k,d)^*\)-choosable.

In this work, we obtain the following result.

**Theorem 1.** Every planar graph without 4-cycles or 7-cycles is \(DP-(1,1,1)\)-colorable.

Theorem 1 generalizes the following result by Lih et al. [6].

**Theorem 2.** Every planar graph without 4-cycles or 7-cycles is \((3,1)^*\)-choosable.

Remark that the proof of \((3,1)^*\)-choosability by Lih et al. cannot be applied to Theorem 1. For example, Lih et al. use the fact that a 3-cycle \(abca\) is \((L,1)^*\)-colorable if \(|L(a)| \geq 2\) and \(|L(b)|, |L(c)| \geq 1\). But this fact is not true for DP-coloring. Let \(L(a) = \{1,2\}, L(b) = \{1\}, L(c) = \{2\}\), and let \((a,1)(b,1), (a,2)(c,2),\) and \((b,1)(c,2)\) be edges of a cover \(H\). One can see that \((H,L)\) has no \(DP-(1,1,1)\)-colorings.

Additionally, we show that every planar graph is \(DP-(0,2,2)\)-colorable. In fact, we present this second main result in a stronger form by using a concept similar to “variable degeneracy” but broader. One immediate consequence of the second main result is that the vertex set of a planar graph without 4-cycles or 7-cycles can be partitioned into three sets such that one set is independent and each of the two remaining sets induces a linear forest.

Some definitions are required to understand the second main result. The concept of variable degeneracy was introduced by Borodin, Kostochka, and Toft [2] as follows. Let \(f\) be a function from \(V(G)\) to the set of positive integers. A graph \(G\) is strictly \(f\)-degenerate if every subgraph \(G'\) has a vertex \(v\) with \(d_{G'}(v) < f(v)\). Let \(f_i\), where \(i \in \{1,\ldots,s\}\), be a function from \(V(G)\) to the set of nonnegative integers. An \((f_1,\ldots,f_s)\)-partition of a graph \(G\) is a partition of \(V(G)\) into \(V_1,\ldots,V_s\) such that the induced subgraph \(G[V_i]\) is strictly \(f_i\)-degenerate for each \(i \in \{1,\ldots,s\}\). Equivalently, the vertices of \(V_i\) can be ordered from left to right such that each vertex in \(V_i\) has less than \(f_i(v)\) neighbors in \(V_i\) on the left.

DP-coloring with variable degeneracy was introduced by Nakprasit and Sittitrai and Nakprasit [8] as follows. Let \(F = (f_1,\ldots,f_s)\) and \(f_i \in \mathbb{Z}^+ \cup \{0\}\), where \(1 \leq i \leq s\). A \(DP-F\)-coloring \(T\) of a cover \((H,L)\) of \(G\) is a transversal \(T\) of \((H,L)\) in which its vertices can be ordered from left to right so that each element \((v,i)\) in \(T\) has less than \(f_i(v)\) neighbors on the left. We say that \(G\) is \(DP-F\)-colorable if \((G,H)\) has a \(DP-F\)-coloring for every cover \(H\).

We observe that the restriction in the previous definition is about the number of neighbors on the left of each element in a transversal. We may employ other restrictions as needed to different applications. This observation inspires us to
define the following concept. Let $B$ be a condition imposed on ordered vertices. A $DP-B$-coloring of $(G, H)$ is a transversal $T$ with ordered vertices from left to right such that each $(v, c) \in T$ satisfies condition $B$ imposed on each element of $H$. In this work, we demonstrate the use of this definition by the condition $B_A$ defined as follows. Let $T$ be a transversal of a cover $(H, L)$ of $G$. We say that $T$ is a $DP-B_A$-coloring if vertices in $T$ can be ordered from left to right such that:

(1) For each $(v, 1) \in T$, $(v, 1)$ has no neighbor on the left.
(2) For each $(v, c) \in T$ where $c \neq 1$, $(v, c)$ has at most one neighbor on the left and that neighbor (if it exists) is adjacent to at most one vertex on the left of $(v, c)$.

We say that $G$ is $DP-B_A$-$k$-colorable if every cover $(H, L)$ of a graph $G$ with $k$-assignment $L$ has a $DP-B_A$-coloring.

**Theorem 3.** Every planar graph without 4-cycles or 7-cycles is $DP-B_A$-$3$-colorable.

**Corollary 4.** If $G$ is a planar graph without 4-cycles or 7-cycles, then

(i) $G$ is $DP$-$(0, 2, 2)$-colorable.
(ii) $V(G)$ can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set.

**Proof.** Suppose Theorem 3 holds. Then the first part of the corollary follows immediately from definitions. To obtain the second part, we define edges on $H$ to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$. One can see that the set of vertices with color 1 is independent and the set of vertices with color $i$ induces a linear forest when $i = 2$ or 3.

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2. **Forbidden Configurations Due to Cycles**

**Lemma 5.** Let $G$ be a graph without 4-cycles and 7-cycles. Then the following statements hold.

(1) There are no adjacent 3-faces.
(2) If a 3-face is adjacent to a 5-face, then they share exactly one edge and two vertices.
(3) A 5-face is not adjacent to two 3-faces.
(4) If $\delta(G) \geq 3$, then each 6-face is not adjacent to a 3-face.
(5) If $\delta(G) \geq 3$, then a 3-vertex is not incident to a 3-face and two 5-faces simultaneously.

**Proof.** (1) If two 3-faces are adjacent, then $G$ has a 4-cycle, a contradiction.
(2) If a 3-face and a 5-face share three vertices (so they share one or two edges), then $G$ has a 4-cycle, a contradiction.

(3) Suppose to the contrary that a 5-faces $C$ is adjacent to two 3-faces. If those two 3-faces share vertex outside $V(C)$, then $G$ has a 4-cycle, for otherwise $G$ has a 7-cycle, a contradiction. Thus a 5-face is not adjacent to two 3-faces.

(4) Suppose to the contrary that a 6-face $f_1$ is adjacent to a 3-face $f_2$. First we suppose $f_1$ is not a simple face. Then its boundary walk forms two 3-cycles with a common vertex. Thus $f_1$ adjacent to $f_2$ yields a 4-cycle, a contradiction.

Now we suppose $f_1$ is a simple face. Since $\delta(G) \geq 3$, $f_1$ and $f_2$ share exactly one edge. If $f_1$ and $f_2$ share exactly two vertices, then $G$ has a 4-cycle or a 7-cycle, a contradiction. Altogether, $f_1$ is not adjacent to $f_2$.

(5) Suppose that $\delta(G) \geq 3$. Observe that if a 5-face is adjacent to a 3-face or another 5-face, then they share exactly one edge and two vertices to avoid a 4-cycle or a 7-cycle. It follows that a 3-vertex incident to a 3-face and two 5-faces yields a 7-cycle.

3. Proof of Theorem 1

3.1. Structure of a minimal counterexample

Lemma 6. Suppose $G$ is a non-DP-$(t_1, \ldots, t_k)$-colorable graph but all of its proper induced subgraphs are DP-$(t_1, \ldots, t_k)$-colorable. Then the following statements hold.

1. $\delta(G) \geq k$.

2. If $t_i = d \geq 1$ for each $i \in \{1, \ldots, k\}$, then every neighbor of a $k$-vertex has degree at least $k + d$.

Proof. (1) Suppose to the contrary that $G$ has a vertex $v$ of degree at most $k - 1$. Let $L$ be a $k$-assignment of $G$ and let $(H, L)$ be a cover of $G$ that does not have a DP-$(t_1, \ldots, t_k)$-coloring. By our assumption, $G' = G - v$ has a DP-$(t_1, \ldots, t_k)$-coloring $T'$. Since $d(v) \leq k - 1$, there exists $(v, i) \in V(H)$ that does not have a neighbor in $T'$. So, we add $(v, i)$ to $T'$ to obtain a desired coloring, a contradiction.

(2) Suppose to the contrary that $u$ and $v$ are adjacent vertices where $d(u) = k$ and $d(v) \leq k + d - 1$. Let $L$ be a $k$-assignment of $G$ and let $(H, L)$ be a cover of $G$ that does not have a DP-$(t_1, \ldots, t_k)$-coloring. By assumption, $G' = G - \{u, v\}$ has a DP-$(t_1, \ldots, t_k)$-coloring $T'$. Then there is $(u, b) \in V(H)$ that does not have a neighbor in $T'$. Suppose $(v, c)$ is adjacent to $(u, b)$ in $H$. If $(v, c)$ has at most $d - 1$ neighbors in $T'$, then we add $(u, b)$ and $(v, c)$ in $T'$ to obtain a desired coloring, a contradiction. Suppose $(v, c)$ has at least $d$ neighbors in $T'$. Then there exists $(v, i) \in V(H)$ that does not have a neighbor in $T'$. So, we add $(u, b)$ and $(v, i)$ to $T'$ to obtain a desired coloring, a contradiction. This completes the proof.
Corollary 7. Suppose $G$ is a non-$DP$-$(1, 1, 1)$-colorable graph but all of its proper induced subgraphs are $DP$-$(1, 1, 1)$-colorable. Then the following statements hold.

1. $\delta(G) \geq 3$.
2. There are no adjacent 3-vertices.

Lemma 8. Suppose $G$ is a counterexample to Theorem 1 but all of its proper induced subgraphs are $DP$-$(1, 1, 1)$-colorable. If $f$ is a face of $G$, then the number of its incident 3-vertices plus the number of its adjacent 3-faces is at most $d(f)$.

Proof. Let $f$ be a face with a boundary walk $v_1, v_2, \ldots, v_k$. Let $f_i$ be a face sharing an edge $v_iv_{i+1}$ with $f$ where subscripts are taken modulo $k$. We claim that if $d(f_i) = d(v_i) = 3$, then $d(f_{i-1}) \geq 4$ and $d(v_{i-1}) \geq 4$. Suppose that $d(f_i) = d(v_i) = 3$. It follows from Corollary 7(2) that $d(v_{i-1}) \geq 4$. If $d(f_{i-1}) = 3$, then there are adjacent 3-cycles, a contradiction. So, the claim holds. It follows from the claim that the average number of $v_i$ and $f_i$ with degree 3 for each $i$ is at most 1. This implies the lemma.

3.2. Discharging procedure

Suppose $G$ is a counterexample to Theorem 1 but all of its proper induced subgraphs are $DP$-$(1, 1, 1)$-colorable. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face $x$ and let $\mu^*(x)$ denote the final charge of $x$ after the discharging process. By the Euler’s formula, $\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = \sum_{v \in V(G)}(d(v) - 4) + \sum_{f \in F(G)}(d(f) - 4) = -8$. We define discharging rules as follows.

Discharging Rules.

(R1) Each 5+-face gives $\frac{1}{3}$ to each adjacent 3-face.
(R2) Each 5-face gives $\frac{1}{3}$ to each incident 3-vertex.
(R3) Each 6+-face gives $\frac{2}{3}$ to each incident 3-vertex.

We aim to show that the final charge $\mu^*(x)$ for each $x \in V(G) \cup F(G)$ is nonnegative. Since the total of charge is not changed by the rules, we obtain a contradiction and prove the main result.

Proof. By Corollary 7(1), every vertex $v$ is a 3+-vertex. If $v$ is a 4+-vertex, then it does not involve in a discharging process and thus $\mu^*(v) = \mu(v) \geq 0$.

Consider a 3-vertex $v$. If $v$ is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{3} = 0$ by (R2) and (R3). If $v$ is incident to a 3-face, then it is incident to two 5+-faces and one of which is a 6+-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + \frac{2}{3} + \frac{2}{3} = 0$ by (R2) and (R3).
Consider a 3-face \( f \). It follows from Lemma 5 that every face adjacent to \( f \) is a \( 5^+ \)-face. Thus \( \mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0 \) by (R1).

If \( f \) is a 4-face, then its charge is not affected by the discharging procedure and thus \( \mu^*(f) = \mu(f) = 0 \).

Consider a 5-face \( f \). Then \( f \) is incident to at most two 3-vertices by Corollary 7(2) and is adjacent to at most one 3-face by Lemma 5(3). Thus \( \mu^*(f) \geq \mu(f) - 3 \times \frac{1}{3} = 0 \) by (R1) and (R2).

Consider a 6-face \( f \). Then \( f \) is incident to at most three 3-vertices by Corollary 7(2) and is not adjacent to a 3-face by Lemma 5(4). Thus \( \mu^*(f) \geq \mu(f) - 3 \times \frac{2}{3} = 0 \) by (R3).

If a 7-face is a simple face, then \( G \) has a 7-cycle, for otherwise \( G \) has a 4-cycle. Thus \( G \) does not contain a 7-face.

Consider a \( k \)-face \( f \) where \( k \geq 8 \). Suppose that \( f \) has \( r \) incident 3-vertices and \( s \) adjacent 3-faces. We have that \( \mu^*(f) = \mu(f) - r \times \frac{1}{3} - s \times \frac{2}{3} \) by (R1) and (R3). Since \( r + s \leq k \) by Lemma 8 and \( r \leq k/2 \) by Corollary 7(2), we have \( r \times \frac{1}{3} + s \times \frac{2}{3} = (r+s) \times \frac{1}{3} + r \times \frac{1}{3} \leq k \times \frac{1}{3} + \frac{k}{2} \times \frac{1}{3} = \frac{k}{2} \). Thus \( \mu^*(f) \geq \mu(f) - \frac{k}{2} = \frac{k}{2} - 4 \geq 0 \).

4. Proof of Theorem 3

4.1. Structure of a minimal counterexample

First, we introduce a concept used in the next two lemmas. Let \( G \) be a graph with a vertex \( v \) and a cover \( H \). Let \( T' \) be a DP-\( B_A \)-coloring of \( G-v \) with an appropriate order \( R \). Adding \( (v, i) \) to the right of \( T' \) is the process to have the transversal \( T' \cup \{(v, i)\} \) of \( G \) with an order such that vertices in \( T' \) are ordered first according to the order \( R \) and then we put \( (v, i) \) at the farthest right. If \( (v, i) \) according to such order satisfies the condition of DP-\( B_A \)-coloring, then \( T' \cup \{(v, i)\} \) is a DP-\( B_A \)-coloring of \( G \) since all remaining vertices in \( T \) satisfy the condition by the order \( R \) already.

Lemma 9. If \( G \) is a non-DP-\( B_A \)-3-colorable graph but all of its proper induced subgraphs are DP-\( B_A \)-3-colorable, then \( \delta(G) \geq 3 \).

Proof. Suppose to the contrary that \( G \) has a vertex \( v \) with degree at most 2. Let \( L \) be a 3-assignment of \( G \) and let \( (H, L) \) be a cover of \( G \) that does not have a DP-\( B_A \)-coloring. By minimality, \( G' = G-v \) has a DP-\( B_A \)-coloring \( T' \). Since \( d(v) \leq 2 \), there exists \( (v, i) \in V(H) \) that does not have a neighbor in \( T' \). We add \( (v, i) \) to the right of \( T' \). Since \( (v, i) \) does not have a neighbor in \( T' \), we obtain a desired coloring. This contradiction completes the proof.

Lemma 10. Suppose \( G \) is a non-DP-\( B_A \)-3-colorable graph but all of its proper induced subgraphs are DP-\( B_A \)-3-colorable. If a 3-vertex \( u \) in \( G \) is adjacent to a
3-vertex, then $u$ has two $5^+$-neighbors. Moreover, if $x$ is a 5-neighbor of $u$, then $x$ has a $4^+$-neighbor.

**Proof.** Let a 3-vertex $u$ be adjacent to $x, y$ and a 3-vertex $v$. By minimality, $G - \{u, v\}$ has a DP-$B_A$-coloring $T$. Choose $(u, c_u) \in V(H)$ such that $(u, c_u)$ is not adjacent to vertices in $T$ and choose $(v, c_v)$ similarly. If $c_u \neq 1$, or $c_u = c_v = 1$ and $(u, 1)$ is not adjacent to $(v, 1)$, then we add $(v, c_v)$ and then $(u, c_u)$ to the right of $T$. Since $(v, c_v)$ is not adjacent to any vertices in $T$ and it is the only vertex that may adjacent to $(u, c_u)$, it follows that $T \cup \{(u, c_u), (v, c_v)\}$ is a desired coloring.

By symmetry, it remains to consider the case that $c_u = c_v = 1$ and $(u, 1)$ and $(v, 1)$ are adjacent, and we call this case unfavorable situation. Note that $(u, 2)$ has exactly one neighbor, say $(x, x_2)$ in $T$, otherwise we can choose 2 or 3 to be $c_u$ and we can avoid unfavorable situation. If $(x, x_2)$ has at most one neighbor in $T$, then we add $(u, 2)$ and subsequently $(v, 1)$ to the right of $T$. By assumption, $(u, 2)$ satisfies the condition of a DP-$B_A$-coloring. Moreover, $(v, 1)$ has no neighbors in $T \cup \{(u, 2), (v, 1)\}$, and thus we have a desired coloring. This contradiction yields that $(x, x_2)$ has at least two neighbors in $T$.

We aim to show that $x$ is a $5^+$-vertex. If we can add $(x, x_1)$ or $(x, x_3)$ where $\{x_1, x_2, x_3\} = \{1, 2, 3\}$ to the right of $T - \{(x, x_2)\}$ to get a DP-$B_A$-coloring $T'$ of $G - \{u, v\}$, then $(u, 2)$ has no neighbors in $T'$. Consequently, we can avoid unfavorable situation by having $c_u = 2$ and then obtain a desired coloring which is a contradiction. Thus we cannot add $(x, x_1)$ or $(x, x_3)$ to the right of $T - \{(x, x_2)\}$ to get a DP-$B_A$-coloring of $G - \{u, v\}$. It follows that each of $(x, x_1)$ and $(x, x_3)$ have neighbors in $T$. Recall that $(x, x_2)$ has at least two neighbors in $T$. Altogether, $x$ in $G$ has at least five neighbors including $u$. By symmetry, $y$ is also a $5^+$-vertex.

Next we show that a 5-vertex $x$ has a $4^+$-neighbor. Suppose $x$ is a 5-vertex. By the above argument, $(x, x_2)$ has exactly two neighbors, $(x, x_1)$ has exactly one neighbor, and $(x, x_3)$ has exactly one neighbor in $T$. By symmetry, assume $x_3 \neq 1$ and $(x, x_3)$ is adjacent to only $(z, c_z)$ in $T$. If we can add $(z, c'_z)$ to the right of $T - \{(x, x_2), (z, c_z)\}$ where $c'_z \neq c_z$ to obtain a DP-$B_A$-coloring $T''$ of $G - \{x, u, v\}$, then we can add $(x, x_3)$ that has no neighbors in $T''$ to the right of $T''$ to obtain a DP-$B_A$-coloring of $G - \{u, v\}$. Recall that $(u, 2)$ is adjacent to only $(x, x_2)$ in $T$. Consequently, $(u, 2)$ has no neighbors in $T'' \cup \{(x, x_3)\}$. It follows that we can avoid unfavorable situation by having $T'' \cup \{(x, 3)\}$ as a DP-$B_A$-coloring of $G - \{u, v\}$ and choosing $c_u = 2$. Thus we assume that we cannot add $(z, c'_z)$ to the right of $T - \{(x, x_2), (z, c_z)\}$ to obtain a DP-$B_A$-coloring of $G - \{x, u, v\}$. One can use a similar argument for the vertex $x$ to prove that $z$ is a $4^+$-vertex. Thus $x$ is a 5-vertex with a $4^+$-neighbor or a $6^+$-vertex, and so is $y$ by symmetry. This completes the proof. ■
4.2. Discharging procedure

Suppose $G$ is a counterexample to Theorem 3 but all of its proper induced subgraphs are DP-$B_A$-3-colorable. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face $x$ and let $\mu^*(x)$ denote the final charge of $x$ after the discharging process. By the Euler’s formula, $\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8$. We call a 3-vertex $v$ a bad 3-vertex if $v$ is adjacent to another 3-vertex, otherwise we call it a good 3-vertex.

We define discharging rules as follows.

**Discharging Rules.**

(R0) Each 5+-vertex gives $\frac{1}{4}$ to each adjacent bad 3-vertex.

(R1) Each 5+-face gives $\frac{1}{3}$ to each adjacent 3-face.

(R2) Each 5-face gives $\frac{1}{6}$ to each incident bad 3-vertex and $\frac{1}{3}$ to each incident good 3-vertex.

(R3) Each 6+-face gives $\frac{1}{3}$ to each incident bad 3-face and $\frac{2}{3}$ to each incident good 3-face.

We aim to show that the final charge $\mu^*(x)$ for each $x \in V(G) \cup F(G)$ is nonnegative. Since the total of charge is not changed by the rules, we obtain a contradiction and prove the main result.

**Proof.** By Lemma 9, every vertex $v$ is a 3+-vertex.

Consider a good 3-vertex $v$. If $v$ is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{4} = 0$ by (R2) and (R3). If $v$ is incident to a 3-face, then it is incident to two 5+-faces and one of which is a 6+-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + \frac{1}{4} + \frac{3}{2} = 0$ by (R2) and (R3).

Consider a bad 3-vertex $v$. By Lemma 10, $v$ is adjacent to two 5+-vertices. If $v$ is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 2 \times \frac{1}{4} + 3 \times \frac{1}{6} = 0$ by (R0), (R2), and (R3). If $v$ is incident to a 3-face, then it is incident to two 5+-faces one of which is a 6+-face by Lemmas 5(1) and 5(1)(5). Thus $\mu^*(v) \geq \mu(v) + 2 \times \frac{1}{4} + \frac{3}{6} + \frac{1}{3} = 0$ by (R0), (R2), and (R3).

If $v$ is a 4-vertex, then it does not involve in a discharging process and thus $\mu^*(v) = \mu(v) = 0$.

Consider a 5-vertex $v$. If $v$ is adjacent to a bad 3-vertex, say $u$, then $v$ has a 4+-neighbor by Lemma 10. Consequently, $v$ is adjacent to at most four bad 3-vertices. Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{4} = 0$ by (R0).

Consider a $k$-vertex $v$ where $k \geq 6$. Then $\mu^*(v) \geq \mu(v) - k \times \frac{1}{4} = (k - 4) - k \times \frac{1}{3} > 0$ by (R0).

Consider a 3-face $f$. It follows from Lemma 5(1) that every face adjacent to $f$ is a 5+-face. Thus $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by (R1).

If $f$ is a 4-face, then its charge is not affected by the discharging procedure and thus $\mu^*(f) = \mu(f) = 0$. 

Consider a 5-face \( f \). From Lemma 5(3), \( f \) is adjacent to at most one 3-face. If \( f \) is incident to at most two 3-vertices, then \( \mu^*(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{3} = 0 \) by (R1) and (R2). If \( f \) is incident to at least three 3-vertices, then \( f \) is incident to exactly three 3-vertices in which two of them are bad 3-vertices by Lemma 10. It follows that \( \mu^*(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{3} - \frac{1}{3} = 0 \) by (R1) and (R2).

Consider a 6-face \( f \). From Lemma 5(4), \( f \) is not adjacent to a 3-face. If \( f \) is incident to at most three 3-vertices, then \( \mu^*(f) \geq \mu(f) - 3 \times \frac{2}{3} = 0 \) by (R3). If \( f \) is incident to at least four 3-vertices, then \( f \) is incident to exactly four 3-vertices in which all of them are bad 3-vertices by Lemma 10. It follows that \( \mu^*(f) = \mu(f) - 4 \times \frac{1}{3} > 0 \) by (R3).

If a 7-face is a simple face, then \( G \) has a 7-cycle, otherwise \( G \) has a 4-cycle. Thus \( G \) does not contain a 7-face.

Finally, consider a \( k \)-face \( f \) where \( k \geq 8 \). Assume that all subscripts are taken modulo \( k \). Let \( v_1, v_2, \ldots, v_k \) be the vertices on the boundary of \( f \), and let \( f_i \) be a face sharing an edge \( v_i v_{i+1} \) with \( f \). We construct a new discharging rule for \( f \) such that each of its incident 3-vertices and adjacent 3-faces gains charge by the new rule not less than it gains by the original rules.

First, let \( f \) send \( \frac{1}{2} \) to each \( v_i \). If \( f_i \) is a 3-face, then let \( \alpha(i) = 1 \), otherwise \( \alpha(i) = 0 \). If \( v_i \) is a 3-vertex, then let \( \beta(i) = 1 \), otherwise \( \beta(i) = 0 \). Let \( v_i \) send charge \( \frac{\alpha(i)}{6} \) to \( f_i \) and \( \beta(i+1)(\frac{1}{2} - \frac{\alpha(i)}{6}) \) to \( v_{i+1} \). Similarly, let \( v_i \) send charge \( \frac{\alpha(i-1)}{6} \) to \( f_{i-1} \) and \( \beta(i-1)(\frac{1}{2} - \frac{\alpha(i-1)}{6}) \) to \( v_{i-1} \). Then each 3-face \( f_i \) gains \( 2 \times \frac{1}{2} \) from \( v_i \) and \( v_{i+1} \), and each 4*-vertex gains a nonnegative charge by the new rule.

Consider a good 3-vertex \( v_i \). Note that at most one of \( f_{i-1} \) and \( f_i \) is a 3-face to avoid a 4-cycle. By symmetry, assume \( f_{i-1} \) is not a 3-face. Then \( v_i \) receives \( \frac{1}{2} \) from \( f \), receives \( \frac{1}{2} \) from \( v_{i-1} \), receives at least \( \frac{1}{2} - \frac{1}{6} = \frac{1}{12} \) from \( v_{i+1} \), and sends at most \( \frac{1}{2} \) to \( f \). Thus \( v_i \) gains charge at least \( \frac{1}{2} + \frac{1}{4} + \frac{1}{12} - \frac{1}{6} = \frac{5}{12} \) by the new rule.

Consider bad 3-vertices \( v_i \) and \( v_{i+1} \). By Lemma 10, \( v_{i-1} \) and \( v_{i+2} \) are 5*-vertices. Since charge sent from \( v_i \) to \( v_{i+1} \) and charge sent from \( v_{i+1} \) to \( v_i \) are the same, we ignore this distribution in the calculation. Note that if \( f_i \) is a 3-face, then none of \( f_{i-1} \) and \( f_{i+1} \) are 3-faces. Assume \( f_i \) is a 3-face. Then \( v_i \) receives \( \frac{1}{2} \) from \( f \), receives \( \frac{1}{4} \) from \( v_{i-1} \), and sends \( \frac{1}{2} \) to \( f_i \). Thus \( v_i \) gains \( \frac{1}{4} + \frac{1}{2} = \frac{3}{2} > \frac{1}{3} \) by the new rule. Assume \( f_i \) is not a 3-face. Then \( v_i \) receives \( \frac{1}{2} \) from \( f_i \), receives at least \( \frac{1}{2} - \frac{1}{6} = \frac{1}{12} \) from \( v_{i-1} \), and sends at most \( \frac{1}{6} \) to \( f_{i-1} \). Thus \( v_i \) gains at least \( \frac{1}{2} + \frac{1}{12} - \frac{1}{6} = \frac{5}{12} > \frac{1}{3} \) by the new rule.

Altogether, let \( f \) send charge at most \( \frac{k}{2} \) with a distribution to its incident 3-faces and adjacent 3-faces that satisfies the original rules. Thus \( \mu^*(f) \geq \mu(f) - \frac{k}{2} = \frac{k}{2} - 4 \geq 0 \).

This completes the proof.

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References


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