HOP DOMINATION IN CHORDAL BIPARTITE GRAPHS

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Abstract

In a graph $G$, a vertex is said to 2-step dominate itself and all the vertices which are at distance 2 from it in $G$. A set $D$ of vertices in $G$ is called a hop dominating set of $G$ if every vertex outside $D$ is 2-step dominated by some vertex of $D$. Given a graph $G$ and a positive integer $k$, the hop domination problem is to decide whether $G$ has a hop dominating set of cardinality at most $k$. The hop domination problem is known to be NP-complete for bipartite graphs. In this paper, we design a linear time algorithm for computing a minimum hop dominating set in chordal bipartite graphs.

Keywords: domination, hop domination, polynomial time algorithm, chordal bipartite graphs.

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1. Introduction

A set $D \subseteq V$ of a graph $G = (V,E)$ is a dominating set of $G$ if every vertex in $V \setminus D$ is adjacent to a vertex in $D$. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. We refer the reader to the two so-called domination books by Haynes, Hedetniemi, and Slater [11, 12] for fundamental concepts in domination in graphs. The distance between two vertices $x$ and $y$ in a connected graph $G$, denoted $d_G(x,y)$, is the length of the shortest $x,y$-path in $G$. For an integer $k \geq 1$, a vertex in a graph $G$ is said to $k$-step dominate itself and all the vertices that are at distance exactly $k$ apart from it. A set $D \subseteq V$ of a graph $G = (V,E)$ is a $k$-step dominating set of $G$ if every vertex in $V \setminus D$ is $k$-step dominated by some vertex of $D$. The $k$-step domination number, $\gamma_{k\text{step}}(G)$, of $G$, is the minimum cardinality of a $k$-step dominating set of $G$. In 1995 Chartrand et al. [6] initiated the concept of 2-step domination in graphs, which was subsequently studied in [4, 9, 15].

The hop domination in graphs is closely related to the 2-step domination number. The concept of hop domination in graphs was introduced by Ayyaswamy and Natarajan [1]. A set $D \subseteq V$ of a graph $G = (V,E)$ is a hop dominating set of $G$ if every vertex of $V \setminus D$ is 2-step dominated by some vertex of $D$. The minimum cardinality of a hop dominating set of a graph $G$ is called the hop domination number of $G$ and is denoted by $\gamma_h(G)$.

Natarajan and Ayyaswamy [19] studied when the hop domination number is equal to other domination parameters. In [20], they also obtained an upper bound on hop domination number of the subdivision graph of any connected graph $G$. Ayyaswamy et al. [2] established upper and lower bounds on the hop domination number of a tree together with the characterization of extremal trees. Natarajan et al. [21] determined the hop domination number in some special family of graphs. Pabilona and Rara [24] characterized the connected hop dominating set in graphs under some binary operations and calculated the connected hop domination number of those graphs. Rakim et al. [26] studied the concept of perfect hop domination in graphs and determined the perfect hop domination number in some graph classes. Henning and Rad [13] presented probabilistic upper bounds for the hop domination number of a graph.

Given a graph $G$ and a positive integer $k$, the hop domination problem is to decide whether $G$ has a hop dominating set of cardinality at most $k$. Henning and Rad [13] proved that the hop domination problem is $\text{NP}$-complete for planar bipartite graphs and planar chordal graphs. Later, Jalalvand and Rad [16] determined the complexity results on $k$-step and $k$-hop dominating sets in graphs. Henning et al. [14] presented some hardness results on the hop domination prob-
lem and designed a linear time algorithm to compute a minimum hop dominating set in bipartite permutation graphs. Chen and Wang [7] investigated the relationship between the total domination number and the hop domination number in diamond-free graphs. Kundu and Majumder [17] gave a linear time algorithm to compute an optimal \( k \)-hop dominating set of a tree for \( k \geq 1 \).

A **chord** of a cycle is an edge joining two nonconsecutive vertices of the cycle. A bipartite graph \( G \) is called a **chordal bipartite** graph if every cycle of length at least 6 has a chord. Most domination problems and their variations are \( \text{NP} \)-hard for chordal bipartite graphs, as illustrated in Table 1 where we consider fundamental domination type parameters including domination, total domination, independent domination, connected domination, locating-domination, locating-total domination, and paired-domination. In Table 1, we have taken the decision versions of the variations of the domination problems.

<table>
<thead>
<tr>
<th>Name of the problem</th>
<th>Complexity Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domination</td>
<td>( \text{NP-complete} ) [18]</td>
</tr>
<tr>
<td>Total domination</td>
<td>Polynomial [8, 23]</td>
</tr>
<tr>
<td>Locating-domination</td>
<td>( \text{NP-complete} ) [10]</td>
</tr>
<tr>
<td>Locating-total domination</td>
<td>( \text{NP-complete} ) [25]</td>
</tr>
<tr>
<td>Connected domination</td>
<td>( \text{NP-complete} ) [18]</td>
</tr>
<tr>
<td>Independent Domination</td>
<td>( \text{NP-complete} ) [8]</td>
</tr>
<tr>
<td>Paired-domination</td>
<td>Polynomial [22]</td>
</tr>
</tbody>
</table>

Table 1. Complexities of variations of domination problems in chordal bipartite graphs.

Chordal bipartite graphs are characterized in terms of weak elimination orderings [27] and strong \( T \)-elimination orderings [5]. Given a weak elimination ordering of a chordal bipartite graph \( G \), a strong \( T \)-elimination ordering of \( G \) can be computed in linear time [23]. In this paper, given a weak elimination ordering of a chordal bipartite graph, we present a linear time algorithm to compute a minimum hop dominating set of the chordal bipartite graph.

2. **Terminology and Notation**

We use the standard notation \( [k] = \{1, \ldots, k\} \). Let \( G = (V, E) \) be a graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The **order** of \( G \) is \( n(G) = |V(G)| \) and the size of \( G \) is \( m(G) = |E(G)| \). Two vertices \( x \) and \( y \) in \( G \) are **adjacent** if they are joined by an edge \( e \), that is, if \( uv \in E(G) \). Two vertices in a graph \( G \) are **independent** if they are not adjacent. A set of pairwise independent vertices in \( G \) is an **independent set** of \( G \). The **open neighborhood** of a vertex \( v \) in \( G \)
is the set $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. The degree of a vertex $v$ is $|N_G(v)|$ and is denoted by $d_G(v)$. We simply use $N(v)$ and $N[v]$ if the context of the graph is clear. A vertex is isolated if the degree of the vertex is 0 and is pendant if the degree of the vertex is 1. For a set $A$ of vertices in $G$, the subgraph of $G$ induced by $A$ is denoted by $G[A]$. The distance between two vertices $x$ and $y$ in a connected graph $G$, denoted by $d_G(x, y)$, is the length of the shortest $x, y$-path in $G$. For a vertex $v$ in $G$, we define $SN(v)$ as the set of vertices at distance exactly 2 from $v$ in $G$, i.e., $SN(v) = \{u \mid d_G(u, v) = 2\}$, and $SN[v] = SN(v) \cup \{v\}$. A vertex $u$ in a graph $G$ is said to 2-step dominate itself and all the vertices that are at distance exactly 2 from $u$.

A walk in a graph is a sequence of vertices in which consecutive vertices are adjacent. A path is a walk in which all the vertices are different, while a cycle is a walk whose first and last vertex are the same and all other vertices are distinct. A chord in a cycle is an edge between two nonconsecutive vertices in the cycle. A graph $G$ is bipartite if $V(G)$ can be partitioned into two independent sets $X$ and $Y$ such that every edge joins a vertex in $X$ to a vertex in $Y$. The partition $(X, Y)$ of $V(G)$ is called a bipartition of $G$. A bipartite graph $G$ with bipartition $(X, Y)$ and edge set $E(G)$ is denoted by $G = (X, Y, E)$. A bipartite graph $G = (X, Y, E)$ is a complete bipartite graph if every vertex of $X$ is adjacent to every vertex of $Y$. For a bipartite graph $G = (X, Y, E)$, we use the notation $n_x = |X|$ and $n_y = |Y|$. A graph $G$ is said to be a chordal bipartite graph if $G$ is bipartite and every cycle of length at least 6 has a chord. Chordal bipartite graphs form a subclass of bipartite graphs and a superclass of bipartite permutation graphs [3].

A vertex $v$ of a graph $G$ is called a weak simplicial vertex if $N_G(v)$ is an independent set of $G$ and for every $u_1, u_2 \in N_G(v)$, either $N_G(u_1) \subseteq N_G(u_2)$ or $N_G(u_2) \subseteq N_G(u_1)$. An ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of the vertices of $G$ is called a weak elimination ordering of $G$ if for every $i \in [n]$, $v_i$ is weak simplicial in $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ and for every $v_j, v_k \in N_G(v_i)$ with $j < k$, $N_G(v_j) \subseteq N_G(v_k)$.

Let $G = (X, Y, E)$ be a bipartite graph, and let $\alpha = (x_1, x_2, \ldots, x_{n_x})$ and $\beta = (y_1, y_2, \ldots, y_{n_y})$ be some orderings of $X$ and $Y$, respectively. The ordering $\alpha$ and $\beta$ is called a strong $T$-elimination ordering of $G$ if for each $i \in [n_y]$ and $j, k \in [n_x]$ with $j < k$, where $x_j, x_k \in N_G(y_i)$, we have that $N_{G'}(x_j) \subseteq N_{G'}(x_k)$, where $G' = G[\{y_i, y_{i+1}, \ldots, y_{n_y}\} \cup \{x_1, x_2, \ldots, x_{n_x}\}]$.

Chordal bipartite graphs are characterized in terms of a weak elimination ordering [27] and are also characterized in terms of a strong $T$-elimination ordering [5]. Given a chordal bipartite graph $G = (V, E)$, a weak elimination ordering of $G$ can be computed in $\mathcal{O}(\min\{m \log n, n^2\})$ time [27].

For notational convenience, for a given set $X = \{x_1, x_2, \ldots, x_{n_x}\}$, we denote $X_i$ as the set $\{x_i, x_{i+1}, \ldots, x_{n_x}\}$ for every $i \in [n_x]$. Similarly given $Y =$
The following relation between a weak elimination ordering and a strong $T$-elimination ordering of a chordal bipartite graph $G = (X,Y,E)$ is established in [23].

**Theorem 1** [23], Given a weak elimination ordering $\sigma$ of a chordal bipartite graph $G = (X,Y,E)$, a strong $T$-elimination ordering $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ of $G$ can be obtained in $O(n)$ time such that

(a) for each $i \in [n_x]$, we have $N_{G'}(y_j) \subseteq N_{G'}(y_k)$, where $G' = G[X \cup Y]$ and $y_j, y_k \in N_{G'}(x_i)$ with $j < k$;

(b) for each $i \in [n_y]$, we have $N_{G''(y_j)} \subseteq N_{G''(y_k)}$, where $G'' = G[X \cup Y_i]$ and $x_j, x_k \in N_{G''(y_i)}$ with $j < k$.

Let $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ be a strong $T$-elimination ordering of $G$ and let $y_j \in Y$. Let $i = \max\{k \mid y_j x_k \in E(G)\}$. Then it can be observed that $y_j$ is a pendant vertex of the graph $G'' = G[X \cup Y_i]$. Therefore, we have the following lemma.

**Lemma 2.** If $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ is a strong $T$-elimination ordering of $G$ and $y_j \in Y$, then there exists $i \in [n_x]$ such that $y_j$ is a pendant vertex in $G'' = G[X \cup Y_i]$.

3. Hop Domination in Chordal Bipartite Graphs

In this section, we present a polynomial time algorithm for computing a minimum hop dominating set in chordal bipartite graphs. Given a weak elimination ordering of a chordal bipartite graph $G = (X,Y,E)$ of order $n$ and size $m$, our algorithm takes $O(n + m)$ time to compute a minimum hop dominating set of $G$. If $G$ is a disconnected graph having components $G_1, G_2, \ldots, G_r$ where $r \geq 2$, then $\gamma_h(G) = \sum_{i=1}^{r} \gamma_h(G_i)$. Hence it suffices for us to consider only connected chordal bipartite graphs for the purpose of designing our algorithm.
Let \( n_x = |X| \) and \( n_y = |Y| \). For \( i \in [n_x] \) and for a vertex \( x_b \in X \), we use the notation \( SN_i(x_b) = X_i \cap \{ x \in X \mid d_G(x_c, x_b) = 2 \} \) and \( SN_i[x_b] = SN_i(x_b) \cup \{ x_b \} \). Similarly, for \( i \in [n_y] \) and for a vertex \( y_a \in Y \), we use the notation \( SN_i(y_a) = Y_i \cap \{ y \in Y \mid d_G(y_c, y_a) = 2 \} \) and \( SN_i[y_a] = SN_i(y_a) \cup \{ y_a \} \).

**Lemma 3.** Let \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) be a strong \( T \)-elimination ordering of a connected chordal bipartite graph \( G = (X, Y, E) \) and \( x_p, x_i \in X \) such that \( d_G(x_p, x_i) = 2 \). If \( a = \max \{ k \mid x_iy_k \in E \} \) and \( b = \max \{ k \mid y_ax_k \in E \} \), then \( SN_i(y_p) \subseteq SN_i[y_b] \).

**Proof.** Let \( x' \in SN_i(x_p) \) be arbitrary. By the definition of \( b \), we have \( b \geq i \). If \( p' = i \), then it is clear that \( x' = x_i \in SN_i[x_b] \). If \( p' = b \), then \( x'_p = x_b \in SN_i[x_b] \). So assume that \( p' \neq i \) and \( p' \neq b \). Since \( d_G(x_p, x_i) = 2 \) and \( d_G(x_p, x'_p) = 2 \), there exist vertices \( y_q \) and \( y'_q \) such that \( y_q \in N(x_p) \cap N(x_i) \) and \( y'_q \in N(x_p) \cap N(x'_p) \), respectively. If \( q' = a \), then \( d_G(x'_p, y_q) = d_G(x_p, y_q) + d_G(y'_q, x'_p) = d_G(x'_p, y'_q) + d_G(y_q, x_i) = 2 \). So \( x'_p \in SN_i[x_b] \) and thus we are done. Hence we may assume that \( q' \neq a \). If \( q' = q \), then \( y'_q \in N(x_i) \); thus \( q' < a \) by the definition of \( a \). Since \( \sigma_X \) and \( \sigma_Y \) form a strong \( T \)-elimination ordering of \( G \), by Theorem 1(a), \( x'_p \in N_G(y'_q) \subseteq N_G(y_a) \), where \( G' = G[X_i \cup Y] \). Now \( d_G(x'_p, x_b) = d_G(x'_p, y_a) + d_G(y_a, x_b) = 2 \). So \( x'_p \in SN_i[x_b] \) and thus we are done. Hence we may assume that \( q \neq q' \). Let \( G' = G[X_i \cup Y] \) and \( G'' = G[X_p \cup Y] \).

Assume that \( p < i \). We now prove that \( x'_p \in N(y_a) \). If \( q < q' \), then, since \( y_q, y'_q \in N_{G'}(x_p) \), by Theorem 1(a), \( x_i \in N_{G''}(y_q) \subseteq N_{G''}(y'_q) \). Now \( y'_p, y_a \in N_{G''}(x_i) \). By Theorem 1(a), \( x'_p \in N_{G''}(y'_q) \subseteq N_{G''}(y_a) \). If \( q' < q \), then by Theorem 1(a), \( x'_p \in N_{G''}(y'_q) \subseteq N_{G''}(y_q) \). Since \( y_q, y_a \in N(x_i) \), by Theorem 1(a), \( x'_p \in N_{G''}(y_q) \subseteq N_{G''}(y_a) \). Now \( d_G(x'_p, x_b) = d_G(x'_p, y_q) + d_G(y_q, x_b) = 2 \). This implies that \( x'_p \in SN_i[x_b] \).

Now assume that \( p > i \). If \( p = b \), then \( d_G(x'_p, x_b) = 2 \). We now prove that \( x'_p \in N(y_a) \). If \( q < q' \), then, since \( y_q, y_a \in N(x_i) \), by Theorem 1(a), \( x_p \in N_{G''}(y_q) \subseteq N_{G''}(y_a) \). Now \( y'_q, y_a \in N(x_p) \), and so by Theorem 1(a), \( x'_p \in N_{G''}(y'_q) \subseteq N_{G''}(y_q) \). This implies that \( x'_p \in SN_i(x_i) \). If \( q' < q \), then \( q' < q \leq a \), by definition of \( a \). If \( q = a \), then \( x_p \in N(y_a) \). If \( q < a \), then since \( y_q, y_a \in N(x_i) \), by Theorem 1(a), \( x_p \in N_{G''}(y_q) \subseteq N_{G''}(y_a) \). Now \( y'_q, y_a \in N(x_p) \) with \( q' < q \leq a \). Thus, by Theorem 1(a), \( x'_p \in N_{G''}(y'_q) \subseteq N_{G''}(y_a) \). Now \( d_G(x'_p, x_b) = d_G(x'_p, y_q) + d_G(y_q, x_b) = 2 \). This implies that \( x'_p \in SN_i[x_b] \).

Similar to Lemma 3, the following lemma can also be proved.

**Lemma 4.** Let \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) be a strong \( T \)-elimination ordering of a connected chordal bipartite graph \( G = (X, Y, E) \) and \( y_p, y_i \in Y \) such that \( d_G(y_p, y_i) = 2 \). If \( a = \max \{ k \mid y_ix_k \in E \} \) and \( b = \max \{ k \mid x_ay_k \in E \} \), then \( SN_i(y_p) \subseteq SN_i[y_b] \).
Now we present our algorithm, namely \text{HDS-CBG}(G) to compute a minimum hop dominating set in a given connected chordal bipartite graph \(G\). The algorithm processes the vertices \(x_1, x_2, \ldots, x_{n_x}\) with respect to a strong \(\mathcal{T}\)-elimination ordering \(\sigma_X = (x_1, x_2, \ldots, x_{n_x})\) and \(\sigma_Y = (y_1, y_2, \ldots, y_{n_y})\) of \(G\) and at each iteration \(i \in [n_x]\), our algorithm selects new vertices to add to the set \(\text{HD}\) if \(x_i\) is not 2-step dominated or there is a pendant vertex \(v \in N(x_i)\) in \(G'\), where \(G' = G[X_i \cup Y]\) such that \(v\) is not 2-step dominated by the set \(\text{HD}\) constructed so far. To achieve this, at each iteration \(i \in [n_x]\), our algorithm proceeds as follows.

- An array \(D\) is maintained on the vertices of \(G\) to track whether a vertex \(v\) of \(G\) is 2-step dominated or not by the set \(\text{HD}\) constructed so far. In particular, \(D[v] = 1\) if \(v\) is 2-step dominated by \(\text{HD}\); otherwise \(D[v] = 0\). Initially, \(D[v] = 0\) for all \(v \in V(G)\).
- In the Lines 6–9, the algorithm checks whether the vertices of the set \(N[x_i]\) in \(G'\) are 2-step dominated or not by the set \(\text{HD}\) constructed so far.
- In the Lines 10–18, the algorithm checks the two conditions (a) \(D[x_i] = 0\) or not, and (b) \(N_G'(x_i)\) has a pendant vertex \(v\) such that \(D[v] = 0\), and adds new vertices to the set \(\text{HD}\) following some rules. All details are described in the algorithm.

For every \(i \in [n_x]\), we define \(\ell(i)\) and \(\rho(i)\) as follows.

\[
\ell(i) = \max\{k \mid x_i y_k \in E(G)\} \quad \text{and} \quad \rho(i) = \max\{k \mid x_k y_{\ell(i)} \in E(G)\}.
\]

In Table 2, we explain the execution of the algorithm \text{HDS-CBG}(G) on the chordal bipartite graph \(G\) shown in Figure 1. Note that in Table 2, we have considered those iterations of the algorithm in which some vertices are selected. In Table 2, we have a column, namely “The set \(\text{HD}\)”. Initially, \(\text{HD} = \emptyset\). Then the updated set \(\text{HD}\) is described for different iterations.

For each \(i \in [n_x]\), let \(\text{HD}_i\) be the set \(\text{HD}\) computed at the end of the \(i\)-th iteration of the algorithm. The following lemmas can be observed from the algorithm \text{HDS-CBG}(G).

**Lemma 5.** If \(x_i\) is the considered vertex for \(i \in [n_x]\) at some point of the algorithm \text{HDS-CBG}(G), then the following are true.

(a) \(D[v] = 1\) for all \(v \in \bigcup_{s \in [i-1]} N[x_s]\).

(b) If \(x_i \notin \text{HD}_{i-1}\) and \(x_i\) has a neighbor \(v\) such that \(N(v) \cap \text{HD}_{i-1} \neq \emptyset\), then \(x_i\) is 2-step dominated by \(\text{HD}_{i-1}\).

(c) If \(N(x_i) \cap \text{HD}_{i-1} \neq \emptyset\), then every \(v \in N(x_i)\) is 2-step dominated by \(\text{HD}_{i-1}\).

Notice that \(\text{HD}_0 = \emptyset\). At the termination of the algorithm \text{HDS-CBG}(G), by Lemma 5, \(\text{HD}_{n_x}\) is a hop dominating set of \(G\). Therefore, to prove that \(\text{HD}_{n_x}\) is a minimum hop dominating set of \(G\), it is sufficient to prove that there is a minimum hop dominating set \(S^*\) of \(G\) such that \(\text{HD}_{n_x} \subseteq S^*\). To prove this, we
**Algorithm 1: HDS-CBG(G)**

**Input:** A connected chordal bipartite graph $G = (X, Y, E)$, where $|X| = n_x$ with a weak elimination ordering of $G$;

**Output:** A hop dominating set HD of $G$;

1. Compute $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$, a strong $\mathcal{T}$-elimination ordering of $G$;

2. $D[v] = 0$ for all $v \in X \cup Y$;

3. Initialize $HD = \emptyset$;

4. **for** $(i = 1$ to $n_x)$ **do**

5. Let $G' = G[X_i \cup Y]$;

6. **if** $(x_i \notin HD$ and $x_i$ has a neighbor $v$ such that $N(v) \cap HD \neq \emptyset$) **then** /* Case 1 */

7. $D[x_i] = 1$;

8. **if** $(N(x_i) \cap HD \neq \emptyset)$ **then** /* Case 2 */

9. $D[u] = 1$ for all $u \in N(x_i)$;

10. **if** $(D[x_i] = 0$ and $N_{G'}(x_i)$ has a pendant vertex $v$ such that $D[v] = 0$) **then** /* Case 3 */

11. $HD = HD \cup \{x_{\rho(i)}, y_{\ell(i)}\}$;

12. $D[u] = 1$ for all $u \in N(x_i)$ and $D[x_{\rho(i)}] = 1$;

13. **else if** $(D[x_i] = 0$ and $N_{G'}(x_i)$ has no pendant vertex $v$ such that $D[v] = 0$) **then** /* Case 4 */

14. $HD = HD \cup \{x_{\rho(i)}\}$;

15. $D[x_i] = 1$ and $D[x_{\rho(i)}] = 1$;

16. **else if** $(D[x_i] \neq 0$ and $N_{G'}(x_i)$ has a pendant vertex $v$ such that $D[v] = 0$) **then** /* Case 5 */

17. $HD = HD \cup \{y_{\ell(i)}\}$;

18. $D[u] = 1$ for all $u \in N(x_i)$;

19. return $HD$;
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use induction on \(i, i \in [nx] \cup \{0\}\), and prove that each \(\text{HD}_i, i \in [nx] \cup \{0\}\), is contained in some minimum hop dominating set of \(G\). Since \(\text{HD}_0 = \emptyset\), the base case is true. Assume that \(i \geq 1\) and that the set \(\text{HD}_{i-1}\) is contained in some minimum hop dominating set \(S'\) of \(G\). We now show that \(\text{HD}_i\) is contained in some minimum hop dominating set of \(G\). For this purpose, we proceed with a series of lemmas. In each lemma, we construct a minimum hop dominating set of \(G\) containing \(\text{HD}_i\) from the minimum hop dominating set \(S'\) of \(G\). We recall our earlier notation \(\rho(i) = \max\{k \mid x_ky_j \in E(G)\}\) and \(\sigma(i) = \max\{k \mid x_ky_{(i)} \in E(G)\}\) which will be used in the following lemmas. In each of the lemma, we assume that \(\sigma_X = (x_1, x_2, \ldots, x_{n_x})\) and \(\sigma_Y = (y_1, y_2, \ldots, y_{n_y})\) is a strong \(T\)-elimination ordering of the graph \(G = (X, Y, E)\).

**Lemma 6.** Let \(S'\) be a minimum hop dominating set of \(G\) such that \(\text{HD}_{i-1} \subseteq S'\) and \(G' = G[\bar{X} \cup Y]\). If \(D[x_i] = 0\) and \(G'\) has a pendant vertex \(y_j \in N(x_i)\) such that \(D[y_j] = 0\), then there is a minimum hop dominating set of \(G\) containing \(\text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}\).

**Proof.** Let \(x_a \in S'\) be the vertex that 2-step dominates \(x_i\) and \(y_b \in S'\) be the vertex that 2-step dominates \(y_j\). Since \(D[x_i] = 0\) and \(D[y_j] = 0\), we note that \(x_a, y_b \notin \text{HD}_{i-1}\). We proceed further with proving the following claims.

**Claim 7.** If \(b \neq j\), then there is a minimum hop dominating set containing \(\text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}\).

**Proof.** Since \(b \neq j\), we have \(d_G(y_j, y_b) = 2\). By Lemma 4, \(SN_j(y_b) \subseteq SN_j[y_{\ell(i)}]\).

By Lemma 5, the vertices from \(\{x_1, x_2, \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{j-1}\}\) are 2-step dominated by \(\text{HD}_{i-1}\).

If \(a = i\) and \(\rho(i) = i\), then \((S' \setminus \{y_b\}) \cup \{y_{\ell(i)}\}\) is a minimum hop dominating set of \(G\) containing \(\text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}\). If \(a = i\) and \(\rho(i) \neq i\), then \(d_G(x_i, x_{\rho(i)}) = 2\) and by Lemma 3, \(SN_i(x_i) \subseteq SN_i[x_{\rho(i)}]\). Thus, \((S' \setminus \{x_a, y_b\}) \cup \{x_{\rho(i)}, y_{\ell(i)}\}\) is a minimum hop dominating set of \(G\) containing \(\text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}\).

If \(a \neq i\), then \(d_G(x_i, x_a) = 2\). By Lemma 3, we have \(SN_i(x_a) \subseteq SN_i[x_{\rho(i)}]\), i.e., the vertices of the set \(\{x_{i-1}, x_{i-1+1}, \ldots, x_{n_x}\}\) that are 2-step dominated by \(x_a\) are 2-step dominated by \(x_{\rho(i)}\). If \(a < i\), then by Lemma 5, \(x_a\) is 2-step dominated by \(\text{HD}_{i-1}\). If \(a > i\), then let \(x_{ik}x_{ik}\) be a shortest path between \(x_i\) and \(x_a\). Then \(y_c, y_{\ell(i)} \in N_G'(x_i)\) with \(c \leq \ell(i)\). By Theorem 1(a), \(x_a \in N_G'(y_c) \subseteq N_G'(y_{\ell(i)})\). This implies that \(x_a \in SN_i[x_{\rho(i)}]\). Hence, \((S' \setminus \{x_a, y_b\}) \cup \{x_{\rho(i)}, y_{\ell(i)}\}\) is a minimum hop dominating set of \(G\) containing \(\text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}\). This completes the proof of Claim 7.

□

**Claim 8.** If \(b = j\), then there is a minimum hop dominating set containing \(\text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}\).
The set $H_{D}^{i} = H_{D}^{i} \cap \{x_{\rho}(i), y_{\ell}(i)\}$.

Table 2. Illustration of the algorithm HDS-CBG($G$) on the graph $G$ shown in Figure 1.

**Proof.** If $\ell(i) \neq j$, then by Lemma 4, $SN_{j}(y_{b}) \subseteq SN_{j}[y_{\ell(i)}]$. Thus, $(S' \setminus \{y_{b}\}) \cup \{y_{\ell(i)}\}$ is a minimum hop dominating set of $G$ containing $H_{D_{i-1}} \cup \{y_{\ell(i)}\}$ in which $y_{j}$ is 2-step dominated by $y_{\ell(i)}$. Hence, by Claim 7, we obtain a minimum hop dominating set of $G$ containing $H_{D_{i-1}} \cup \{x_{\rho}(i), y_{\ell(i)}\}$.

So we may assume that $\ell(i) = j$, implying that $b = \ell(i) = j$. If $a = i$ and $\rho(i) = i$, then $S'$ is the minimum hop dominating set of $G$ containing $H_{D_{i-1}} \cup \{x_{\rho}(i), y_{\ell(i)}\}$. If $a = i$ and $\rho(i) \neq i$, then $d_{G}(x_{i}, x_{\rho(i)}) = 2$ and by Lemma 3, $SN_{i}(x_{i}) \subseteq SN_{i}[x_{\rho(i)}]$. So $(S' \setminus \{x_{a}\}) \cup \{x_{\rho(i)}\}$ is a minimum hop dominating set of $G$ containing $H_{D_{i-1}} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$.
If \( a \neq i \), then \( d_G(x_i, x_a) = 2 \). By Lemma 3, we have \( SN_i(x_a) \subseteq SN_i[x_{\rho(i)}] \), i.e., the vertices of the set \( \{x_i, x_{i+1}, \ldots, x_{n_a}\} \) that are 2-step dominated by \( x_a \) are 2-step dominated by \( x_{\rho(i)} \). If \( a < i \), then by Lemma 5, \( x_a \) is 2-step dominated by \( HD_{i-1} \). If \( a > i \), then let \( x_i y_c x_a \) be a shortest path between \( x_i \) and \( x_a \). Then, \( y_c, y_{\ell(i)} \in N_G(x_i) \) with \( c \leq \ell(i) \). By Theorem 1(a), \( x_a \in N_G(y_c) \subseteq N_G'(y_{\ell(i)}) \). This implies that \( x_a \in SN_i[x_{\rho(i)}] \). Therefore, \( (S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\} \) is a minimum hop dominating set of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\} \). This completes the proof of Claim 8.

We now return to the proof of Lemma 6. If \( b \neq j \), then by Claim 7, we obtain a minimum hop dominating set of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\} \). If \( b = j \), then by Claim 8, we obtain a minimum hop dominating set \( S'' \) of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\}, \{y_{\ell(i)}\} \). This completes the proof of Lemma 6.

**Lemma 9.** Let \( S' \) be a minimum hop dominating set of \( G \) such that \( HD_{i-1} \subseteq S' \) and \( G' = G'[X_i \cup Y] \). If \( D[x_i] = 0 \) and \( N_G(x_i) \) has no pendant vertex \( v \) such that \( D[v] = 0 \), then there is a minimum hop dominating set of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\} \).

**Proof.** Let \( x_a \in S' \) be a vertex that 2-step dominates \( x_i \). Since \( D[x_i] = 0 \), we have \( x_a \notin HD_{i-1} \). Clearly, by definition of \( \rho(i) \), we have \( \rho(i) \geq i \).

First assume that \( a = i \). If \( \rho(i) = i \), then \( a = i = \rho(i) \) and hence \( S' \) is a minimum hop dominating set of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\} \). If \( \rho(i) \neq i \), then \( d_G(x_i, x_{\rho(i)}) = 2 \) and by Lemma 3, \( SN_i(x_a) \subseteq SN_i[x_{\rho(i)}] \). Since by Lemma 5, every vertex of the set \( \{x_i, x_{i+1}, \ldots, x_{n_a}\} \cup \{y_1, y_2, \ldots, y_{\ell(i)}\} \) is 2-step dominated by \( HD_{i-1} \), \( (S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\} \) is a minimum hop dominating set of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\} \).

Now assume \( a \neq i \). Then, \( d_G(x_i, x_a) = 2 \) and by Lemma 3, \( SN_i(x_a) \subseteq SN_i[x_{\rho(i)}] \), i.e., the vertices of the set \( \{x_i, x_{i+1}, \ldots, x_{n_a}\} \) that are 2-step dominated by \( x_a \) are 2-step dominated by \( x_{\rho(i)} \). Also by Lemma 5, the vertices from \( \{x_1, x_2, \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{\ell(i)-1}\} \) are 2-step dominated by \( HD_{i-1} \), where \( k = \min\{d \mid y_d \in N(x_i) \text{ and } D[y_d] = 0\} \). If \( a < i \), then by Lemma 5, \( x_a \) is 2-step dominated by \( HD_{i-1} \). If \( a > i \), then let \( x_i y_c x_a \) be a shortest path between \( x_i \) and \( x_a \). Then \( y_c, y_{\ell(i)} \in N_G(x_i) \) with \( c \leq \ell(i) \). By Theorem 1(a), \( x_a \in N_G'(y_c) \subseteq N_G'(y_{\ell(i)}) \). This implies that \( x_a \in SN_i[x_{\rho(i)}] \). Hence, \( (S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\} \) is a hop dominating set of \( G \) containing \( HD_{i-1} \cup \{x_{\rho(i)}\} \).

**Lemma 10.** Let \( S' \) be a minimum hop dominating set of \( G \) such that \( HD_{i-1} \subseteq S' \) and \( G' = G'[X_i \cup Y] \). If \( D[x_i] \neq 0 \) and \( G' \) has a pendant vertex \( y_j \in N(x_i) \) such that \( D[y_j] = 0 \), then there is a minimum hop dominating set of \( G \) containing \( HD_{i-1} \cup \{y_{\ell(i)}\} \).
Proof. Let \( y_b \in S' \) be a vertex that 2-step dominates \( y_j \). Since \( D[y_j] = 0 \), we have \( y_b \notin \text{HD}_{i-1} \). If \( b = j \), then by Claim 8 of Lemma 6, we get a minimum hop dominating set of \( G \) containing \( \text{HD}_{i-1} \cup \{y_{\ell(i)}\} \). Hence we may assume that \( b \neq j \). Thus, \( d_G(y_j, y_b) = 2 \). By Lemma 4, we have \( SN_j(y_b) \subseteq SN_j(y_{\ell(i)}) \), i.e., the vertices of the set \( \{y_j, y_j, \ldots, y_{n_y}\} \) that are 2-step dominated by \( y_b \) are now 2-step dominated by \( y_{\ell(i)} \). By Lemma 5, all the vertices from \( \{x_1, x_2, \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{j-1}\} \) are 2-step dominated by \( \text{HD}_{i-1} \). If \( b < j \), then by Lemma 5, \( y_b \) is 2-step dominated by \( \text{HD}_{i-1} \). If \( b > j \), then, since \( y_j \) is a pendant neighbor of \( x_i \), the path \( y_j x_i y_b \) is a shortest path between \( y_j \) and \( y_b \). Then \( y_b, y_{\ell(i)} \in N_{G'}(x_i) \) with \( b \leq \ell(i) \). This implies that \( y_b \in SN_j(y_{\ell(i)}) \). Again since \( D[x_i] = 0 \), \( x_i \in \text{HD}_{i-1} \) or \( x_i \) is also 2-step dominated by \( \text{HD}_{i-1} \). Hence, \( (S' \setminus \{y_b\}) \cup \{y_{\ell(i)}\} \) is a hop dominating set of \( G \) containing \( \text{HD}_{i-1} \cup \{y_{\ell(i)}\} \).

We now return to the proof of the statement that \( \text{HD}_{\alpha_x} \) is a minimum hop dominating set of the chordal bipartite graph \( G \). Recall that by the induction hypothesis, for \( i \geq 1 \), \( \text{HD}_{i-1} \) is contained in a minimum hop dominating set \( S' \) of \( G \). Notice that the algorithm considers the vertex \( x_i \) and its neighbors at the \( i \)-th iteration of the algorithm. Further, the algorithm first updates whether any vertex \( N(x_i) \) can be 2-step dominated or not. For this, it checks two conditions. The first condition is whether \( N(x_i) \cap \text{HD}_{i-1} \neq \emptyset \), and the second condition is whether \( x_i \) has a neighbor \( v \) such that \( N(v) \cap \text{HD}_{i-1} \neq \emptyset \). If \( N(x_i) \cap \text{HD}_{i-1} = \emptyset \), say \( v \in N(x_i) \cap \text{HD}_{i-1} \), then every vertex of \( N(x_i) \setminus \{v\} \) is 2-step dominated by \( v \). So in this case, \( \text{HD}_i = \text{HD}_{i-1} \), and hence \( \text{HD}_i \) is contained in \( S' \). Similarly if \( x_i \) has a neighbor \( v \) such that \( N(v) \cap \text{HD}_{i-1} \neq \emptyset \), then the vertex \( x_i \) is 2-step dominated by \( v \). Thus in this case \( \text{HD}_i = \text{HD}_{i-1} \), and hence \( \text{HD}_i \) is contained in \( S' \).

After this, the algorithm checks whether any vertex of \( N[x_i] \) is not 2-step dominated by \( \text{HD}_{i-1} \). If \( x_i \) is not 2-step dominated by \( \text{HD}_{i-1} \) (i.e., if \( D[x_i] = 0 \) and \( N_{G'}(x) \) has a pendant neighbor \( v \) that is not 2-step dominated by \( \text{HD}_{i-1} \), then the algorithm selects \( x_{\rho(i)} \) and \( y_{\ell(i)} \). Thus, \( \text{HD}_i = \text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\} \).

By Lemma 6, there is a minimum hop dominating set of \( G \) containing \( \text{HD}_i = \text{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\} \). If \( x_i \) is not 2-step dominated by \( \text{HD}_{i-1} \) (i.e., if \( D[x_i] = 0 \) and \( N_{G'}(x) \) has no pendant neighbor \( v \) that is not 2-step dominated by \( \text{HD}_{i-1} \), then the algorithm selects \( x_{\rho(i)} \). Thus, \( \text{HD}_i = \text{HD}_{i-1} \cup \{x_{\rho(i)}\} \).

By Lemma 9, there is a minimum hop dominating set of \( G \) containing \( \text{HD}_i = \text{HD}_{i-1} \cup \{x_{\rho(i)}\} \). If \( x_i \) is 2-step dominated by \( \text{HD}_{i-1} \) (i.e., if \( D[x_i] 
eq 0 \) and \( N_{G'}(x) \) has a pendant neighbor \( v \) that is not 2-step dominated by \( \text{HD}_{i-1} \), then the algorithm selects \( y_{\ell(i)} \). Hence, \( \text{HD}_i = \text{HD}_{i-1} \cup \{y_{\ell(i)}\} \).

By Lemma 10, there is a minimum hop dominating set of \( G \) containing \( \text{HD}_i = \text{HD}_{i-1} \cup \{y_{\ell(i)}\} \). Therefore, by induction \( \text{HD}_{\alpha_x} \) is a minimum hop dominating set of \( G \). We record this formally in the following lemma.
Lemma 11. HD\(_n\_\) is a minimum hop dominating set of the given chordal bipartite graph \(G\).

We now discuss how the algorithm HDS-CBG\((G)\) can be implemented in \(O(n + m)\) time for a given chordal bipartite graph \(G\) having \(n\) vertices and \(m\) edges. Given a chordal bipartite graph with a weak elimination ordering, by Theorem 1, a strong \(\mathcal{T}\)-elimination ordering of \(G\) can be computed in \(O(n)\) time.

We maintain an array \(D\) to track whether a vertex is 2-step dominated or not by the hop dominating set constructed thus far. Initially, \(D[v] = 0\) for all \(v \in V(G)\). At the \(i\)-th iteration, the algorithm checks the set \(\{v \in N(x_i) \mid D[v] = 0\}\) and the \(D\)-label of the vertex \(x_i\) which can be done in \(O(|N[x_i]|)\) time, i.e., in \(O(d_G(x_i) + 1) = O(d_G(x_i))\) time. Once new vertices are selected by the algorithm, the vertices in \(N(x_i)\) are updated to be 2-step dominated, i.e., the \(D\)-label is made 1 for the selected vertices and the 2-step dominated vertices of \(N[x_i]\).

To select new vertices, the algorithm checks the conditions as mentioned in Case 1–5 of the algorithm. In Case 1 of the algorithm at the \(i\)-iteration it checks the condition “\(x_i \notin \text{HD}\) and \(x_i\) has neighbor \(v\) such that \(N(v) \cap \text{HD} \neq \emptyset\)”. To do this, we maintain two arrays \(L\) and \(A\) on the vertices of \(G\). Initially, \(L[v] = A[v] = 0\) for every \(v \in V(G)\). Once a vertex \(v\) is selected to the set \(\text{HD}\), \(L[v]\) is made 1 and \(A[u]\) is made 1 for every \(u \in N(v)\). Hence we can conclude that if \(L[v] = 1\) at any iteration of the algorithm, then \(v\) belongs to \(\text{HD}\) and if \(A[u] = 1\), then a neighbor of \(u\) is already present in \(\text{HD}\). Throughout the algorithm, the arrays \(L\) and \(A\) can be maintained in at most

\[
\sum_{v \in V(G)} O(d_G(v)) = O(n + m)
\]

time. In other cases, the algorithm looks for a pendant neighbor of \(x_i\) in \(G' = G[X_i \cup Y]\). For this, we maintain an array \(B\) on the vertices of \(Y\). Initially, \(B[y] = d_G(y)\) for every \(y \in Y\). For every \(i \in [n_x]\), after the end of the \(i\)-th iteration \(B[y]\) is decremented by 1 for every \(y \in N(x_i)\). Hence, at the beginning of the \(i\)-th iteration if \(B[y'] = 1\) for some \(y' \in N(x_i)\), then \(y'\) is a pendant neighbor of \(x_i\) in \(G'\). Throughout the algorithm, the array \(B\) can therefore be maintained in at most \(\sum_{v \in V(G)} O(d_G(v)) = O(n + m)\) time. Moreover, at the \(i\)-th iteration, the arrays \(D, L, A\), and \(B\) can be maintained in at most \(O(d_G(x_i))\) time. As before, all other conditions at the \(i\)-th iteration can be checked in at most \(O(d_G(x_i))\) time. Thus the degrees of the vertices of the graph are scanned a constant number of times throughout the algorithm. Therefore, the algorithm in total takes at most \(O(n + m)\) time.

Due to the above discussion and Lemma 11, we have the following theorem.

Theorem 12. Given a weak elimination ordering of a chordal bipartite graph \(G\) having \(n\) vertices and \(m\) edges, a minimum hop dominating set of \(G\) can be computed in \(O(n + m)\) time.
4. Conclusion

The hop domination problem is known to be NP-complete for planar bipartite graphs, planar chordal graphs, and perfect elimination bipartite graphs. In this paper, we present a linear time algorithm for computing a minimum hop dominating set in chordal bipartite graphs if a weak elimination ordering of the graph is given. Since the hop domination problem is NP-complete for chordal graphs, it would be very interesting to decide the complexity of the minimum hop domination problem in subclasses of chordal graphs such as block graphs and interval graphs.

References


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