Note

TWO SUFFICIENT CONDITIONS FOR COMPONENT FACTORS IN GRAPHS

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Abstract

Let G be a graph. For a set \( \mathcal{H} \) of connected graphs, a spanning subgraph \( H \) of a graph \( G \) is called an \( \mathcal{H} \)-factor of \( G \) if each component of \( H \) is isomorphic to a member of \( \mathcal{H} \). An \( \mathcal{H} \)-factor is also referred as a component factor. If \( G - e \) admits an \( \mathcal{H} \)-factor for any \( e \in E(G) \), then we say that \( G \) is an \( \mathcal{H} \)-factor deleted graph. Let \( k \geq 2 \) be an integer. In this article, we verify that (i) a graph \( G \) admits a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor if and only if its binding number \( \text{bind}(G) \geq \frac{2}{2k+1} \); (ii) a graph \( G \) with \( \delta(G) \geq 2 \) is a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor deleted graph if its binding number \( \text{bind}(G) \geq \frac{2}{2k-1} \).

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1. Introduction

In this article, we discuss only finite simple graphs without loops or multiple edges. Given a graph $G$, we let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For a vertex $x$ of a graph $G$, we use $N_G(x)$ to denote the set of vertices adjacent to $x$ in $G$, and use $d_G(x)$ to denote the degree of $x$ in $G$. We write $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. For a vertex subset $X$ of a graph $G$, we denote by $G[X]$ the subgraph of $G$ induced by $X$, and write $G - X = G[V(G) \setminus X]$ and $N_G(X) = \bigcup_{x \in X} N_G(x)$. We denote by $I(G)$ the set of isolated vertices of $G$, and write $i(G) = |I(G)|$. The binding number $\text{bind}(G)$ of a graph $G$ is the minimum, taken over all $X \subseteq V(G)$ with $X \neq \emptyset$ and $N_G(X) \neq V(G)$, of the ratio $|N_G(X)|/|X|$. We denote by $C_n$ the cycle with $n$ vertices, by $K_n$ the complete graph with $n$ vertices, and by $K_{n,m}$ the complete bipartite graph with partite sets $X$ of size $n$ and $Y$ of size $m$, where $V(K_{n,m}) = X \cup Y$. For a tree $T$, we use $\text{Leaf}(T)$ to denote the set of leaves. An edge of $T$ incident with a leaf is said to be a pendant edge.

We define a special class of trees $T(2k+1)$, where $k \geq 2$ is an integer. Let $R$ be a tree that satisfies the following conditions: for any $x \in V(R) - \text{Leaf}(R)$,

\begin{enumerate}
    \item $d_{R - \text{Leaf}(R)}(x) \in \{1,3,\ldots,2k+1\}$
    \item $2 \cdot (\text{the number of leaves adjacent to } x \text{ in } R) + d_{R - \text{Leaf}(R)}(x) \leq 2k+1$
\end{enumerate}

For such a tree $R$, we derive a new tree $T_R$ as follows:

\begin{enumerate}
    \item insert a new vertex of degree 2 into each edge of $R - \text{Leaf}(R)$ and
    \item for every vertex $x$ of $R - \text{Leaf}(R)$ with $d_{R - \text{Leaf}(R)}(x) = 2r+1 < 2k+1$, add $k - r$ (the number of leaves adjacent to $x$ in $R$) pendant edges to $x$.
\end{enumerate}

Then the set of such trees $T_R$ for all trees $R$ satisfying conditions (a) and (b) is denoted by $T(2k+1)$.

For a set $\mathcal{H}$ of connected graphs, a spanning subgraph $H$ of a graph $G$ is called an $\mathcal{H}$-factor of $G$ if every component of $H$ is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. If $G - e$ admits an $\mathcal{H}$-factor for any $e \in E(G)$, then we say that $G$ is an $\mathcal{H}$-factor deleted graph.

Tutte [8] derived a characterization for a graph admitting a $\{K_2,C_n : n \geq 3\}$-factor. Amahashi and Kano [2] showed a necessary and sufficient condition for the existence of a $\{K_{1,j} : 1 \leq j \leq k\}$-factor in a graph. Kano, Lu and Yu [4] presented a sufficient condition for a graph to have a $\{K_{1,2},K_{1,3},K_5\}$-factor. Kano and Saito [6] obtained a result on the existence of a $\{K_{1,j} : k \leq j \leq 2k\}$-factor in a graph. Zhang, Yan and Kano [11] posed a sufficient condition for the existence of $\{K_{1,j},K_{2k} : k \leq j \leq 2k - 1\}$-factors in graphs. Zhou [14] derived some results on the existence of component factors in graphs. For the relationships between
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binding number and graph factors, we refer the reader to [3, 7, 9, 10, 12, 13, 15–17].

Kano, Lu and Yu [5] gave a criterion for a graph having a \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\}-factor, which is shown in the following.

**Theorem 1** (Kano, Lu and Yu [5]). Let \( k \) be an integer with \( k \geq 2 \). Then a graph \( G \) admits a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor if and only if

\[
i(G - X) \leq \left( k + \frac{1}{2} \right) |X|
\]

for every \( X \subseteq V(G) \).

In this article, we establishes some relationships between binding numbers and \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factors in graphs.

**Theorem 2.** Let \( k \) be an integer with \( k \geq 2 \). Then a graph \( G \) admits a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor if and only if its binding number \( \text{bind}(G) \geq \frac{2}{2k+1} \).

**Theorem 3.** Let \( k \) be an integer with \( k \geq 2 \). Then a graph \( G \) with \( \delta(G) \geq 2 \) is a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor deleted graph if its binding number \( \text{bind}(G) \geq \frac{2}{2k+1} \).

**Remark 4.** For two graphs \( H_1 \) and \( H_2 \), \( H_1 \cup H_2 \) denotes the union of \( H_1 \) and \( H_2 \), and \( H_1 \vee H_2 \) denotes the join of \( H_1 \) and \( H_2 \). We show that the condition \( \text{bind}(G) \geq \frac{2}{2k+1} \) in Theorem 3 cannot be replaced by \( \text{bind}(G) \geq \frac{2}{2k} \). To explain this, we construct a graph \( G = H_1 \vee ((2kK_1) \cup (2H_2)) \), where \( H_1 = K_2, H_2 = K_2 \), and \( k \geq 2 \) is an integer. Choose \( Y = V(2kK_1) \). Obviously, \( Y \neq \emptyset \) and \( N_G(Y) \neq V(G) \). Furthermore, we easily see that \( \text{bind}(G) = \frac{|N_G(Y)|}{|Y|} = \frac{2}{2k} \). Set \( e \in E(2H_2) \) and \( G' = G - e \). We choose \( X = V(H_1) \). Thus, we derive

\[
i(G' - X) = 2k + 2 > 2k + 1 = 2 \left( k + \frac{1}{2} \right) = \left( k + \frac{1}{2} \right) |X|.
\]

By Theorem 1, \( G' \) has no \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor, and so, \( G \) is not a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor deleted graph.

2. The Proof of Theorem 2

We first verify the following lemma, which is very useful in the proof of Theorem 2.

**Lemma 5.** Let \( G \) be a graph and \( \lambda \geq 1 \) be a real number. Then the following three statements are equivalent.

(i) \( i(G - S) \leq \lambda |S| \) for all \( S \subseteq V(G) \).
(ii) $\lambda |N_G(X)| \geq |X|$ for all independent set $X$ of $G$.

(iii) $\lambda |N_G(Y)| \geq |Y|$ for all $Y \subset V(G)$.

**Proof.** The equivalence of (i) and (ii) of Lemma 5 is known and easy (See Lemma 7.1 [1]). In what follows, we prove only that (ii) implies (iii).

We may assume that $G$ is connected. Let $\emptyset \neq Y \subset V(G)$, and let $X = Y \cap N_G(Y)$. Then $Y - X$ is an independent set of $G$, $N_G(X) \supseteq X$ and $N_G(Y - X) \cap Y = \emptyset$. Then by (ii) and $\lambda \geq 1$, we have

$$\lambda |N_G(Y)| \geq \lambda (|N_G(Y - X)| + |X|) \geq |Y - X| + |X| = |Y|.$$ 

Hence, (iii) holds. Lemma 5 is verified.

**Proof of Theorem 2.** Set $\lambda = \frac{2k+1}{2}$ in Lemma 5. According to Theorem 1, Lemma 5 and the definition of $\text{bind}(G)$, we see that

$$\text{bind}(G) \geq \frac{2}{2k+1}$$

$$\iff \frac{2k+1}{2} |N_G(Y)| \geq |Y| \text{ for all } Y \subset V(G)$$

$$\iff i(G - S) \leq \frac{2k+1}{2} |S| \text{ for all } S \subset V(G)$$

$$\iff G \text{ admits a } \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k+1)\}\text{-factor.}$$

We finish the proof of Theorem 2.

3. THE PROOF OF THEOREM 3

**Proof of Theorem 3.** Let $G' = G - e$ for any fixed $e = xy \in E(G)$ and $\mathcal{H} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k+1)\}$. To prove Theorem 3, it suffices to verify that $G'$ admits an $\mathcal{H}$-factor. On the contrary, we assume that $G'$ has no $\mathcal{H}$-factor. Then it follows from Theorem 1 that

$$i(G' - X) > \left(k + \frac{1}{2}\right) |X|$$

for some vertex subset $X$ of $G$.

We first demonstrate the following claim.

**Claim 1.** $|X| \geq 2$.

**Proof.** Since $\delta(G) \geq 2$, $\delta(G') \geq 1$ and so $G'$ has no isolated vertex. Assume that $|X| = 1$. Then it follows from (1) and $k \geq 2$ that $i(G' - X) > (k + \frac{1}{2})|X| \geq \frac{5}{2}$, which implies

$$i(G' - X) \geq 3.$$
It is obvious that \( i(G' - X) = i(G - e - X) \leq i(G - X) + 2 \). Combining this with (2), we derive \( i(G - X) \geq i(G' - X) - 2 \geq 3 - 2 = 1 \), which implies that there exists at least one vertex \( u \) in \( G - X \) with \( d_{G - X}(u) = 0 \). Thus, we have \( d_G(u) \leq d_{G - X}(u) + |X| = 0 + 1 = 1 \), which contradicts \( \delta(G) \geq 2 \). Therefore, we obtain \( |X| \geq 2 \). We finish the proof of Claim 1. \( \square \)

It follows from \( k \geq 2 \), \( \text{bind}(G) \geq \frac{2k - 1}{2} \), Lemma 5 and Claim 1 that

\[
i(G' - X) = i(G - e - X) \leq i(G - X) + 2 \leq \frac{2k - 1}{2}|X| + 2 \leq \frac{2k + 1}{2}|X|,
\]

which contradicts (1). Hence, \( G - e \) has an \( H \)-factor by Theorem 1, which implies that \( G \) is an \( H \)-factor deleted graph. This completes the proof of Theorem 3. \( \blacksquare \)

Finally, we present an open problem.

**Problem.** Find a criterion for a graph to be a \( \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, T(2k + 1)\} \)-factor deleted graph.

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**References**


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