ZERO AND TOTAL FORCING DENSE GRAPHS

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Abstract

If $S$ is a set of colored vertices in a simple graph $G$, then one may allow a colored vertex with exactly one non-colored neighbor to force its non-colored neighbor to become colored. If by iteratively applying this color change rule, all of the vertices in $G$ become colored, then $S$ is a zero forcing set of $G$. The minimum cardinality of a zero forcing set in $G$, written $Z(G)$, is the zero forcing number of $G$. If in addition, $S$ induces a subgraph of $G$ without isolated vertices, then $S$ is a total forcing set of $G$. The total forcing number of $G$, written $F_t(G)$, is the minimum cardinality of a total forcing set in $G$. In this paper we introduce, and study, the notion of graphs for which all vertices are contained in some minimum zero forcing set, or some minimum total forcing set; we call such graphs ZF-dense and TF-dense, respectively. A graph is ZTF-dense if it is both ZF-dense and TF-dense. We determine various classes of ZTF-dense graphs, including among others, cycles, complete multipartite graphs of order at least three that are not stars, wheels, $n$-dimensional hypercubes, and diamond-necklaces. We show that no tree of order at least three is ZTF-dense. We show that if $G$ and $H$ are connected graphs of order at least two that are both ZF-dense, then the join $G + H$ of $G$ and $H$ is ZF-dense.

Keywords: zero forcing sets, zero forcing number, ZF-dense.

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1. Introduction

Coloring the vertices of a graph $G$ and allowing this initial coloring to propagate throughout the vertex set of $G$ is known as a dynamic coloring of $G$. In this paper, we will focus on the dynamic coloring due to the forcing process, which we recall the definition from [10] as follows. Let $G$ be a finite and simple graph with vertex set $V(G)$, and let $S \subseteq V(G)$ be a set of initially “colored” vertices, all remaining vertices being “uncolored”. All vertices contained in $S$ are said to be $S$-colored, while all vertices not in $S$ are $S$-uncolored. At each discrete time step, if a colored vertex has exactly one uncolored neighbor, then this colored vertex forces its uncolored neighbor to become colored. If $v$ is such a colored vertex, then we call $v$ a forcing vertex, and say that $v$ has been played. The initial set of vertices $S$ is a zero forcing set, if by iteratively applying this forcing process all of $V(G)$ becomes colored. We call such a set, an $S$-forcing set. If $S$ is a zero forcing set of $G$ and $v$ is an $S$-colored vertex which has been played, then $v$ is called an $S$-forcing vertex. The zero forcing number of $G$, written $Z(G)$, is the cardinality of a minimum forcing set in $G$. If $S$ is an $S$-forcing set of $G$ which also induces a subgraph without isolated vertices, then $S$ is a total forcing set, abbreviated TF-set, of $G$. The total forcing number of $G$, written $F_t(G)$, is the cardinality of a minimum TF-set in $G$.

Zero forcing in graphs was first introduced and studied in an AIM Special Work Group [3] in 2008, and has subsequently been extensively studied in the literature; MathSciNet lists over 140 papers to date on the topic. For a small sample of recent (2020) papers on zero forcing we refer the reader to [4, 12, 15, 16, 20]. The notion of total forcing in graphs was first introduced by Davila [7] in 2015 as a strengthening of zero forcing in graphs, and has been studied, for example, in [8–11].

Motivation. There is a close connection between power domination in a graph and zero forcing in graphs, as discussed, for example, by Benson et al. [5] in 2018. The power domination process in a graph $G$ can be described as choosing a set $S$ of vertices in $G$ and applying the zero forcing process to the closed neighborhood $N[S]$ of $S$. Indeed, as first observed by Aazemi [1,2], a set $S$ is a power dominating set of a graph $G$ if and only if $N[S]$ is a zero forcing set of $G$. Benson et al. [5] proved that the zero forcing number of a graph is at most its maximum degree times its power domination number.

There is also a close connection with domination and zero forcing, and with total domination and total forcing in graphs (see, for example, [11, 13]). A frequently studied problem with domination type parameters is to determine the set of vertices that belong to every or to some minimum dominating set, as this is important in obtaining algorithmic and complexity results, as well as bounds
on the parameters. We refer the reader to [6, 18, 21, 23] for a small sample of such papers.

In this paper, we study an analogous concept for zero forcing and total forcing in graph. We call a graph with the property that every vertex is contained in some minimum zero forcing set (respectively, minimum TF-set) a ZF-dense graph (respectively, a TF-dense graph). If a graph is both ZF-dense and TF-dense, then we say that the graph is ZTF-dense.

**Definitions and notation.** For notation and graph terminology, we will typically follow the monograph [19]. Specifically, this paper will only consider finite and simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of $G$ will be denoted by $n(G) = |V(G)|$ and $m(G) = |E(G)|$, respectively. A nontrivial graph is a graph of order at least 2. Two vertices $v, w \in V(G)$ are said to be neighbors, or adjacent, if $vw \in E(G)$. The open neighborhood of a vertex $v \in V(G)$, written $N_G(v)$, is the set of all neighbors of $v$, whereas the closed neighborhood of $v$ is $N_G(v) = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$, written $d_G(v)$, is the number of neighbors of $v$ in $G$; and so, $d_G(v) = |N_G(v)|$. The complete graph, path, and cycle, on $n$ vertices will be denoted by $K_n$, $P_n$, and $C_n$, respectively.

A graph $G$ is connected if for all vertices $v$ and $w$ in $G$, there exists a $(v, w)$-path. The length of a shortest $(v, w)$-path in $G$ is the distance between $v$ and $w$, and is written $d_G(v, w)$ or simply $d(v, w)$ if $G$ is clear from context.

A tree is a connected graph which contains no cycle as a subgraph. A vertex of degree 1 in a tree is called a leaf and a vertex with a leaf neighbor is a support vertex. A strong support vertex is a vertex with at least two leaf neighbors. A branch vertex of a tree is a vertex of degree at least 3 in the tree. A star is a non-trivial tree with at most one vertex which is not a leaf, and if the star in question has $n$ leaves, we denote the star by $K_{1, n}$.

For $k \geq 2$ a graph $G$ is $k$-partite if its vertex set $V(G)$ can be partitioned into $k$ subsets $V_1, V_2, \ldots, V_k$ (called partite sets) in such a way that no two vertices of $V_i$ are adjacent for all $i \in [k]$. Further, if every vertex of $V_i$ is adjacent to every vertex not in $V_i$ for all $i \in [k]$, then $G$ is a complete $k$-partite graph. A 2-partite graph is called a bipartite graph. A graph is a multipartite graph if it is a $k$-partite graph for some $k \geq 2$.

The Cartesian product $G \square H$ of two graphs $G$ and $H$ is the graph whose vertex set is $V(G) \times V(H)$. Two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent in $G \square H$ if either $g_1 = g_2$ and $h_1h_2$ is an edge in $H$, or $h_1 = h_2$ and $g_1g_2$ is an edge in $G$.

The join graph of two graphs $G$ and $H$, written $G + H$ (also written $G \vee H$ or $G \sqcup H$ in the literature), is the graph obtained from the disjoint union of $G$ and $H$ by joining each vertex of $G$ to every vertex of $H$.

We use the standard notation $[k] = \{1, \ldots, k\}$.
2. Known Results and Preliminary Observations

In this section we provide preliminary observations and results on both ZF-dense and TF-dense graphs. First, we recall some known elementary result. As defined earlier, a nontrivial graph is a graph of order at least 2.

**Observation 1.** If $G$ is a nontrivial connected graph, then $Z(G) \leq F_t(G)$.

Any initially forcing vertex in the forcing process must be colored along with all but one of its neighbors. Thus the forcing number of a graph is always bounded from below by the minimum degree, as first observed in [3].

**Observation 2** [3]. If $G$ is a graph with minimum degree $\delta$, then $Z(G) \geq \delta(G)$.

The zero forcing number and total forcing number of paths, cycles, complete graphs and stars is easy to compute.

**Observation 3** [10]. The following holds.

(a) For $n \geq 2$, $Z(P_n) = 1$ and $F_t(P_n) = 2$.
(b) For $n \geq 3$, $Z(C_n) = F_t(C_n) = 2$.
(c) For $n \geq 3$, $Z(K_n) = F_t(K_n) = n - 1$.
(d) For $n \geq 3$, $Z(K_{1,n-1}) = n - 2$ and $F_t(K_{1,n-1}) = n - 1$.

By Observation 3, for $n \geq 2$, $Z(P_n) = 1$ and $F_t(P_n) = 2$. Every minimum zero forcing set of a path consists of a leaf of the path. In particular for $n \geq 3$, no internal vertex of a path $P_n$ belongs to a minimum zero forcing set of the path. Thus, $P_n$ is ZF-dense if and only if $n \in \{1, 2\}$. For $n \geq 2$, coloring any two consecutive vertices on the path $P_n$ produces a minimum TF-set in the path. Thus, $P_n$ is TF-dense if and only if $n \geq 2$. By Observation 3 for $n \geq 3$, $Z(C_n) = F_t(C_n) = 2$. Coloring any two consecutive vertices of the cycle $C_n$ results in both a minimum zero forcing set and a minimum TF-set. Thus, $C_n$ is both ZF-dense and TF-dense.

By Observation 3 for $n \geq 3$, $Z(K_n) = F_t(K_n) = n - 1$. Coloring any arbitrary $(n - 1)$-element subset of vertices in $K_n$ results in both a minimum forcing set and a minimum TF-set. Thus, $K_n$ is both ZF-dense and TF-dense.

By Observation 3 for $n \geq 3$, $Z(K_{1,n-1}) = n - 2$. Every minimum forcing set of the star $K_{1,n-1}$ consists of $n - 2$ leaves of the star. Since the central vertex of such a star does not belong to a minimum forcing set of the star, $K_{1,n-1}$ is not ZF-dense. By Observation 3 for $n \geq 3$, $F_t(K_{1,n-1}) = n - 1$. Coloring the central vertex of $K_{1,n-1}$ and any combination of $n - 2$ leaves of the star results in a minimum TF-set. Thus, $K_{1,n-1}$ is TF-dense. We state the above observations formally as follows.
Observation 4. The following holds.
(a) All paths of order at least 3 are TF-dense, but not ZF-dense.
(b) All cycles are ZTF-dense.
(c) All complete graphs on at least three vertices are ZTF-dense.
(d) All stars of order at least 3 are TF-dense, but not ZF-dense.

3. Families of Zero and Total Forcing Dense Graphs

In this section, we present several families of ZTF-dense graphs, such as certain multipartite graphs, wheel graphs, and the n-dimensional hypercube. If we exclude stars, then all multipartite graphs of order at least 3 are ZTF-dense, as the following result shows.

Proposition 5. If $G$ is a complete multipartite graph of order $n \geq 3$ that is not a star, then $G$ is ZTF-dense. Moreover, if $G \neq K_n$, then $Z(G) = F_t(G) = n - 2$.

Proof. Let $G$ be a complete multipartite graph of order $n \geq 3$ that is not a star. Thus, $G$ is a complete $k$-partite graph for some $k \geq 2$. Let $V_1, V_2, \ldots, V_k$ be the partite sets of $G$, where $1 \leq |V_1| \leq |V_2| \leq \cdots \leq |V_k|$. If $|V_k| = 1$, then $n = k$ and $G = K_k$, and by Observation 4, $G$ is ZTF-dense. Further, by Observation 3, $Z(K_n) = F_t(K_n) = n - 1$. Hence, we may assume that $|V_k| \geq 2$. Since $G$ is not a star, either $k = 2$ and $|V_1| \geq 2$ or $k \geq 3$.

We show firstly that $Z(G) \geq n - 2$. Suppose, to the contrary, that $Z(G) \leq n - 3$. Let $S$ be a minimum forcing set of $G$, and so $|S| = Z(G) \leq n - 3$. If some partite set of $G$ contains two or more $S$-uncolored vertices, then it would not be possible to color these vertices. Hence, every partite set of $G$ contains at most one $S$-uncolored vertex. This implies that there are three distinct partite sets of $G$, each of which contains an $S$-uncolored vertex. But then every vertex of $G$ has at least two $S$-uncolored neighbors, implying that $S$ is not a forcing set of $G$, a contradiction. Therefore, $Z(G) \geq n - 2$.

We show next that $Z(G) = F_t(G) = n - 2$. If $u$ is an arbitrary vertex in $V_k$ and $v$ is an arbitrary vertex in $V(G) \setminus V_k$, then the set $S = V(G) \setminus \{u, v\}$ is a TF-set of $G$, implying that $n - 2 \leq Z(G) \leq F_t(G) \leq |S| = n - 2$. Consequently, $Z(G) = F_t(G) = n - 2$. Moreover if $w$ is an arbitrary vertex of $G$, then we can choose the vertices $u$ and $v$ distinct from $w$, and therefore we can choose the set $S$ to contain the vertex $w$. Thus, every vertex of $G$ belongs to some minimum forcing set of $G$ and to some minimum TF-set of $G$. Hence, $G$ is ZTF-dense. ■

The wheel graph $W_n$ of order $n \geq 4$ is the graph obtained from a cycle $C_{n-1}$ by adding a new vertex and joining it to every vertex on the cycle. We call the cycle $C_{n-1}$ the outer cycle of the wheel, and we call the new added vertex
adjacent to every vertex on the outer cycle the central vertex of the wheel. We show next that all wheel graphs are ZTF-dense.

**Proposition 6.** If \( G \) is a wheel graph of order \( n \geq 4 \), then \( G \) is ZTF-dense. Moreover, \( Z(G) = F_t(G) = 3 \).

**Proof.** Let \( G \) be the wheel graph \( W_n \) of order \( n \geq 4 \) obtained from a cycle \( v_1 v_2 \cdots v_{n-1} v_1 \) by adding a new vertex \( v \) and joining it to every vertex on the cycle. Thus, \( v \) has degree \( n-1 \) in \( G \), while every vertex of the wheel different from \( v \) has degree 3 in \( G \). By Observations 1 and 2, we have \( F_t(G) \geq Z(G) \geq \delta(G) = 3 \).

We show next that \( F_t(G) \leq 3 \). Let \( S \) consist of the central vertex \( v \) of the wheel together with any two consecutive vertices on the outer cycle. Renaming vertices if necessary, we may assume that \( S = \{v, v_{n-1}, v_1\} \). We note that \( G[S] \cong K_3 \), and so \( S \) induces a graph without isolated vertices. Further, the set \( S \) is a forcing set of \( G \) since if \( x_i = v_i \) for \( i \in [n-3] \), then the sequence \( x_1, \ldots, x_{n-3} \) of played vertices in the forcing process results in all vertices of \( G \) colored, where \( x_i \) denotes the forcing vertex played in the \( i \)th step of the forcing process. More precisely, when the vertex \( x_i \) is played in the forcing process, it forces the vertex \( v_{i+1} \) to be colored for \( i \in [n-3] \). Therefore, \( S \) is a TF-set of \( G \), and so \( F_t(G) \leq |S| = 3 \). Consequently, \( F_t(G) = Z(G) = 3 \). Further, since we can choose the set \( S \) to contain the central vertex \( v \) of the wheel together with any two consecutive vertices on the outer cycle, every vertex of \( G \) belongs to a minimum forcing set and a minimum TF-set of \( G \). Thus, the wheel graph \( G \) of order \( n \geq 4 \) is ZTF-dense. \( \blacksquare \)

We next consider the \( n \)-dimensional hypercube \( Q_n \). From our perspective it is important that \( Q_n \) can be represented as the \( n^{th} \) power of \( K_2 \) with respect to the Cartesian product operation \( \Box \), that is, \( Q_1 = K_2 \) and \( Q_n = Q_{n-1} \Box K_2 \) for \( n \geq 2 \). The 4-dimensional hypercube \( Q_4 \) is illustrated in Figure 1, where the darkened vertices form a TF-set of \( Q_4 \). Peters [22] determined the forcing number of the hypercube \( Q_n \), and showed that \( Z(Q_n) = 2^{n-1} \). We show next that all hypercubes \( Q_n \) are ZTF-dense.

![Figure 1](image.png)

*Figure 1. Total forcing (and zero forcing) in the hypercube \( Q_4 \).*

**Proposition 7.** For \( n \geq 3 \), if \( G \) is the \( n \)-dimensional hypercube \( Q_n \), then \( G \) is ZTF-dense. Moreover, \( Z(G) = F_t(G) = 2^{n-1} \).
Proof. For \( n \geq 3 \), let \( G \) be the \( n \)-dimensional hypercube \( Q_n \). Thus, \( G = Q_{n-1} \Box K_2 \). By Peters’s result (see [22]), \( Z(Q_n) = 2^{n-1} \). Let \( S \) be the set of initially colored vertices obtained by coloring one of the copies of \( Q_{n-1} \) in the product \( Q_{n-1} \Box K_2 \). We note that \( G[S] \cong Q_{n-1} \), and so since \( n \geq 3 \), the set \( S \) induces a subgraph without isolated vertices. Each vertex in \( S \) is adjacent with exactly one vertex outside of \( S \), and thus, each \( S \)-colored vertex is \( S \)-forcing. Allowing these vertices to force results in all of \( V(Q_n) \) becoming colored. Thus, \( S \) is a TF-set with \( |S| = 2^{n-1} \). Hence, \( Z(G) \leq F_t(G) \leq 2^{n-1} = Z(Q_n) \). Consequently, we must have equality throughout this inequality chain. In particular, \( Z(G) = F_t(G) = 2^{n-1} \). Further, since we can choose the set \( S \) to contain the vertices from either copy of \( Q_{n-1} \) in \( G \), every vertex of \( G \) belongs to a minimum forcing set and a minimum TF-set of \( G \). Thus, the hypercube \( Q_n \) is ZTF-dense for all \( n \geq 3 \).

We remark that the ZTF-dense graphs described in the statements of Propositions 5, 6, and 7 all have equal forcing number and total forcing numbers. However in general, if \( G \) is a ZTF-dense graph, then it is possible that \( Z(G) < F_t(G) \). We illustrate this with the family of cubic graphs known as diamond-necklaces. Following the notation in [17], for \( k \geq 2 \) an integer, let \( N_k \) be the connected cubic graph constructed as follows. Take \( k \) disjoint copies \( D_1, D_2, \ldots, D_k \) of a diamond, where \( V(D_i) = \{a_i, b_i, c_i, d_i\} \) and where \( a_ib_i \) is the missing edge in \( D_i \). Let \( N_k \) be obtained from the disjoint union of these \( k \) diamonds by adding the edges \( \{b_ia_{i+1} \mid i \in [k-1]\} \) and adding the edge \( b_ka_1 \). Let \( A = \{a_1, a_2, \ldots, a_k\} \), \( B = \{b_1, b_2, \ldots, b_k\} \), \( C = \{c_1, c_2, \ldots, c_k\} \) and \( D = \{d_1, d_2, \ldots, d_k\} \). We call \( N_k \) a diamond-necklace with \( k \) diamonds. Let \( N_{\text{cubic}} = \{N_k \mid k \geq 2\} \). A diamond-necklace, \( N_6 \), with six diamonds is illustrated in Figure 2, where the darkened vertices form a TF-set in the graph.

![Figure 2. A diamond-necklace N₆.](image-url)

The authors in [9] determined the forcing and total forcing numbers of a diamond-necklace

Proposition 8. If \( G \in N_{\text{cubic}} \) has order \( n \), then \( F_t(G) = \frac{1}{2}n \) and \( Z(G) = \frac{1}{4}n + 2 \).
We show next that all diamond-necklaces are ZTF-dense. In the proof of the following result, we adopt our earlier notation for a diamond-necklace. This result shows that there exist ZTF-dense graphs $G$ with the difference $F_t(G) - Z(G)$ arbitrarily large.

**Proposition 9.** If $G \in \mathcal{N}_{\text{cubic}}$, then $G$ is ZTF-dense.

**Proof.** Let $G \in \mathcal{N}_{\text{cubic}}$ have order $n$. Thus, $G = N_k$ is a diamond-necklace with $k$ diamonds for some $k \geq 2$, where $n = 4k$. By Proposition 8, $F_t(G) = 2k$. As observed in [9], the set $S_1 = (A \setminus \{a_k\}) \cup C \cup \{b_k\}$ is a TF-set of $G$ since the sequence $x_1, x_2, \ldots, x_{2k}$ of played vertices in the forcing process result in all vertices of $G$ colored, where $x_i$ denotes the forcing vertex played in the $i$th step of the process and where $x_{2i-1} = a_i$ and $x_{2i} = d_i$ for $i \in [k-1]$ and where $x_{2k-1} = b_{k-1}$ and $x_{2k} = a_k$; that is, the sequence of played vertices is given by $a_1, d_1, a_2, d_2, \ldots, a_{k-1}, d_{k-1}, b_{k-1}, a_k$. Since $|S_1| = 2k = F_t(G)$, the set $S_1$ is a minimum TF-set of $G$. Analogously, for all $i \in [k]$ the set $S_i = (A \setminus \{a_i\}) \cup C \cup \{b_i\}$ is a minimum TF-set of $G$, as is the set $S'_i = (A \setminus \{a_i\}) \cup D \cup \{b_i\}$. Thus, every vertex of $G$ belongs to some minimum TF-set of $G$. Hence, $G$ is TF-dense.

Let $G \in \mathcal{N}_{\text{cubic}}$ have order $n$. Thus, $G = N_k$ is a diamond-necklace with $k$ diamonds for some $k \geq 2$, where $n = 4k$. By Proposition 8, $F_t(G) = 2k$. As observed in [9], the set $S_1 = (A \setminus \{a_1\}) \cup C \cup \{b_1\}$ is a TF-set of $G$ since the sequence $x_1, x_2, \ldots, x_{2k}$ of played vertices in the forcing process result in all vertices of $G$ colored, where $x_i$ denotes the forcing vertex played in the $i$th step of the process and where $x_1 = b_1, x_2 = d_1$, and $x_{2i+1} = a_{i+1}$ and $x_{2i+2} = d_{i+1}$ for $i \in [k-1]$; that is, the sequence of played vertices is given by $b_1, d_1, a_2, d_2, \ldots, a_{k-1}, b_{k-1}$. Since $|S_1| = 2k = F_t(G)$, the set $S_1$ is a minimum TF-set of $G$. Analogously, for all $i \in [k]$ the set $S_i = (A \setminus \{a_i\}) \cup C \cup \{b_i\}$ is a minimum TF-set of $G$, as is the set $S'_i = (A \setminus \{a_i\}) \cup D \cup \{b_i\}$. Thus, every vertex of $G$ belongs to some minimum TF-set of $G$. Hence, $G$ is TF-dense.

We show next that $G$ is ZF-dense. By Proposition 8, $Z(G) = k + 2$. As observed in [9], the set $D_1 = C \cup \{b_1, a_2\}$ is a forcing set of $G$ since the sequence $x_1, x_2, \ldots, x_{3k-2}$ of played vertices in the forcing process result in all vertices of $G$ colored, where $x_i$ denotes the forcing vertex played in the $i$th step of the process and where $x_{3i-2} = a_{i+1}, x_{3i-1} = d_{i+1}$, and $x_{3i} = b_{i+1}$ for $i \in [k-1]$, and where $x_{3k-2} = a_1$; that is, the sequence of played vertices is given by $a_2, d_2, b_2, a_3, d_3, \ldots, a_k, b_k, a_1$. Since $|D_1| = k + 2 = Z(G)$, the set $D_1$ is a minimum forcing set of $G$. Analogously, for all $i \in [k]$ the set $D_i = C \cup \{b_i, a_{i+1}\}$ is a minimum forcing set of $G$, as is the set $D'_i = D \cup \{b_i, a_{i+1}\}$, where $a_{k+1} = a_1$. Thus, every vertex of $G$ belongs to some minimum forcing set of $G$. Hence, $G$ is ZF-dense. As observed earlier, $G$ is TF-dense. Therefore, $G$ is ZTF-dense. ■
4. ZTF-Dense Trees

Recall that by Observation 4, no path of order at least 3 and no star of order at least 3 is ZF-dense. In this section, we show that the only ZF-dense trees are $K_1$ and $K_2$. For this purpose, we first prove the following lemma.

**Lemma 10.** If $T$ is a ZF-dense tree, then the following holds.

(a) $T$ has no strong support vertex.

(b) There is no path $P: v_1v_2v_3v_4$ in $T$ where $d_T(v_1) = 2$, $d_P(v_1) = 1$ and $d_T(v_2) = 2$. We note that $d_T(v_3) \geq 2 = d_P(v_3)$.

(c) There is no path $P: v_1v_2v_3v_4v_5$ in $T$ where $d_T(v_1) = 1$ and $d_T(v_2) = d_T(v_3) = 2$, then $d_T(v_5) \geq 3$.

**Proof.** Let $T$ be a ZF-dense tree. For an arbitrary vertex $v$ of $T$, let $S_v$ be a minimum zero forcing set of $T$ that contains the vertex $v$. Thus, $|S_v| = Z(T)$ and $v \in S_v$. Further, let $S'_v = S_v \setminus \{v\}$.

(a) Suppose, to the contrary, that $T$ has a strong support vertex $v$. Let $v_1$ and $v_2$ be two distinct leaf neighbors of $v$. Since $S_v$ is a zero forcing set, at least one of $v_1$ and $v_2$ belong to the set $S_v$. Renaming $v_1$ and $v_2$, if necessary, we may assume that $v_1 \in S_v$. But then the set $S'_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S'_v$, we play the vertex $v_1$, which results in the set $S_v$ of colored vertices after the first step. Thereafter, we continue with the exact same sequence of vertices in the forcing process starting with the zero forcing set $S_v$ to color the remaining uncolored vertices of $T$. Thus contradicts the minimality of the zero forcing set $S_v$.

(b) Suppose, to the contrary, that $P: v_1v_2v_3v_4$ is a path in $T$ where $d_T(v_1) = 2$, $d_T(v_3) = 1$ and $d_T(v_2) = 2$. We note that $d_T(v_3) \geq 2 = d_P(v_3)$. Let $v = v_3$ and consider the minimum zero forcing set $S_v$ of $T$ that contains the vertex $v$. Since $S_v$ is a zero forcing set, at least one of $v_1$, $v_2$ and $v_4$ belong to the set $S_v$. If $v_4 \in S_v$, then analogously as in the proof of part (a), the set $S''_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$, we play the vertex $v_1$. If $v_2 \in S_v$, then the set $S''_v = (S'_v \setminus \{v_2\}) \cup \{v_1\}$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$ we play the vertex $v_1$. If $v_1 \in S_v$, then the set $S''_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$ we play the vertex $v_1$. In all three cases, we produce a zero forcing set of $T$ of cardinality strictly less than $|S_v|$, contradicting the minimality of the zero forcing set $S_v$.

(c) Suppose, to the contrary, that $P: v_1v_2v_3v_4v_5$ is a path in $T$ where $d_T(v_1) = 2$ and $d_T(v_2) = 2$. We note that $d_T(v_3) \geq 2 = d_P(v_3)$. Let $v = v_3$ and consider the minimum zero forcing set $S_v$ of $T$. Since $S_v$ is a zero forcing set, at least one of $v_1$, $v_2$, $v_4$ and $v_5$ belong to the set $S_v$. If $v_1 \in S_v$, then the set $S''_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$ we play the vertex $v_1$. If $v_2 \in S_v$, then the set $S''_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$ we play the vertex $v_1$. If $v_4 \in S_v$, then the set $S''_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$ we play the vertex $v_1$. If $v_5 \in S_v$, then the set $S''_v$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S''_v$ we play the vertex $v_1$. In all three cases, we produce a zero forcing set of $T$ of cardinality strictly less than $|S_v|$, contradicting the minimality of the zero forcing set $S_v$.


process starting with the set $S'_v$, we play the vertex $v_1$. Analogously, if $v_5 \in S_v$, then the set $S'_v$ is a zero forcing set. If $v_2 \in S_v$, then the set $S''_v = (S'_v \setminus \{v_2\}) \cup \{v_1\}$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S'_v$, we play the vertex $v_1$. Analogously, if $v_4 \in S_v$, then the set $(S'_v \setminus \{v_4\}) \cup \{v_3\}$ is a zero forcing set. In all four cases, we produce a zero forcing set of $T$ of cardinality strictly less than $|S_v|$, contradicting the minimality of the zero forcing set $S_v$.

(d) Let $v_1v_2v_3$ be a path in $T$ where $d_T(v_1) = 1$ and $d_T(v_2) = 2$, and suppose, to the contrary, that $d_T(v_3) \leq 2$. By Observation 4, the tree $T$ is not a path, since no path of order at least 3 is ZF-dense. This implies that $d_T(v_3) = 2$ and there exists a vertex of degree at least 3 in $T$. Let $w$ be the vertex of degree at least 3 in $T$ that is at minimum distance from $v_1$ in $T$. Further, let $d_T(v_i, w) = k_i$ and so $k \geq 3$ and let $Q: v_1v_2v_3\cdots v_{k+1}$ be the path from $v_1$ to $w$ in $T$, where $w = v_{k+1}$. We now consider the vertex $v = v_2$. If $v_i \in S_v$ for some $i \in [k+1] \setminus \{2\}$, then the set $S''_v = (S'_v \setminus \{v_i\}) \cup \{v_1\}$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set $S'_v$, we play the vertex $v_1$. Thus, $v = v_2$ is the only vertex of $Q$ that belongs to the set $S_v$. This implies that the set $S_v \subseteq V(T) \setminus V(Q)$ is necessarily a zero forcing set of $T$, contradicting the minimality of the set $S_v$. This completes the proof of Lemma 10.

We are now in a position to prove that the only ZF-dense trees are $K_1$ and $K_2$.

**Theorem 11.** The only ZF-dense trees are $K_1$ and $K_2$.

**Proof.** Let $T$ be a ZF-dense tree. We show firstly that $T$ is a path. Suppose, to the contrary, that $T$ is not a path. Let $P: v_1v_2v_3\cdots v_d$ be a longest path in $T$, where $d \geq 3$. Necessarily, $v_1$ is a leaf in $T$. If $d = 3$, then $T$ is a star, contradicting Observation 4. Hence, $d \geq 4$. If $d_T(v_2) \geq 3$, then by the maximality of the path $P$ every neighbor of $v_2$ different from $v_3$ is a leaf, implying that $v_2$ is a strong support vertex, contradicting Lemma 10(a). Thus, $d_T(v_2) = 2$. By Lemma 10(d), $d_T(v_3) \geq 3$. Let $u_2$ be a neighbor of $v_3$ not on $P$. By Lemma 10(b), $u_2$ is not a leaf, and so $d_T(u_2) \geq 2$. By the maximality of the path $P$, every neighbor of $u_2$ different from $v_3$ is a leaf. Thus if $d_T(u_2) \geq 3$, then $u_2$ is a strong support vertex of $T$, contradicting Lemma 10(a). Hence, $d_T(u_2) = 2$. Let $u_1$ be the neighbor of $u_2$ different from $v_3$. Thus, $u_1$ is a leaf of $T$. Hence, $u_1u_2v_3v_2v_1$ is a path in $T$, where $u_1$ and $v_1$ are leaves of $T$ and $u_2$ and $v_2$ are support vertices of $T$ of degree 2, contradicting Lemma 10(c). Therefore, $T$ is a path. As observed earlier, a path $P_n$ is ZF-dense if and only if $n \in \{1, 2\}$. Hence, $T \in \{K_1, K_2\}$.

Recall that a nontrivial graph is a graph of order at least 2. As an immediate consequence of Observation 4(a) and Theorem 11, the only non-trivial tree that is ZTF-dense is the tree $K_2$. 


5. The Join of Two Graphs

In this section we study ZF-dense and ZTF-dense graphs under the join operation. We remark that Theorem 12(a) appears in the Ph.D. Dissertation of Taklimi [24]. Acknowledging this result, we also present a proof of this result as it helps clarify the proof of Theorem 12(b).

**Theorem 12.** If $G$ and $H$ are non-trivial connected graphs, then the following holds.

(a) $Z(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}$.

(b) If both $G$ and $H$ are ZF-dense, then $G + H$ is ZF-dense.

**Proof.** Let $S$ be an arbitrary minimum zero forcing set of $G + H$. Starting with the colored set $S$, let $x_1, \ldots, x_t$ be the sequence of played vertices in the forcing process that results in all vertices of $G + H$ colored, where $x_i$ denotes the forcing vertex played in the $i$th step of the process. Let $X$ be the resulting set of played vertices, and so $X = \{x_1, \ldots, x_t\}$. We proceed further by establishing some properties of the set $X$.

**Claim 13.** We can choose the set $X$ so that $X \subseteq V(G)$ or $X \subseteq V(H)$.

**Proof.** Suppose that $X$ contains vertices from both $G$ and $H$. Renaming $G$ and $H$ if necessary, we may assume that $x_1 \in V(G)$. Let $x_i$ be the vertex in $X$ with smallest subscript that belongs to $H$; that is, $x_i \in V(H)$ and $i \in [t] \setminus \{1\}$. By definition, the first vertex $x_1$ played in the sequence of played vertices has exactly one uncolored neighbor immediately before it is played. This implies that all vertices of $H$, except possibly for one vertex, are $S$-colored (and therefore belong to the set $S$). After the vertex $x_1$ is played, all vertices of $H$ are colored. When the vertex $x_i \in V(H)$ is played in the $i$th step of the forcing process, all vertices of $G$, except for exactly one vertex, say $v$, are colored. This implies that after the vertex $x_i$ is played, all vertices of $G + H$ are colored. Thus, $x_i$ is the final vertex played, and so $x_i = x_t$. By our choice of the index $i$, we note therefore that $x_t$ is the only vertex in $X$ that belongs to $H$. However, replacing $x_t$ in $X$ with an arbitrary neighbor, $v'$ say, of $v$ in $G$ produces a new sequence $x_1, \ldots, x_{t-1}, v'$ of played vertices in the forcing process that results in all vertices of $G + H$ colored. The resulting set $(X \setminus \{x_t\}) \cup \{v'\}$ of played vertices in this sequence belong entirely to the graph $G$.

Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$.

**Claim 14.** $Z(G + H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$.

**Proof.** By Claim 13, we can choose the set $X$ so that $X \subseteq V(G)$ or $X \subseteq V(H)$. Renaming the graphs $G$ and $H$ if necessary, we may assume that $X \subseteq V(G)$. As
observed in the proof of Claim 13, immediately before the first vertex $x_1 \in X$ is played, all vertices of $H$, except possibly for one vertex, belong to the set $S$. Thus, $S_H = V(H)$ or $S_H = V(H) \setminus \{v\}$ for some vertex $v \in V(H)$.

Suppose that all $S_H = V(H)$. In this case, the set $S_G$ is a zero forcing set of $G$ and $x_1, \ldots, x_t$ is a sequence of played vertices in the forcing process in the graph $G$ that results in all vertices of $G$ colored. Thus, $Z(G) \leq |S_G|$, and so $Z(G+H) = |S| = |S_G| + |S_H| \geq Z(G) + n(H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$. Hence, we may assume that $S_H = V(H) \setminus \{v\}$ for some vertex $v \in V(H)$, for otherwise the desired result hold.

With this assumption, immediately before the first vertex $x_1$ is played in the forcing sequence in $G+H$, all neighbors of $x_1$ in $G+H$ are colored, except for the vertex $v \in V(H)$ which becomes colored when the vertex $x_1$ is played. In particular, we note that in this case all neighbors of $x_1$ in $G$ belong to the set $S$; that is, $N_G(x_1) \subseteq S$.

We now consider the set $S' = S_G \setminus \{x_1\}$. If $S'$ is a zero forcing set in $G$, then $Z(G) \leq |S'| = |S_G| - 1$. Thus, $Z(G+H) = |S| = |S_G| + |S_H| = (|S'| + 1) + (n(H) - 1) = |S'| + n(H) \geq Z(G) + n(H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$. Hence we may assume that $S'$ is not a zero forcing set in $G$. This implies that at least one vertex of $X$ is a neighbor of $x_1$ in $G$. Let $x_j$ be a vertex of $X$ of smallest subscript such that $x_j$ is a neighbor of $x_1$ in $G$; that is, $x_j \in N_G(x_1)$ and $j \in [t] \setminus \{1\}$.

We now consider the set $S^* = S_G \setminus \{x_j\}$. We show that $S^*$ is a zero forcing set in $G$ by showing that the sequence of vertices $x_1, \ldots, x_t$ in the forcing process colors all vertices of $G$. When we play the vertex $x_1$, its unique $S^*$-uncolored neighbor, namely the vertex $x_j$, becomes colored. After the vertex $x_1$ is played, the set of colored vertices in $G$ with respect to the coloring $S^*$ is exactly the same as the set of colored vertices in $G$ with respect to the coloring $S_G$. We note that after the vertex $x_j$ is played in $G+H$ all vertices in $H$ are colored, implying that each vertex $x_i$ where $i \in [t] \setminus \{1\}$ forces an unique vertex in $G$ to be colored. These observations imply that once the vertex $x_1$ is played in the set $S^*$, we may proceed exactly as before by playing the vertices $x_2, \ldots, x_t$ in turn in the forcing process to color all vertices of $G$. Thus, $S^*$ is a zero forcing set in $G$, and so $Z(G) \leq |S^*| = |S_G| - 1$. Therefore, $Z(G+H) = |S| = |S_G| + |S_H| = (|S^*| + 1) + (n(H) - 1) = |S^*| + n(H) \geq Z(G) + n(H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$. This completes the proof of Claim 14.

**Claim 15.** $Z(G+H) \leq \min\{n(G) + Z(H), n(H) + Z(G)\}$.

**Proof.** Coloring all vertices in $G$ and coloring the vertices in a minimum zero forcing set in $H$ we form a zero forcing set in $G+H$. Analogously, coloring all vertices in $H$ and coloring the vertices in a minimum zero forcing set in $G$ we form a zero forcing set in $G+H$. Thus, $Z(G+H) \leq \min\{n(G) + Z(H), n(H) + Z(G)\}$. □
As an immediate consequence of Claims 14 and 15, the zero forcing number of the join $G + H$ of $G$ and $H$ is determined, and is given by

$$Z(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}.$$  

This completes the proof of part (a). To prove part (b), suppose that both $G$ and $H$ are ZF-dense. Renaming $G$ and $H$ if necessary, we may assume that $n(H) + Z(G) = \min\{n(G) + Z(H), n(H) + Z(G)\}$. Coloring all vertices in $H$ and coloring the vertices in a minimum zero forcing set in $G$ we form a minimum zero forcing set in $G + H$. Since $G$ is ZF-dense, we can choose the minimum zero forcing set in $G$ to contain any specified vertex in $G$. Therefore, there exists a minimum zero forcing set in $G + H$ that contains any specified vertex in $G + H$, implying that $G + H$ is ZF-dense. This completes the proof of part (b), and completes the proof of Theorem 12.

We close with the following result.

**Theorem 16.** If $G$ and $H$ are non-trivial connected graphs, then the following holds.

(a) $F_t(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}$.

(b) If both $G$ and $H$ are ZF-dense, then $G + H$ is TF-dense.

**Proof.** In view of Theorem 12, we may assume, renaming the graphs $G$ and $H$ if necessary, that $Z(G + H) = n(G) + Z(H)$. Notice that $V(G) \cup S_H$ is a zero forcing set of $G + H$, where $S_H$ denotes a minimum zero forcing set of $H$. By definition of the join operation, the set $V(G) \cup S_H$ induces a subgraph of $G + H$ without isolated vertices, and is therefore a TF-set of $G + H$. Hence, $F_t(G + H) \leq Z(G + H)$. By Observation 1, $F_t(G + H) \leq Z(G + H)$. Consequently, $F_t(G + H) = Z(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}$. This proves part (a). Part (b) follows readily.

As an immediate consequence of Theorems 12 and 16, we have the following result.

**Corollary 17.** Let $G$ and $H$ be non-trivial connected graphs. If both $G$ and $H$ are ZF-dense, then $G + H$ is ZTF-dense.

**6. Concluding Remarks**

In this paper, several classes of ZTF-dense graphs are given, such as complete multipartite graphs, wheels, $n$-dimensional hypercubes, diamond-necklaces, as well as constructions to build ZF-dense graphs. It would be interesting to obtain other classes of ZTF-dense or ZF-dense graphs.
As mentioned in the introductory section, vertices that belong to every or to some minimum dominating set of a graph are well studied, and such sets are often useful in obtaining algorithmic and complexity results, as well as bounds on domination type parameters. Since there is a connection between domination type parameters and zero forcing, it would be interesting to obtain algorithmic and complexity results, and to determine upper bounds on the zero forcing number in classes of ZTF-dense or ZF-dense or TF-dense graphs.

For $r \geq 3$, let $G_r$ be the class of connected $r$-regular graphs. If $G \in G_r$ is a ZF-dense graph of sufficiently large order, determine or estimate the best possible constant $a_r$ such that $Z(G) \leq a_r \times n(G)$, and if $G \in G_r$ is a TF-dense graph of sufficiently large order, determine or estimate the best possible constant $b_r$ such that $F_t(G) \leq b_r \times n(G)$. These constants are given by

$$a_r = \sup_{G \in G_r} \frac{Z(G)}{n(G)} \quad \text{and} \quad b_r = \sup_{G \in G_r} \frac{F_t(G)}{n(G)}.$$

As shown in Propositions 8 and 9 we know that $a_3 \geq \frac{1}{4}$ and $b_3 \geq \frac{1}{2}$. It would be interesting to determine the exact values of $a_3$ and $b_3$.

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