DAISY HAMMING GRAPHS

TANJA DRAVEC AND ANDREJ TARANENKO

University of Maribor
Faculty of Natural Sciences and Mathematics
Koroska cesta 160, SI-2000 Maribor, Slovenia
and
Institute of Mathematics, Physics and Mechanics
Jadranska 19, SI-1000 Ljubljana, Slovenia

e-mail: tanja.dravec@um.si
andrej.taranenko@um.si

Abstract

Daisy graphs of a rooted graph $G$ with the root $r$ were recently introduced as a generalization of daisy cubes, a class of isometric subgraphs of hypercubes. In this paper we first address a problem posed in [A. Taranenko, Daisy cubes: A characterization and a generalization, European J. Combin. 85 (2020) 103058] and characterize rooted graphs $G$ with the root $r$ for which all daisy graphs of $G$ with respect to $r$ are isometric in $G$, assuming the graph $G$ satisfies the rooted triangle condition. We continue the investigation of daisy graphs $G$ (generated by $X$) of a Hamming graph $H$ and characterize those daisy graphs generated by $X$ of cardinality 2 that are isometric in $H$. Finally, we give a characterization of isometric daisy graphs of a Hamming graph $K_{k_1} \square \cdots \square K_{k_n}$ with respect to $0^n$ in terms of an expansion procedure.

Keywords: daisy graphs, expansion, isometric subgraphs.

2010 Mathematics Subject Classification: 05C75.

1. Introduction and Preliminary Results

All graphs $G = (V, E)$ in this paper are undirected and without loops or multiple edges. The distance $d_G(u, v)$ between two vertices $u$ and $v$ is the length of a shortest $u, v$-path, and the interval $I_G(u, v)$ between $u$ and $v$ consists of all the vertices on all shortest $u, v$-paths, that is, $I_G(u, v) = \{x \in V(G) | d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. For a set $U$ of vertices of a graph $G$ we denote by $\langle U \rangle_G$
the subgraph of $G$ induced by the set $U$. The index $G$ may be omitted when the graph will be clear from the context. A subgraph $H$ of $G$ is called isometric if $d_H(u, v) = d_G(u, v)$, for all $u, v \in V(H)$.

The Cartesian product $G = G_1 \square \cdots \square G_n$ of $n$ graphs $G_1, \ldots, G_n$ has the $n$-tuples $(x_1, \ldots, x_n)$ as its vertices (with vertex $x_i$ from $G_i$) and an edge between two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ if and only if, for some $i$, the vertices $x_i$ and $y_i$ are adjacent in $G_i$, and $x_j = y_j$, for the remaining $j \neq i$ [6]. The Cartesian product of $n$ copies of $K_2$ is a hypercube or $n$-cube $Q_n$. If all the factors in a Cartesian product are complete graphs then $G$ is called a Hamming graph. The Hamming graph $H = K_{k_1} \square \cdots \square K_{k_n}$ will be denoted by $H_{k_1, \ldots, k_n}$. Isometric subgraphs of hypercubes are called partial cubes and isometric subgraphs of Hamming graphs are called partial Hamming graphs. Note, a tuple $(x_1, \ldots, x_n)$ may be written in a shorter form as $x_1 \cdots x_n$.

For any positive integer $n$ the set $\{1, \ldots, n\}$ is denoted by $[n]$ and the set $\{0, 1, \ldots, n-1\}$ by $[n]_0$. Let $k_1, \ldots, k_n$ be positive integers and let $V = \prod_{i=1}^n [k_i]_0$. The Hamming distance, $H(u, v)$, of two vectors $u, v \in V$ is the number of coordinates in which they differ. Note, a Hamming graph $H_{k_1, \ldots, k_n}$ is the graph with the vertex set $\prod_{i=1}^n [k_i]_0$, such that the Hamming distance and the distance function of the graph coincide. Let $v = v_1 \cdots v_n \in V(H_{k_1, \ldots, k_n})$. If $x_1 \cdots x_n \in I_{H_{k_1, \ldots, k_n}}(v, 0^n)$, then $x_i \in \{0, v_i\}$, for any $i \in [n]$.

A recent paper by Klavžar and Mollard [8] introduced a new family of graphs called daisy cubes. The daisy cube $Q_n(X)$ is the subgraph of $Q_n$ induced by the union of the intervals $I(x, 0^n)$ over all $x \in X \subseteq V(Q_n)$. Daisy cubes are shown to be partial cubes (i.e., isometric subgraphs of hypercubes) and include some other previously well known classes of cube-like graphs, e.g. Fibonacci cubes [7] and Lucas cubes [11, 12]. Regarding daisy cubes, several results have already appeared in the literature. Vesel [14] has shown that a cube-complement of a daisy cube is also a daisy cube. Moreover, daisy cubes also appear in chemical graph theory in connection with resonance graphs. Žigert Pleteršek has shown in [16] that resonance graphs of the so-called kinky benzenoid systems are daisy cubes and Brezovnik et al. [3] characterized catacondensed even ring systems of which resonance graphs are daisy cubes.

Taranenko [13] characterized daisy cubes by means of special kind of peripheral expansions and thus proved that daisy cubes are tree-like partial cubes [2]. In the same paper a generalization of daisy cubes to arbitrary rooted graphs was introduced. These graphs are called daisy graphs of rooted graphs with respect to the root. A sufficient but not a necessary condition for a rooted graph $G$ in which every daisy graph of $G$ with respect to the root is isometric in $G$ was presented. We improve this result with another sufficient condition for this and also prove that both conditions together with an additional one provide a characterization of such graphs. We present these and related results in Section 2. In Section 3
we focus on daisy graphs of Hamming graphs (with respect to a chosen root),
called daisy Hamming graphs. Since hypercubes are a special case of Hamming
graphs and daisy cubes are a special case of daisy graphs, a natural question that
arises is: what properties do isometric daisy Hamming graphs have. Studying
the properties of these graphs we obtain a characterization of isometric daisy
Hamming graphs in terms of a specific kind of expansion.

We continue this section with some notations and preliminary results.

**Definition.** [9] Let $G$ be a graph and $(u, v, w)$ a triple of vertices of $G$. A triple
$(x, y, z)$ of vertices of $G$ is a pseudo-median of the triple $(u, v, w)$ if it satisfies all
of the following conditions.

1. (i) There is a shortest $u, v$-path in $G$ that contains both $x$ and $y$;
   (ii) There is a shortest $v, w$-path in $G$ that contains both $y$ and $z$;
   (iii) There is a shortest $u, w$-path in $G$ that contains both $x$ and $z$;
2. $d(x, y) = d(y, z) = d(x, z)$;
3. $d(x, y)$ is minimal under the first two conditions.

The distance $d(x, y)$ is called the size of the pseudo-median $(x, y, z)$.

Pseudo-median of a triple $(u, v, w)$ of size 0, is called a median of $(u, v, w)$. Let $G$ be a graph and $(u, v, w)$ a triple of vertices of $G$. A triple $(x, y, z)$ of vertices of $G$ is a quasi-median of the triple $(u, v, w)$ if it is a pseudo-median of $(u, v, w)$ and if $(u, v, w)$ has no pseudo-median different from $(x, y, z)$. Note that any triple $(u, v, w)$ of vertices $u = u_1 \cdots u_n, v = v_1 \cdots v_n, w = w_1 \cdots w_n$ of a Hamming graph $H_{k_1, \ldots, k_n}$ has a quasi-median $(x, y, z)$, that can be obtained in
the following way. If $u_i, v_i$ and $w_i$ are pairwise distinct, then $x_i = u_i, y_i = v_i, z_i = w_i$. If $u_i, v_i$ and $w_i$ are not all pairwise distinct with at least two of $u_i, v_i, w_i$ equal to $p_i$, then $x_i = y_i = z_i = p_i$. The size of this quasi-median is the number
of coordinates in which $u, v$ and $w$ are all distinct [9].

A binary expansion was first defined in [10] and a generalization of binary
expansion using more covering sets was first introduced in [9]. We will use the
definition of general expansion introduced by Chepoi [4], as follows.

**Definition.** [4] Let $G$ be a connected graph and let $W_1, W_2, \ldots, W_n$ be subsets
of $V(G)$ such that

1. $W_i \cap W_j \neq \emptyset$, for all $i, j \in [n]$;
2. $\bigcup_{i=1}^n W_i = V(G)$;
3. There are no edges between sets $W_i \setminus W_j$ and $W_j \setminus W_i$, for all $i, j \in [n]$;
4. Subgraphs $\langle W_i \rangle, \langle W_i \cup W_j \rangle$ are isometric in $G$, for all $i, j \in [n]$.

Then to each vertex $x \in V(G)$ we associate a set $\{i_1, i_2, \ldots, i_u\}$ of all indices
$i_j$, where $x \in W_{i_j}$. A graph $G'$ is called an expansion of $G$ relative to the sets
$W_1, W_2, \ldots, W_n$ if it is obtained from $G$ in the following way.
1. Replace each vertex $x$ of $G$ with a clique with vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$;
2. If an index $i_s$ belongs to both sets \{${i_1, \ldots, i_t}$, ${i_1', \ldots, i_t'}$\} corresponding to adjacent vertices $x$ and $y$ in $G$, then let $x_{i_s}y_{i_s} \in E(G')$.

An expansion of $G$ relative to the sets $W_1, W_2, \ldots, W_n$ is called peripheral if there exists $i \in [n]$ such that $W_i = V(G)$. The peripheral expansion of $G$ relative to the sets $W_1, W_2, \ldots, W_n$ will be denoted by $pe(G; W_1, \ldots, W_n)$.

An illustration of an expansion can be seen in Figure 1. In the left-hand side one can see a cycle $C_6$ (with the vertices $a, b, c, d, e, f$) and three subsets of vertices $W_1 = \{a, b, c, d, e, f\}$, $W_2 = \{a, b, c\}$ and $W_3 = \{e, d\}$. It is easy to verify that $W_1, W_2$ and $W_3$ satisfy the conditions of the definition of expansion. The expansion of the cycle $C_6$ with respect to the sets $W_1, W_2$ and $W_3$ is obtained in the following way. Since $a$ and $b$ both belong to $W_1$ and $W_2$, they are each replaced with a clique on two vertices ($a_1$ and $a_2$, and $b_1$ and $b_2$, respectively). The vertex $c$ belongs to all three sets ($W_1, W_2$ and $W_3$) and is therefore replaced by a clique on three vertices ($c_1$, $c_2$ and $c_3$). The vertex $d$ belongs to $W_1$ and $W_3$ and is replaced with a clique on two vertices ($d_1$ and $d_3$). The vertices $e$ and $f$ both belong to only one vertex set, namely $W_1$, they are both replaced by $e_1$ and $f_1$, respectively. Finally, edges between vertices with the same index are added, if the corresponding vertices from the original graph are adjacent. The resulting expansion is shown in the right-hand side of Figure 1. Note, since $W_1 = V(C_6)$ the depicted expansion is also a peripheral expansion.

![Figure 1. A graph $G$ (left-hand side) and its expansion (right-hand side) with respect to the sets $W_1, W_2$ and $W_3$.](image-url)
Daisy Hamming Graphs

\[
W_{uv} = \{x \in V(G) \mid d(u, x) < d(v, x)\};
\]
\[
U_{uv} = \{x \in W_{uv} \mid \text{there exists } z \in W_{vu} \text{ such that } xz \in E(G)\};
\]
\[
F_{uv} = \{xz \in E(G) \mid x \in U_{uv} \land z \in U_{vu}\}.
\]

With these sets we can define Djoković relation \(\sim\) as follows [5]. For \(uv, xy \in E(G)\)
\[uv \sim xy\] if and only if \(x \in W_{uv} \land y \in W_{vu}\).

It follows from the definition that \(F_{uv}\) is precisely the set of edges from \(E(G)\) that are in relation \(\sim\) with \(uv \in E(G)\). Note also that the relation \(\sim\) is reflexive and symmetric but not transitive in general. In [1] Brešar introduced relation \(\triangle\) on the edge set of a connected graph as follows.

**Definition** [1]. Let \(G\) be a connected graph and \(uv, xy \in E(G)\). Then \(uv \triangle xy\) if and only if \(uv \sim xy\) or there exists a clique with edges \(e, f \in E(G)\) such that \(xy \sim e\) and \(uv \sim f\).

Note that the relation \(\triangle\) is also reflexive and symmetric but it is not necessarily transitive. Brešar proved that the relation \(\triangle\) is transitive in partial Hamming graphs [1]. He also proved that each \(\triangle\)-class is a union of some \(\sim\)-classes. For edges \(ab, cd \in E(G)\) the \(\sim\)-classes \(F_{ab}\) and \(F_{cd}\) are in the same \(\triangle\)-class if and only if there is a clique containing edges \(a'b' \in F_{ab}\) and \(c'd' \in F_{cd}\).

### 2. Isometric Daisy Graphs

In [13] a generalization of daisy cubes was defined in the following way.

**Definition** [13]. Let \(G\) be a rooted graph with the root \(r\). For \(X \subseteq V(G)\) the daisy graph \(G_r(X)\) of the graph \(G\) with respect to \(r\) (generated by \(X\)) is the subgraph of \(G\) where
\[
G_r(X) = \langle\{u \in V(G) \mid u \in I_G(r, v) \text{ for some } v \in X\}\rangle.
\]

If \(H = G_r(X)\) is an isometric subgraph of \(G\) we say that \(H\) is an isometric daisy graph of a graph \(G\) with respect to \(r\). Note that it follows from the definition of daisy graphs, that \(V(G_r(X)) = \bigcup_{v \in X} I_G(v, r)\). Moreover, if \(u \in V(G_r(X))\), then \(I_G(u, r) \subseteq V(G_r(X))\). Therefore any convex subgraph \(H\) of a rooted graph \(G\) with root \(r\), such that \(H\) contains \(r\), is a daisy graph of \(G\) with respect to \(r\).

In [13] Taranenko presented a sufficient condition for a rooted graph \(G\) with the root \(r\) in which any daisy graph with respect to \(r\) is isometric. He also proved that the mentioned condition is not necessary.
Proposition 1 [13]. Let $G$ be a rooted graph with the root $r$. If for any two vertices of $G$, say $u$ and $v$, it holds that there exists a pseudo-median of $(u,v,r)$ of size 0, then every daisy graph of $G$ with respect to $r$ is isometric in $G$.

We give another sufficient condition for a rooted graph $G$ with respect to the root $r$ in which any daisy graph with respect to $r$ is isometric and prove that both conditions yield a characterization of rooted graphs $G$ satisfying rooted triangle condition in which all daisy graphs with respect to the root are isometric.

Theorem 2. Let $G$ be a rooted graph with the root $r$. If for any two vertices of $G$, say $u$ and $v$, there exists a pseudo-median of size 1 of the triple of vertices $u, v$ and $r$, then every daisy graph of $G$ with respect to $r$ is isometric in $G$.

Proof. Let $H$ be an arbitrary daisy graph of $G$ with respect to $r$. Also, let $u$ and $v$ be two arbitrary vertices of $H$, and let $(x,y,z)$ be a pseudo-median of $(u,v,r)$ of size 1. Hence there exists a shortest $u,v$-path in $G$ that contains $x$ and $y$, where $x \in I_G(u,r)$ and $y \in I_G(v,r)$. Thus $I_G(u,x) \subseteq I_G(u,r) \subseteq V(H)$, as $H$ is a daisy graph of $G$ with respect to $r$ and analogously $I_G(v,y) \subseteq I_G(v,r) \subseteq V(H)$. Therefore $d_H(u,x) = d_G(u,x)$ and $d_H(v,y) = d_G(v,y)$. Since $x$ and $y$ lie on a shortest $u,v$-path we get

\[
d_G(u,v) = d_G(u,x) + d_G(x,y) + d_G(y,v) = d_G(u,x) + d_G(y,v) + 1 = d_H(u,x) + d_H(y,v) + 1 \geq d_H(u,v).
\]

Moreover, $H$ is a subgraph of $G$ and therefore $d_G(u,v) \leq d_H(u,v)$ and consequently $H$ is an isometric subgraph of $G$.

Definition. A graph $G$ satisfies the triangle condition if for any three vertices $u,v,w \in V(G)$, such that $d(v,w) = 1$ and $d(u,v) = d(u,w) \geq 2$, there exists a vertex $x \in V(G)$ adjacent to $v$ and $w$ with $d(x,u) = d(u,v) - 1$.

Definition. A rooted graph $G$ with the root $r$ satisfies the rooted triangle condition if for any two adjacent vertices $v,w \in V(G)$, such that $d(r,v) = d(r,w) \geq 2$ there exists a vertex $x \in V(G)$ adjacent to $v$ and $w$ with $d(x,r) = d(r,v) - 1$.

Theorem 3. Let $G$ be a rooted graph with the root $r$ such that $G$ satisfies the rooted triangle condition. If every daisy graph of $G$ with respect to $r$ is isometric in $G$, then for any $u,v \in V(G)$ there exists a pseudo-median in $G$ of size 0 or 1 for the triple $u,v$ and $r$.

Proof. Let $u$ and $v$ be two arbitrary vertices of a rooted graph $G$ with the root $r$. Let $H = G_r(\{u,v\})$. Hence $V(H) = I_G(u,r) \cup I_G(v,r)$. Since $H$ is an isometric subgraph of $G$, there exists a shortest $u,v$-path $P$ in $G$ which is entirely contained in $H$. Denote $P : u = u_0, u_1, \ldots, u_{k-1}, u_k = v$. As $P \subseteq V(H)$,
$u_i \in I_G(u, r) \cup I_G(v, r)$, for any $i \in \{0, 1, \ldots, k\}$. If $v \in I_G(u, r)$, then $(v, v, v)$ is a pseudo-median of $(u, v, r)$ of size 0 and the proof is completed. Similarly, $(u, u, u)$ is a pseudo-median of $(u, v, r)$ of size 0 if $u \in I_G(v, r)$, and the proof is also completed in this case. Hence we may assume that $u \notin I_G(v, r) \text{ and } v \notin I_G(u, r)$.

Let $j \in [k]_0$ be the largest index such that $u_j \in I_G(u, r)$. Hence $u_l \in I_G(v, r)$ for any $l \in \{j + 1, \ldots, k\}$. If $u_j \in I_G(v, r)$, then $u_j \in I_G(u, r) \cap I_G(v, r)$ and hence $(u_j, u_j, u_j)$ is a pseudo-median of $(u, v, r)$ of size 0. Next, we assume that $u_j \notin I_G(v, r)$. Since $u_{j+1} \notin I_G(u, r)$, $d_G(u_j, r) = d_G(u_{j+1}, r) = l$. If $l = 1$, then $(u_j, u_{j+1}, r)$ is a pseudo-median of $(u, v, r)$ of size 1. If $l > 1$, then by the rooted triangle condition, there exists $x \in V(G)$ that is adjacent to $u_j$ and $u_{j+1}$ and $x \in I_G(r, u_j) \cap I_G(r, u_{j+1})$. Hence $(u_j, u_{j+1}, x)$ is a pseudo-median of $(u, v, r)$ of size 1, which completes the proof.

From the proof of Theorem 3 we get the following.

Corollary 4. Let $G$ be a rooted graph with the root $r$ such that $G$ satisfies the rooted triangle condition and let $\{u, v\} \subseteq V(G)$. If $H = G, (\{u, v\})$ is isometric in $G$, then there exists a pseudo-median in $G$ of size 0 or 1 for the triple $u, v$ and $r$.

Proposition 1, Theorem 2 and Theorem 3 give the following characterization of rooted graphs $G$ with the root $r$ satisfying the rooted triangle condition, such that every daisy graphs of $G$ with respect to $r$ is isometric in $G$.

Corollary 5. Let $G$ be a rooted graph with the root $r$ such that $G$ satisfies the rooted triangle condition. Every daisy graph of $G$ with respect to $r$ is isometric in $G$, if and only if for any $u, v \in V(G)$ there exists a pseudo-median of size 0 or 1 of the triple of vertices $u, v$ and $r$.

Lemma 6. If $H$ is a Hamming graph, then $H$ satisfies the triangle condition.

Proof. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be two adjacent vertices of $H$ and $w = (w_1, \ldots, w_n) \in V(H)$ such that $d(u, w) = d(v, w) = k$, with $k \geq 2$. Since $uv \in E(H)$, there exists $i \in \{1, \ldots, n\}$ such that $v_i \neq u_i$ and $v_j = u_j$, for all $j \neq i$. Moreover, since $d(u, w) = d(v, w) = k$, it follows that $w_i \neq u_i$ and $w_i \neq v_i$. Let $x = (u_1, \ldots, u_{i-1}, w_i, u_{i+1}, \ldots, u_n)$. Clearly, $xu \in E(H)$ and $xv \in E(H)$ and $x \in I_H(u, w) \cap I_H(v, w)$. The assertion follows.

Lemma 6 and Corollary 5 imply the following.

Corollary 7. Let $H$ be a Hamming graph with the root $r$. Every daisy graph of $H$ with respect to $r$ is isometric in $H$, if and only if for any $u, v \in V(H)$ there exists a pseudo-median of size 0 or 1 for the triple $u, v$ and $r$. 
The above results refer to rooted graphs \( G \) for which all daisy graphs with respect to the root are isometric. Now we chose one daisy graph \( H \) of \( G \) with respect to the root of \( G \) and study when is \( H \) isometric in \( G \).

Note that one can easily deduce from the proofs of Proposition 1 and Theorem 2 that if \( G \) is a rooted graph with the root \( r \) and \( H \) a daisy graph of \( G \) with respect to \( r \) such that for any \( u \) and \( v \) in \( H \), there exists a pseudo-median of size 0 or 1 of the triple of vertices \( u, v \) and \( r \), then \( H \) is isometric in \( G \). It is clear that the reverse statement is not necessarily true. For example, let \( G \) be the cycle \( C_6 \) and \( u \) and \( r \) two antipodal vertices of \( C_6 = u, x_1, x_2, r, y_1, y_2, u \). Then \( G_r(\{u\}) \) is the whole graph \( G \) and thus isometric in \( G \), but there clearly exists a triple of vertices in \( G \), for example \((x_1, y_2, r)\) having no pseudo-median of size 0 or 1 in \( G \).

**Problem 8.** Let \( G \) be a rooted graph with the root \( r \). Characterize daisy graphs of \( G \) with respect to \( r \) (generated by \( X \)) that are isometric in \( G \).

Let \( G \) be a rooted graph with the root \( r \). For \( X = \{v\} \subseteq V(G) \) the above problem is equivalent to the characterization of intervals \( I_G(v, r) \) that are isometric in \( G \).

In the rest of this section we will consider Hamming graphs and study properties of isometric daisy subgraphs. Thus let \( H = H_{k_1 \ldots k_n} \) be a Hamming graph with the root \( r = 0^n \). Let \( G = H_r(X) \) be a daisy graph of \( H \) with respect to \( r \) (generated by \( X \)). Note that if \( |X| = 1 \), then \( G \) is a daisy cube. Moreover, if \( x = x_1 \cdots x_n \) is the vertex of \( X \), then \( G \cong Q_n((y_1 \cdots y_n)) \), where \( y_i = \min \{x_i, 1\} \), for any \( i \in \{1, \ldots, n\} \). For \( |X| = 2 \) we have the following characterization of isometric daisy graphs of a Hamming graph.

**Theorem 9.** Let \( H = H_{k_1 \ldots k_n} \) be a Hamming graph with the root \( 0^n \) and let \( G = H_0(X) \) be a daisy graph of \( H \) generated by the set \( X = \{x, y\} \) of cardinality 2. Then \( G \) is an isometric subgraph of \( H \) if and only if there exists a pseudo-median of \((x, y, 0^n)\) of size 0 or 1 in \( G \).

**Proof.** Let \( G = H_0(\{x, y\}) \). Denote \( x = x_1 \cdots x_n \), \( y = y_1 \cdots y_n \) and \( r = 0^n = r_1 \cdots r_n \).

Suppose first, \( G \) is an isometric subgraph of \( H \). By Lemma 6, the graph \( H \) satisfies the triangle condition and consequently also the rooted triangle condition. Using the same line of thought as in the proof of Theorem 3 one can easily check that there exists a pseudo-median of \((x, y, 0^n)\) of size 0 or 1 in \( G \).

For the converse suppose that there is a pseudo-median of size 0 or 1 of \((x, y, 0^n)\) in \( G \). Since the size of the pseudo-median in a Hamming graph is the number of coordinates in which \( x, y \) and \( r \) are all distinct, there is at most one coordinate in which \( x, y \) and \( r \) are all pairwise distinct. To simplify, permute factors of \( H \) such that \( x \) has the first \( i - 1 \) coordinates equal to 0 and all other coordinates different from 0 (i.e., \( i - 1 \) is the number of coordinates of \( x \) that
are equal to 0), and if there exists a coordinate in which $x, y$ and $r$ are pairwise
distinct, let this be the $i$th coordinate. Since $(x, y, r)$ has a pseudo-median of size
0 or 1, $y_j \in \{x_j, 0\}$, for any valid index $j > i$.

Let $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ be two arbitrary vertices of $G$. Note,
$V(G) = I_H(x, 0^n) \cup I_H(y, 0^n)$. We will prove that there exists a $u, v$-path in $G$
with $d_G(u, v) = H(u, v) = d_H(u, v)$.

Suppose first that $u, v \in I_H(x, 0^n)$ (the case when $u, v \in I_H(y, 0^n)$ is
proved in a similar way). Then $u_j = v_j = 0$, for any $j < i$, and for any $j \geq i$, it holds
that $u_j \in \{x_j, 0\}$ and $v_j \in \{x_j, 0\}$. We construct $u, v$-path of length $H(u, v)$
in $G$ in the following way. Start in $u$ and continue with $u^{(1)}$ which is obtained
from $u$ by replacing the first coordinate of $u$, say $u_j$, in which $u$ and $v$ differ,
with $v_j$. Since $v_j \neq u_j$ and $u, v \in I_H(x, 0^n)$, $\{v_j, u_j\} = \{x_j, 0\}$ and consequently
$u^{(1)} \in I_H(x, 0^n) \subseteq V(G)$. We continue in the same way step by step, such that
at the step $k$ we replace the first coordinate of $u^{(k)}$, say $u^{(k)}_j$, in which $u^{(k)}$ and $v$
differ, with $v_j$. Since all the vertices $u^{(k)}$, for any valid $k$, are contained in $V(G)$
and the constructed path $P$ is of length $H(u, v)$, $P$ is a $u, v$-path of $G$ of length
d examined in $v$.

Finally, let $u \in I_H(x, 0^n)$ and $v \in I_H(y, 0^n) \setminus I_H(x, 0^n)$.

Let $I_D$ be the set of indices in which $u$ and $v$ differ. We will also use the
following sets. The set $I_M = \{i' \in I_D \mid u_{i'} \neq 0 \land v_{i'} \neq 0\}$, this is an empty set,
if $(x, y, r)$ has a pseudo-median of size 0, otherwise it contains the index $i$. Let
$I_u = \{i' \in I_D \mid u_{i'} = 0\}$ and $I_v = \{i' \in I_D \mid v_{i'} = 0\}$. Note that $I_M, I_u$ and $I_v$
form a partition of $I_D$.

We construct a $u, v$-path in the following way. The first part of the path is constructed by using all the indices from the set $I_v = \{i_1, i_2, \ldots, i_{|I_v|}\}$. Let
$u^{(0)} = u$ be the first vertex of this path. The next vertex of the path, $u^{(1)}$, is
obtained from $u^{(0)}$ by replacing the coordinate $u^{(0)}_{i_1}$ with 0. The vertex $u^{(2)}$, is
obtained from $u^{(1)}$ by replacing the coordinate $u^{(1)}_{i_2}$ with 0. Assume we have
already obtained the vertex $u^{(j)}$, then we obtain the vertex $u^{(j+1)}$ from $u^{(j)}$ by
replacing the coordinate $u^{(j)}_{i_{j+1}}$ with 0. We do this for every index in $I_v$, so the last
vertex we obtain is $u^{(|I_v|)}$. It is easy to see, that these vertices indeed form a path
(two consecutive vertices differ in exactly one coordinate). Since we only change
coordinates to 0, it is also clear that every vertex constructed so far belongs to
$I_H(u, 0^n) \subseteq I_H(x, 0^n) \subseteq V(G)$.

If $I_M$ is not an empty set, we form the next vertex in our path, say $v^{(0)}$, from
$u^{(|I_v|)}$ by replacing the coordinate $u^{(|I_v|)}_{i_1}$ to $v_1$. Again, since $u^{(0)}$ and $v$ differ only
in indices of the set $I_u$ and the values of coordinates at those indices in $v^{(0)}$ is 0,
it is clear that $v^{(0)} \in I_H(v, 0^n) \subseteq I_H(y, 0^n) \subseteq V(G)$. If $I_M$ is an empty set, we
denote the vertex $u^{(|I_v|)}$ by $v^{(0)}$.

We continue with the construction of our $u, v$-path by using all the indices
from the set $I_u = \{j_1, j_2, \ldots, j_{|I_u|}\}$. The next vertex of the path, $v^{(1)}$, is obtained from $v^{(0)}$ by replacing the coordinate $v^{(0)}_{j_1}$ with $v_{j_1}$. The vertex $v^{(2)}$, is obtained from $v^{(1)}$ by replacing the coordinate $v^{(1)}_{j_2}$ with $v_{j_2}$. Assume we have already obtained the vertex $v^{(k)}$, then we obtain the vertex $v^{(k+1)}$ from $v^{(k)}$ by replacing the coordinate $v^{(k)}_{j_{k+1}}$ with $v_{j_{k+1}}$. We do this for every index in $I_u$, so the last vertex we obtain is $v^{(|I_u|)}$. It is easy to see, that these vertices indeed form a path (two consecutive vertices differ in exactly one coordinate). Since we only change coordinates, say at index $j'$, from 0 to $v^{(j')}$, it is also clear that every vertex constructed in this part of the path belongs to $I_{H}^{o}(v,0^{n}) \subseteq I_{H}^{o}(y,0^{n}) \subseteq V(G)$. Note, that the vertex $v^{(|I_u|)}$ is actually the vertex $v$. The fact, that the sets $I_M, I_u$ and $I_v$ form a partition of $I_D$ implies that the length of the constructed path is $H(u, v)$. This concludes our proof. 

In section 3 we give a constructive characterization of isometric daisy graphs of a Hamming graph. The above characterization of isometric daisy graphs of a Hamming graph generated by a set of cardinality at most 2, rises the question about a non-constructive characterization of isometric daisy graphs of a Hamming graph generated by a set of cardinality at least 3. Note, this is a specific case of Problem 8.

3. Characterization of Isometric Daisy Hamming Graphs

Let $G'$ be a daisy graph of a Hamming graph $H' = H_{k_1, \ldots, k_n-1}$ with respect to $0^{n-1}$. Let $G$ be a peripheral expansion of $G'$ relative to $W'_0, W'_1, \ldots, W'_k$. If for any $i \in \{1, \ldots, k\}$, the graph $(W'_i)_H'$ is a daisy graph of $H'$ with respect to $0^{n-1}$, then the peripheral expansion $pe'(G'; W'_0, \ldots, W'_k)$ is called daisy peripheral expansion of $G'$ relative to $W'_0, \ldots, W'_k$.

In this section we prove that isometric daisy graphs of a Hamming graph are precisely the graphs that can be obtained from $K_1$ by a sequence of daisy peripheral expansions.

**Theorem 10.** Let $H = H_{k_1, \ldots, k_n}$ be a Hamming graph with the root $0^n$. If $G$ is an isometric daisy graph of $H$ with respect to the root $0^n$, then the daisy peripheral expansion of $G$ relative to the sets $V(G) = W_0, \ldots, W_l$, is an isometric daisy graph of $H' = K_{l+1} \square H$ with respect to $0^{n+1}$.

**Proof.** Let $G'$ be the daisy peripheral expansion of $G$ relative to $W_0, W_1, \ldots, W_l$. Therefore, $G'$ consists of a disjoint union of a copy of $G = \langle W_0 \rangle$ and a copy of $\langle W_i \rangle$, for any $i \in \{1, \ldots, l\}$. We define the labels of the vertices of $G'$ as follows. Prepend $i$ to each vertex of $G'$ corresponding to the copy of $\langle W_i \rangle$, for all
i ∈ \{0, \ldots, l\}. Hence the labels of the vertices of \(G'\) are vectors of length \(n + 1\) and the first coordinate is an integer from \(\{0, \ldots, l\}\).

First, we prove that two vertices of \(G'\) are adjacent if and only if the corresponding vectors differ in exactly one position. Since \(G'\) is the expansion of \(G\) relative to \(W_0, \ldots, W_l\), it follows from the definition of expansion that two vertices \(u' = u_1 \cdots u_n u_{n+1}\) and \(v' = v_1 \cdots v_n v_{n+1}\) of \(G'\) are adjacent in \(G'\) if and only if \(u = u_2 \cdots u_{n+1}\) and \(v = v_2 \cdots v_{n+1}\) are adjacent in \(G\) and both belong to the same set \(W_i\), or if \(u = u_2 \cdots u_{n+1} = v = v_2 \cdots v_{n+1}\) and \(u\) belongs to two different sets \(W_{u_1}\) and \(W_{v_1}\). The last condition directly implies that \(u'\) and \(v'\) differ in exactly one coordinate, namely the first coordinate. If \(u = u_2 \cdots u_{n+1}\) and \(v = v_2 \cdots v_{n+1}\) are adjacent in \(G\) and contained in the same set \(W_i\), then \(u\) and \(v\) differ in exactly one coordinate. But then, since they are both in \(W_i\), \(u_1 = v_1 = i\) and hence \(u'\) and \(v'\) differ in exactly one coordinate. Hence \(G'\) is an induced subgraph of \(H' = K_{l+1} \square H\).

In the second step we prove that \(G'\) is a daisy graph of \(H'\) with respect to \(0^{n+1}\). Let \(v' = v_0 v_1 \cdots v_n \in V(G')\) and let \(x' = x_0 \cdots x_n \in I_{H'}(v', 0^{n+1})\). Hence \(x_i \in \{0, v_i\}\), for any \(i \in \{0, \ldots, n\}\). Since \(v' = v_0 v_1 \cdots v_n\), it follows that \(v = v_1 \cdots v_n \in W_{v_0}\). We know that the graph \(\langle W_{v_0}\rangle\) is a daisy graph of \(H\) with respect to \(0^n\) and \(x = x_1 \cdots x_n \in I_H(v, 0^n)\), therefore \(x \in V(\langle W_{v_0}\rangle)\). Hence if \(x' = 0 x_1 \cdots x_n\), then \(x'\) is in the copy of \(G\) in \(G'\). If \(x' = v_0 x_1 \cdots x_n\), then \(x'\) is in the copy of \(\langle W_{v_0}\rangle\) in \(G'\). In both cases we deduce that \(x' \in V(G')\), which completes this part of the proof.

It remains to prove that \(G'\) is an isometric subgraph of \(H'\). Let \(u' = u_0 \cdots u_n\) and \(v' = v_0 \cdots v_n\) be two arbitrary vertices of \(G'\). If \(u_0 = v_0\), then \(u = u_1 \cdots u_n \in W_{u_0}\) and \(v = v_1 \cdots v_n \in W_{u_0}\). Since \(G'\) is an expansion of \(G\), relative to \(W_0, \ldots, W_l\), the definition of expansion implies that \(\langle W_{u_0}\rangle\) is isometric in \(G\). As \(G\) is isometric in \(H\),

\[
d_{\langle W_{u_0}\rangle}(u, v) = d_G(u, v) = d_H(u, v) = H(u, v).
\]

Hence

\[
d_{G'}(u', v') = d_{\langle W_{u_0}\rangle}(u, v) = H(u, v) = H(u', v') = d_{H'}(u', v'),
\]

where the penultimate equality holds because \(u_0 = v_0\).

Finally, consider the case where \(u_0 \neq v_0\). Hence \(u = u_1 \cdots u_n \in W_{u_0}\) and \(v = v_1 \cdots v_n \in W_{v_0}\). Since \(\langle W_{u_0} \cup W_{v_0}\rangle\) is isometric in \(G\) (by the definition of expansion), there exists a shortest \(u, v\)-path \(P : u = u^0, u^1, \ldots, u^k = v\) in \(G\) (note that each \(u^i\) is a vertex in \(G\) and hence has the form \(u^i = u_1^i \cdots u_n^i\)) which is entirely contained in \(\langle W_{u_0} \cup W_{v_0}\rangle\). Since \(G\) is isometric in \(H\), we get

\[
d_{\langle W_{u_0} \cup W_{v_0}\rangle}(u, v) = d_G(u, v) = d_H(u, v) = H(u, v).
\]
Let $i \in \{0, \ldots, k\}$ be the smallest index such that $u^i \in W_0$. Since there are no edges between $W_{u_0} \setminus W_i$ and $W_{v_0} \setminus W_i$, $u^i \in W_{u_0}$. Then the path $u^i = u^0, u^1, \ldots, u^i = v'$, where $u^i = v_0u^i$, for any $l \in \{0, \ldots, i\}$ and $v^l = v_0u^l$, for any $l \in \{i, \ldots, k\}$, is a $u^i, v'$-path in $G'$. Hence

$$d_{G'}(u^i, v') \leq d_G(u, v) + 1 = H(u, v) + 1 = H(u^i, v') = d_{G'}(u^i, v'),$$

where the penultimate equality holds because $u_0 \neq v_0$. Since $G'$ is a subgraph of $H'$, the assertion follows. 

Let $G$ be an isometric daisy graph of a Hamming graph $H = H_{k_1, \ldots, k_n}$ with respect to $0^n$, where $H$ is the smallest possible. We introduce the following terminology which will be used throughout this section. For any $j \in [n]$, we define the following sets.

\[
\begin{align*}
W_i^j &= \{u = u_1 \cdots u_n \in V(G) \mid u_j = i\}, \text{ for any } i \in [k_j]_0; \\
U_i^j &= \{x \in W_i^j \mid \exists y \in W_0^j : xy \in E(G)\}, \text{ for any } i \in [k_j]_0; \\
U_{0^j}^j &= \{x \in W_0^j \mid \exists y \in W_0^j : xy \in E(G)\}, \text{ for any } i \in \{1, \ldots, k_j - 1\}; \\
U_0^j &= \bigcup_{i=1}^{k_j-1} U_{0^j}^j.
\end{align*}
\]

Also, for any $j \in [n]$ and any $i \in [k_j]_0$ denote by $e_i^j$ the vertex of the Hamming graph $H$ labeled by $0^{j-1}i0^{n-j}$.

**Lemma 11.** Let $G$ be an isometric daisy graph of a Hamming graph $H = H_{k_1, \ldots, k_n}$ with respect to $0^n$, where $H$ is the smallest possible. For any $j \in [n]$ and any $i \in [k_j]_0$, if $W_i^j \neq \emptyset$, then there exists $uv \in E(G)$ such that $W_i^j = W_{uv}$.

**Proof.** Let $j \in [n]$ and $i \in [k_j]_0$ be arbitrary, with $W_i^j \neq \emptyset$, and $x = x_1 \cdots x_n \in W_i^j$. Hence $x_j = i$. Since $G$ is a daisy graph of $H$ with respect to $0^n$ and $x' = 0^{j-1}0^{n-j} \in I_H(0^n, x)$, it follows that $x' \in V(G)$. Since $x_j' = i$, $x' \in W_i^j$. Then $W_{x'0^n}$ contains exactly all the vertices of $G$, that are closer to $x'$ than $0^n$, i.e., all vertices of $G$ with $j$-th coordinate equal to $i$. Hence $W_{x'0^n} = W_i^j$. 

For the edge $uv$ of a Partial Hamming graph, the sets $W_{uv}$ have many nice properties [1, 4, 15]. Since our graph $G$ is a partial Hamming graph, it follows from Lemma 11 that the sets $W_i^j$ also have these properties.

**Lemma 12.** Let $G$ be an isometric daisy graph of a Hamming graph $H = H_{k_1, \ldots, k_n}$ with respect to $0^n$, where $H$ is the smallest possible. For any $\Delta$-class $F$ of $G$, there exists an edge $f \in F$ with $0^n$ as an endpoint.

**Proof.** Let $F$ be an arbitrary $\Delta$-class of $G$ and $uv \in F$, where $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$. Hence, $u_i \neq v_i$, for some $i \in [k_j]$, and $u_j = v_j$, for any $j \in [n] \setminus \{i\}$. 

First, suppose that one of \( u_i \) and \( v_i \) equals 0, say \( u_i \). It follows that \( 0^n \in W_{uv} \). Since \( e_{v_i}^i \in I_H(v, 0^n) \) and \( G \) is a daisy graph of \( H \) with respect to \( 0^n \), it follows that \( e_{v_i}^i \in V(G) \). Since the \( i \)-th coordinate of \( e_{v_i}^i \) is \( v_i \), the vertex \( e_{v_i}^i \in W_{vu} \). Hence, \( 0^n e_{v_i}^i \sim uv \) and therefore \( 0^n e_{v_i}^i \in F \).

Finally, suppose neither \( u_i \) nor \( v_i \) equals 0. Since \( x = u_1 \cdots u_{i-1}0u_{i+1} \cdots u_n \in I_H(u, 0^n) \) and \( G \) is a daisy graph of \( H \) with respect to \( 0^n \), the vertex \( x \in V(G) \). Note that \( u, v \) and \( x \) induce \( K_3 \) in \( G \). Hence, \( vx \triangle uv \) and consequently the edge \( vx \) belongs to \( F \). Now, consider the vertex \( e_{v_i}^i \), which belongs to \( I_H(v, 0^n) \) and therefore is a vertex of \( G \). Similarly to the first case, we deduce that \( e_{v_i}^i \in W_{vx} \). Clearly, \( 0^n \in W_{xv} \) and \( 0^n e_{v_i}^i \) is an edge of \( G \). It follows that \( 0^n e_{v_i}^i \sim xv \) and therefore \( 0^n e_{v_i}^i \in F \).

From the definition of the relation \( \triangle \) it follows that the \( \triangle \)-class \( F_j \) generated by the edge \( 0^n e_{i}^j \), for some \( i \neq 0 \), contains exactly all edges between \( U_k^j \) and \( U_l^j \), for any \( 0 \leq k < l \leq k_j - 1 \). Thus using Lemma 12 we deduce the following.

**Corollary 13.** Let \( G \) be an isometric daisy graph of a Hamming graph \( H = H_{k_1, \ldots, k_n} \) with respect to \( 0^n \), where \( H \) is the smallest possible. There are exactly \( n \) \( \triangle \)-classes \( F_1, \ldots, F_n \) of \( E(G) \), where for any \( j \in [n] \) the \( \triangle \)-class \( F_j \) is generated by the edge \( 0^n e_{i}^j \), for some \( 0 < i \leq k_j - 1 \).

Let \( G \) be an isometric daisy graph of a Hamming graph \( H = H_{k_1, \ldots, k_n} \) with respect to \( 0^n \), where \( H \) is the smallest possible. Let \( j \in [n] \) and \( i \in [k_j] \). A subgraph \( \langle W_i^j \rangle \) of a graph \( G \) is called **peripheral** if \( U^j_i = W_i^j \). The \( \triangle \)-class \( F \) generated by the edge \( 0^n e_{i}^j \), for some \( 0 < l \leq k_j - 1 \), of the graph \( G \) is called **peripheral** if \( U^j_{l'} = W_i^j \), for any \( l' \in \{1, \ldots, k_j - 1\} \).

**Lemma 14.** If \( G \) is an isometric daisy graph of a Hamming graph \( H = H_{k_1, \ldots, k_n} \) with respect to \( 0^n \), where \( H \) is the smallest possible, then every \( \triangle \)-class \( F \) of the graph \( G \) is peripheral.

**Proof.** Let \( F \) be an arbitrarily chosen \( \triangle \)-class of \( G \), such that \( 0^n e_{i}^j \in F \). Let \( i \in \{1, \ldots, k_j - 1\} \) be arbitrary. To prove the assertion, we will show that any vertex of \( W_i^j \) has a neighbour in \( W_0^j \) (which means \( W_i^j = U^j_i \)). Take any \( x = x_1 \cdots x_n \in W_i^j \), hence \( x_j = i \). Now, consider \( x' = x_1 \cdots x_{j-1}0x_{j+1} \cdots x_n \). Note, that \( x' \in I_H(0^n, x) \subseteq V(G) \) and therefore \( x' \in W_0^j \). Since \( xx' \in E(G) \), the assertion follows.

**Lemma 15.** Let \( G \) be an isometric daisy graph of a Hamming graph \( H = H_{k_1, \ldots, k_n} \) with respect to \( 0^n \), where \( H \) is the smallest possible. For every \( j \in [n] \) and any \( i \in [k_j] \) the subgraph \( \langle W_i^j \rangle \) of the graph \( G \) is a daisy graph of \( H' = H_{k_1, \ldots, k_j-1, k_{j+1}, \ldots, k_n} \) with respect to \( 0^{n-1} \).

**Proof.** Define \( X_i^j = \{ x_1 \cdots x_{j-1}x_{j+1} \cdots x_n | x_1 \cdots x_n \in W_i^j \} \). Let \( r : W_i^j \rightarrow X_i^j \)
be the projection defined by \( r : x_1 \cdots x_n \mapsto x_1 \cdots x_{j-1}x_{j+1} \cdots x_n \), which is clearly a bijection between \( W_i^j \) and \( X_i^j \).

Let \( u = u_1 \cdots u_{n-1} \in X_i^j \) be arbitrary and \( w \in I_{H'}(0^{n-1}, u) \). We claim that \( w \in X_i^j \). Since \( u \in X_i^j \), it follows from the definition of \( X_i^j \) that \( u' = u_1 \cdots u_{j-1}iu_{j} \cdots u_{n-1} \in W_i^j \). Since \( w \in I_{H'}(0^{n-1}, u) \), it follows that \( w_l = u_l \) or \( w_l = 0 \), for all \( 1 \leq l \leq n - 1 \). Let \( w' = w_1 \cdots w_{j-1}w_{j} \cdots w_{n-1} \). Since \( w' \in I_{H}(0^n, u') \), it follows that \( w' \in V(G) \) and as the \( i \)th coordinate of \( w' \) is \( i \), the vertex \( w' \) belongs to \( W_i^j \). By the definition of \( X_i^j \), \( w \in X_i^j \). Therefore \( \langle X_i^j \rangle_{H'} \) is a daisy graph of \( H' \) with respect to \( 0^{n-1} \). Since \( \langle W_i^j \rangle_{H} \cong \langle X_i^j \rangle_{H'} \), the assertion follows.

In [1] the contraction of a partial Hamming graph \( G \) was defined in the following way. Let \( uv \in E(G) \) and let \( \Delta \)-class with respect to \( uv \in E(G) \), denote it by \( \Delta_{uv} \), be the union of \( k \) distinct \( \sim \)-classes \( F_{xixj} \). A graph \( G' \) is a contraction of a partial Hamming graph \( G \) with respect to the edge \( uv \in E(G) \) if each clique induced by edges belonging to \( \Delta_{uv} \) is contracted to a single vertex. For all \( i \in [k] \), let \( W_i^j \) be the set of vertices in \( G' \) that corresponds to \( W_{x_i} = \{ w \in V(G) \mid d(w, x_i) < d(w, x_j) \}, \) for any \( j \neq i \). Brešar proved that the expansion of \( G' \) relative to \( W_1^1, \ldots, W_k^j \) is exactly the graph \( G \) [1].

**Theorem 16.** Let \( G \) be an isometric daisy graph of a graph \( H = H_{k_1, \ldots, k_n} \) with respect to \( 0^n \), where \( H \) is the smallest possible. Then there exists a daisy graph \( G^r \subseteq G \) such that \( G \) can be obtained from \( G' \) by a daisy peripheral expansion.

**Proof.** Let \( F \) be an arbitrary \( \Delta \)-class of the graph \( G \). By Corollary 13 there exist \( j \in [n] \) and \( i \in \{1, \ldots, k_j - 1 \} \) such that \( F \) is generated by the edge \( 0^ne_i^j \). Let the graph \( G' \) be obtained from the graph \( G \) by a contraction with respect to the edge \( 0^ne_i^j \). For any \( l \in [k_j] \), denote by \( X_l \) the set of vertices in \( G' \) that corresponds to \( W^j_l \) in \( G \). By the definition of a contraction, the graph \( G \) is the expansion of \( G' \) relative to sets \( X_0, \ldots, X_{k_j-1} \). By Lemma 14, it follows that \( F \) is a peripheral \( \Delta \)-class. Using the fact that \( F \) is generated by the edge \( 0^ne_i^j \), it follows from the definition of peripheral classes, that \( U^j_l = W^j_i \), for any \( i' \in \{1, \ldots, k_j - 1 \} \) (every vertex of \( W^j_i \) has a neighbour in \( W^j_{i'} \)). Since \( \bigcup_{i=0}^{k_j-1} X_i = V(G) \) (definition of expansion) we obtain that \( X_0 = V(G') \). By Lemma 15, it follows that the subgraphs \( \langle X_i \rangle_{G'} \) are daisy graphs which proves that \( G \) is obtained from \( G' \) by daisy peripheral expansion.

From Theorem 10 and Theorem 16 we immediately obtain the following characterization.

**Theorem 17.** A graph \( G \) is an isometric daisy graph of a graph \( H = H_{k_1, \ldots, k_n} \) with respect to \( 0^n \), where \( H \) is the smallest possible, if and only if it can be obtained from the one vertex graph by a sequence of daisy peripheral expansions.
Acknowledgments

The authors are grateful to the referees for their careful reading and suggestions for improvement. This work was supported by the Slovenian Research Agency under the grants P1-0297, J1-1693 and J1-9109.

References


doi:10.1016/j.disc.2019.01.008

doi:10.1016/0095-8956(90)90073-9


Received 27 May 2020
Revised 20 October 2020
Accepted 20 October 2020