ON THE RAMSEY NUMBERS OF NON-STAR TREES
VERSUS CONNECTED GRAPHS OF ORDER SIX

ROLAND LORTZ

Technische Universität Braunschweig
Institut für Analysis und Algebra, AG Algebra
38092 Braunschweig, Germany

e-mail: r.lortz@tu-braunschweig.de

AND

INGRID MENGERSEN

Moorhüttenweg 2d
38104 Braunschweig, Germany

e-mail: ingrid.mengersen@t-online.de

Abstract

This paper completes our studies on the Ramsey number \( r(T_n, G) \) for trees \( T_n \) of order \( n \) and connected graphs \( G \) of order six. If \( \chi(G) \geq 4 \), then the values of \( r(T_n, G) \) are already known for any tree \( T_n \). Moreover, \( r(S_n, G) \), where \( S_n \) denotes the star of order \( n \), has been investigated in case of \( \chi(G) \leq 3 \). If \( \chi(G) = 3 \) and \( G \neq K_{2,2,2} \), then \( r(S_n, G) \) has been determined except for some \( G \) and some small \( n \). Partial results have been obtained for \( r(S_n, K_{2,2,2}) \) and for \( r(S_n, G) \) with \( \chi(G) = 2 \). In the present paper we investigate \( r(T_n, G) \) for non-star trees \( T_n \) and \( \chi(G) \leq 3 \). Especially, \( r(T_n, G) \) is completely evaluated for any non-star tree \( T_n \) if \( \chi(G) = 3 \) where \( G \neq K_{2,2,2} \), and \( r(T_n, K_{2,2,2}) \) is determined for a class of trees \( T_n \) with small maximum degree. In case of \( \chi(G) = 2 \), \( r(T_n, G) \) is investigated for \( T_n = P_n \), the path of order \( n \), and for \( T_n = B_{2,n-2} \), the special broom of order \( n \) obtained by identifying the centre of a star \( S_3 \) with an end-vertex of a path \( P_{n-2} \). Furthermore, the values of \( r(B_{2,n-2}, S_m) \) are determined for all \( n \) and \( m \) with \( n \geq m - 1 \). As a consequence of this paper, \( r(F, G) \) is known for all trees \( F \) of order at most five and all connected graphs \( G \) of order at most six.

Keywords: Ramsey number, Ramsey goodness, tree, star, path, broom, small graph.

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1. Introduction

**Ramsey number and Ramsey goodness.** For graphs $F$ and $G$ the Ramsey number $r(F,G)$ is the smallest integer $p$ such that in every 2-coloring of the edges of $K_p$ there is a copy of $F$ in the first color or a copy of $G$ in the second color. The chromatic surplus $s(G)$ is defined to be the smallest number of vertices in a color class under any $\chi(G)$-coloring of the vertices of $G$, where $\chi(G)$ denotes the chromatic number of $G$. It is well-known (see [6] or [7]) that for any connected graph $F$ with $n$ vertices and any graph $G$ with $s(G) \leq n$ the Ramsey number $r(F,G)$ satisfies
\begin{equation}
    r(F,G) \geq (n-1)(\chi(G) - 1) + s(G).
\end{equation}
If equality occurs, then $F$ is said to be $G$-good. Chvátal [3] has proved that every tree $T_n$ of order $n$ is $K_m$-good, i.e., $r(T_n, K_m) = (n-1)(m-1) + 1$. Moreover, several classes of non-complete graphs $G$ are known where every tree $T_n$ is $G$-good, but there are also graphs $G$ and trees $T_n$ such that $r(T_n, G)$ differs even considerably from the lower bound given in (1) — a survey on results for $r(T_n, G)$ can be found in [17].

**Our contribution.** Faudree, Rousseau and Schelp [7] initiated the systematic study of $r(T_n, G)$ for graphs $G$ of small order $p(G)$ and investigated the case $p(G) \leq 5$. In [11] and [12] we started to extend these investigations to graphs $G$ with $p(G) = 6$. Using the result on $r(T_n, K_m)$ due to Chvátal and results on $r(T_n, G)$ for nearly complete graphs $G$ due to Chartrand, Gould and Polimeni [2] and Gould and Jacobson [8] it was not difficult to derive that any tree $T_n$ with $n \geq 5$ is $G$-good for all graphs $G$ with $p(G) = 6$ and $\chi(G) \geq 4$. In [11] our main focus was on $r(S_n, G)$ where $S_n$ denotes the star of order $n$ and $G$ is a connected graph of order six with $G \neq K_{2,2,2}$ and $\chi(G) \leq 3$. In [12] we studied $r(S_n, K_{2,2,2})$. Especially we proved that in case of $\chi(G) = 3$ and $G \neq K_{2,2,2}$ the star $S_n$ is $G$-good or, in a few cases, $r(S_n, G)$ differs by 1 or 2 from the lower bound (1). In contrast, for $n$ sufficiently large, $r(S_n, K_{2,2,2}) > 2n - 2 + \lfloor \sqrt{n} - 1 - 6(n-1)^{11/40} \rfloor$, i.e., $r(S_n, K_{2,2,2})$ differs considerably from the lower bound $2n$ given in (1).

In this paper we study $r(T_n, G)$ for non-star trees $T_n$ and connected graphs $G$ with $p(G) = 6$ and $\chi(G) \leq 3$. We prove that every non-star tree $T_n$ is $G$-good for every connected graph $G \notin \{K_{1,1,4}, K_{2,2,2}\}$ with $p(G) = 6$ and $\chi(G) = 3$. A more general result on $r(T_n, K_{1,1,m})$ due to Erdős, Faudree, Rousseau and Schelp [6] and our results from [13] show that, except for $n \leq 5$, every non-star tree $T_n$ is also $K_{1,1,4}$-good. The case $G = K_{2,2,2}$ remains to a great extent unsolved. We present several $K_{2,2,2}$-good non-star trees $T_n$ with small maximum degree, but the behavior of $r(S_n, K_{2,2,2})$ implies that non-star trees $T_n$ with sufficiently large $n$ and maximum degree close to $n-1$ cannot be $K_{2,2,2}$-good.

To determine $r(T_n, G)$ for every tree $T_n$ and all connected graphs $G$ of order six with $\chi(G) = 2$, i.e., the star $S_6$ and the connected spanning subgraphs of $K_{2,4}$
and $K_{3,3}$, seems to be a hard problem. Partial results on $r(S_n, G)$ were obtained in [11]. In this paper we investigate $r(T_n, G)$ for two non-star trees $T_n$, namely $T_n = P_n$, the path on $n$ vertices, and $T_n = B_{2,n-2}$, a special case of a broom $B_{k,n-k}$ defined as a tree of order $n \geq 5$ obtained by identifying the centre of a star $S_{k+1}$, $k \geq 2$, with an end-vertex of a path $P_{n-k}$. The choice of these two non-star trees is due to the project to evaluate $r(F,G)$ for graphs $F$ of order at most five and graphs $G$ of order six — the only non-star trees on at most five vertices are the paths $P_n$ with $4 \leq n \leq 5$ and the broom $B_{3,3}$. Instead of $r(T_n, S_6)$ we consider the more general case $r(T_n, S_m)$. Parsons [14] has already determined $r(P_n, S_m)$ for all $n$ and $m$ by explicit formulas and a recurrence, and we evaluate $r(B_{2,n-2}, S_m)$ for all $n$ and $m$ with $n \geq m - 1$. The results in this paper together with the results in [11] and [12] imply that $r(F, G)$ is known for all trees $T_n$ of order at most five and all connected graphs $G$ of order six.

**Notation and terminology.** Some specialized notation and terminology will be used. The vertex set of a graph $G$ is denoted by $V(G)$. We write $G' \subseteq G$ if $G'$ is a subgraph of $G$ and, for $U \subseteq V(K_n)$, $[U]$ is the subgraph induced by $U$. A coloring of a graph here always means a 2-coloring of its edges with colors red and green. An $(F_1, F_2)$-coloring is a coloring containing neither a red copy of $F_1$ nor a green copy of $F_2$. Given a coloring of $K_n$, we define the $r$-degree $d_r(v)$ to be the number of red edges incident to $v \in V(K_n)$. Moreover, $\Delta_r = \max_{v \in V(K_n)} d_r(v)$. The set of vertices joined red to $v$ is denoted by $N_r(v)$. Similarly we define $d_g(v)$, $\Delta_g$ and $N_g(v)$. Furthermore, $[U]_r$ and $[U]_g$ are the red and the green subgraphs induced by $U$. For disjoint subsets $U_1, U_2 \subseteq V(K_n)$, $q_r(U_1, U_2)$ denotes the number of red edges between $U_1$ and $U_2$, and $q_g(U_1, U_2)$ is defined similarly. The vertex of degree $n - 1$ in a star $S_n$ with $n \geq 3$ is called the centre of the star. We write $P_k = u_1u_2 \cdots u_k$ for the path $P_k$ with vertices $u_1, \ldots, u_k$ and edges $u_iu_{i+1}$ for $i = 1, \ldots, k - 1$. Moreover, $(u_1u_2 \cdots u_k)$ means the cycle $C_k$ obtained from $P_k = u_1u_2 \cdots u_k$ by adding the edge $u_1u_k$, and an edge $u_iu_j$ is called a diagonal of length $\ell$ of $C_k$ if $u_i$ and $u_j$ are vertices with distance $\ell$ on $C_k$. The bristles of a broom $B_{k,n-k}$ are the $k$ edges joining the vertex $v^*$ of degree $k + 1$ to a vertex of degree 1 and the path $P_{n-k}$ with end-vertex $v^*$ is said to be the handle of the broom. The complement $\overline{K}_n$ of $K_n$ is denoted by $E_n$, and for the complete $k$-partite graph $K_{n_1,n_2,\ldots,n_k} = E_{n_1} + E_{n_2} + \cdots + E_{n_k}$ with $V(E_{n_i}) = U_i$ we write $U_1 + U_2 + \cdots + U_k$.

2. **Non-Star Trees $T_n$ and the Graphs $G$ with $\chi(G) = 3$**

First we consider the graphs $G$ of order six with chromatic number $\chi(G) = 3$ and $G \notin \{K_{1,1,4}, K_{2,2,2}\}$. The following theorem states that for all these graphs $G$ every non-star tree $T_n$ is $G$-good.
\textbf{Theorem 2.1.} Let \( n \geq 4 \), \( T_n \neq S_n \), and let \( G \) be a graph of order six with \( \chi(G) = 3 \) where \( G \neq K_{1,1,4} \) and \( G \neq K_{2,2,2} \). Then

\[
 r(T_n, G) = \begin{cases} 
 2n - 1 & \text{if } G \subseteq K_{1,2,3}, \\
 2n & \text{otherwise.}
\end{cases}
\]

To prove Theorem 2.1 by induction on \( n \) the following properties of trees \( T_n \) are essential.

\textbf{Lemma 2.2.} (i) If \( n \geq 6 \) and \( T_n \notin \{ S_n, B_{n-3,3}, B_{n-3,3,3} \} \), then \( T_n \) contains vertices \( v_1 \) and \( v_2 \) of degree 1 with distance \( d(v_1, v_2) \geq 3 \) such that \( T_n - \{v_1, v_2\} \) is a non-star tree of order \( n - 2 \).

(ii) If \( n \geq 5 \) and \( T_n \neq S_n \), then \( T_n \) contains a vertex \( v \) of degree 1 such that \( T_n - \{v\} \) is a non-star tree of order \( n - 1 \).

\textbf{Proof.} Let \( P = u_0 u_1 \cdots u_\ell \) be a path of maximum length \( \ell \) in \( T_n \). Clearly, \( d(u_0) = d(u_\ell) = 1 \). Moreover, \( T_n \neq S_n \) implies \( \ell \geq 3 \).

(i) Since in a tree any two vertices are connected by a unique path, \( d(u_0, u_\ell) = \ell \geq 3 \). Consider the tree \( T^* = T_n - \{u_0, u_\ell\} \) of order \( n - 2 \). Obviously, \( T^* \neq S_{n-2} \) for \( \ell \geq 5 \). In case of \( \ell = 3 \), \( T^* \neq S_{n-2} \) also holds, since otherwise one of the vertices \( u_1 \) and \( u_2 \) has to be the centre of \( S_{n-2} \), and this yields \( T_n = B_{n-3,3} \), a contradiction. It remains \( \ell = 4 \). Then we are done if \( T^* \neq S_{n-2} \). In case of \( T^* = S_{n-2} \), \( u_2 \) has to be the centre of \( S_{n-2} \) and among the \( n - 3 \geq 3 \) vertices of degree 1 in \( T^* \) adjacent to \( u_2 \) we find a vertex \( w \) of degree 1 in \( T_n \). But then \( u_0 \) and \( w \) are vertices of degree 1 with \( d(u_0, w) \geq 3 \) such that \( T_n - \{u_0, w\} \) is a non-star tree of order \( n - 2 \).

(ii) Consider the tree \( T' = T_n - \{u_0\} \) of order \( n - 1 \). Clearly, \( T' \neq S_{n-1} \) for \( \ell \geq 4 \). It remains \( \ell = 3 \). Then we are done if \( T' \neq S_{n-1} \). In case of \( T' = S_{n-1} \), \( u_2 \) has to be the centre of \( S_{n-1} \) forcing \( T_n = B_{n-3,3,3} \) where \( n - 3 \geq 2 \). But then \( T_n - \{u_3\} \) is a non-star tree of order \( n - 1 \).

Besides Lemma 2.2 the values of \( r(T_n, P_3) \) and \( r(T_n, P_4) \) for \( T_n \neq S_n \) will be used to prove Theorem 2.1. Chvátal and Harary [4] obtained a formula to derive \( r(G, P_3) \) for any graph \( G \) depending on the edge independence number \( \beta_1(\overline{G}) \) of the complement \( \overline{G} \) of \( G \).

\textbf{Theorem 2.3} (Chvátal and Harary [4]). Let \( G \) be a graph of order \( n \). Then

\[
 r(G, P_3) = \begin{cases} 
 n & \text{if } \overline{G} \text{ contains a 1-factor}, \\
 2n - 2\beta_1(\overline{G}) - 1 & \text{otherwise.}
\end{cases}
\]

For every tree \( T_n \neq S_n \), \( \beta_1(T_n) = \lfloor n/2 \rfloor \). Applying Theorem 2.3 we obtain the following result.
Corollary 2.4. Let \( n \geq 4 \) and \( T_n \neq S_n \). Then \( r(T_n, P_3) = n \).

The next result on \( r(T_n, P_4) \) was already mentioned without proof by Faudree, Rousseau and Schelp in [7].

Theorem 2.5. Let \( n \geq 4 \) and \( T_n \neq S_n \). Then \( r(T_n, P_4) = n + 1 \).

Proof. Since \( \chi(P_4) = 2 \) and \( s(P_4) = 2 \) we obtain \( r(T_n, P_4) \geq n + 1 \) from (1). To prove that \( r(T_n, P_4) = n + 1 \) we use induction on \( n \). It is easy to check that \( r(T_n, P_4) \leq n + 1 \) holds for \( 4 \leq n \leq 5 \) if \( T_n \neq S_n \), i.e., \( T_n \in \{ P_4, P_5, B_{2,2,3} \} \) (cf. also [4] and [5]). Now let \( n \geq 6 \). By the induction hypothesis, \( r(T_k, P_4) \leq k + 1 \) for every tree \( T_k \neq S_k \) with \( 4 \leq k < n \). Suppose that a \((T_n, P_4)\)-coloring of \( K_{n+1} \) with vertex set \( V \) exists for some tree \( T_n \neq S_n \) of order \( n \).

Case 1. \( K_3 \subseteq |V|_g \). Let \( U = \{ u_1, u_2, u_3 \} \) be the vertex set of a green \( K_3 \) and \( W = V \setminus U \). Since \( P_4 \not\subseteq |V|_g \), all edges between \( U \) and \( W \) have to be red. Thus \( K_n \not\subseteq |V|_r \). By Lemma 2.2(i), \( T_n \) contains two vertices \( v_1 \) and \( v_2 \) of degree 1 with \( d(v_1, v_2) \geq 3 \) such that the tree \( T' = T_n - \{ v_1, v_2 \} \) of order \( n - 2 \) is not a star. The induction hypothesis yields \( r(T', P_4) \leq n - 1 \). Consider \( V' = V \setminus \{ u_1, u_2 \} \). Since \( |V'| = n - 1 \) and \( P_4 \not\subseteq |V'|_g \), we obtain that \( T' \subseteq |V'|_r \). Let \( a_1 \) and \( a_2 \) be the two vertices in \( T' \) such that \( a_i \) is adjacent to \( v_i \) in \( T_n \). Since \( d(v_1, v_2) \geq 3 \), \( a_1 \neq a_2 \). If \( a_1 \neq a_2 \) \( \subseteq W \), then the edges \( a_1 u_1 \) and \( a_2 u_2 \) together with \( T' \) would yield a red \( T_n \), a contradiction. If \( a_1 = u_3 \) or \( a_2 = u_3 \), say \( a_1 = u_3 \), then a vertex \( w \in W \) exists where \( w \notin V(T') \). But then the edges \( a_1 w \) and \( a_2 u_2 \) together with \( T' \) again yield a red \( T_n \).

Case 2. \( K_3 \not\subseteq |V|_g \). Let \( v \) be a vertex in \( V \) with \( d_g(v) = \Delta_g \). Corollary 2.4 and \( T_n \not\subseteq |V|_r \) force \( P_3 \subseteq |V|_g \), and this implies \( \Delta_g \geq 2 \). Let \( W = V \setminus \{ v \} \). As \( K_3 \not\subseteq |V|_g \) and \( P_4 \not\subseteq |V|_g \), in \( |W| \) every \( w \in N_g(v) \) is incident to red edges only. By Lemma 2.2(ii), \( T_n \) must contain a vertex \( u \) of degree 1 such that \( T' = T_n - \{ u \} \) is a tree of order \( n - 1 \) different from \( S_{n-1} \). Let \( w \in V(T') \) be the neighbor of \( u \) in \( T_n \). By the induction hypothesis, \( r(T', P_4) \leq n \). Since \( |W| = n \) and \( P_4 \not\subseteq |W|_g \), a red \( T' \) occurs in \( |W| \). If \( w \in N_r(v) \), then \( T' \) together with \( uw \) yields a red \( T_n \), a contradiction. It remains that \( w \in N_g(v) \). We already know that in \( |W| \) every \( w \in N_g(v) \) is incident to red edges only. Since \( |W| = n \), there is a vertex \( w' \in W \) with \( w' \notin V(T') \). But then \( T' \) together with \( ww' \) yields a red \( T_n \) and the proof is complete.

With these preparations we can now prove Theorem 2.1.

Proof of Theorem 2.1. By (1), \( r(T_n, G) \geq 2n - 1 \) for any graph \( G \) with \( \chi(G) = 3 \). If \( G \neq K_{1,1,4} \) and \( G \not\subseteq K_{1,2,3} \), then \( s(G) = 2 \), and (1) yields \( r(T_n, G) \geq 2n \). Moreover, \( s(G) = 2 \) and \( G \neq K_{2,2,2} \) imply \( G \subseteq K_{2,2,2} - e \). Thus, it suffices to
prove \( r(T_n, K_{1,2,3}) \leq 2n - 1 \) and \( r(T_n, K_{2,2,2} - e) \leq 2n \) for every tree \( T_n \neq S_n \) where \( n \geq 4 \). We use that the join \( E_2 + P_4 \) is isomorphic to \( K_{2,2,2} - e \) and we write \( \{v_1, v_2\} + P_4 \) if \( V(E_2) = \{v_1, v_2\} \). The proof consists of two parts: in (i) we derive the desired results for \( T_n = B_{n-3,3} \), and in (ii) we consider the trees \( T_n \notin \{S_n, B_{n-3,3}\} \).

(i) Let \( T_n = B_{n-3,3} \) where the degenerated broom \( B_{1,3} = P_4 \) is included. Suppose we have a \((B_{n-3,3}, K_{1,2,3})\)-coloring of \( K_{2n-1} \) or a \((B_{n-3,3}, K_{2,2,2} - e)\)-coloring of \( K_{2n} \). Let \( V \) denote the vertex sets of the complete graphs.

**Claim 2.6.** \( S_{n-1} \subseteq [V]_r \).

**Proof.** From [11] we know that \( r(S_{n-1}, G) \leq 2n - 1 \) if \( G = K_{1,2,3} \) or \( G = K_{2,2,2} - e \) and \( n \geq 5 \). Because of \( S_3 = P_3 \), \( r(P_3, G) = r(G, P_3) \) and Theorem 2.3 this upper bound also holds for \( n = 4 \). Thus, if \( K_{1,2,3} \not\subseteq [V]_g \) or \( K_{2,2,2} - e \not\subseteq [V]_g \), then \( S_{n-1} \subseteq [V]_r \). \( \square \)

**Claim 2.7.** \( S_n \not\subseteq [V]_r \).

**Proof.** Assume that \( S_n \subseteq [V]_r \), and let \( U \) be the vertex set of a red \( S_n \) with centre \( u_0 \). Since a red \( B_{n-3,3} \) is forbidden, \([U \setminus \{u_0\}]\) has to be a green \( K_{n-1} \). Moreover, all edges between \( W = V \setminus U \) and \( U \setminus \{u_0\} \) have to be green. This gives a green \( K_6 - K_3 \) in case of \( |V| = 2n - 1 \), i.e., \( |W| = n - 1 \), contradicting \( K_{1,2,3} \not\subseteq [V]_g \). In case of \( |V| = 2n \), i.e., \( |W| = n \), Corollary 2.4 and \( B_{n-3,3} \not\subseteq [V]_r \) imply that a green \( P_3 \) must occur in \( W \). This yields a green \( K_6 - e \), a contradiction to \( K_{2,2,2} - e \not\subseteq [V]_g \). \( \square \)

Now we use Claim 2.6 and consider a red \( S_{n-1} \) with vertex set \( U \) and centre \( u_0 \). By Claim 2.7 and \( B_{n-3,3} \not\subseteq [V]_r \), all edges between \( U \) and \( V = W \setminus U \) have to be green. In case of \( |V| = 2n - 1 \) it follows that \( |W| = n \), and Corollary 2.4 together with \( B_{n-3,3} \not\subseteq [V]_r \) imply that a green \( P_3 = w_1w_2w_3 \) occurs in \( W \). But then \( \{u_2\} + \{w_1, w_3\} + \{u_0, u_1, u_2\} \) where \( \{u_1, u_2\} \subseteq U \setminus \{u_0\} \) is a green \( K_{1,2,3} \), a contradiction. In case of \( |V| = 2n \) we obtain \( |W| = n + 1 \), and Theorem 2.5 together with \( B_{n-3,3} \not\subseteq [V]_r \) guarantee a green \( P_3 \) in \( W \). But this forces \( \{u_1, u_2\} + P_4 \) to be a green \( K_{2,2,2} - e \), a contradiction, and we are done for \( T_n = B_{n-3,3} \).

(ii) It remains that \( T_n \notin \{S_n, B_{n-3,3}\} \). We use induction on \( n \) to prove \( r(T_n, K_{1,2,3}) \leq 2n - 1 \) and \( r(T_n, K_{2,2,2} - e) \leq 2n \) for every tree \( T_n \notin \{S_n, B_{n-3,3}\} \) with \( n \geq 4 \).

First we derive the desired results for \( 4 \leq n \leq 5 \). There is only one tree \( T_n \notin \{S_n, B_{n-3,3}\} \) with \( 4 \leq n \leq 5 \), namely \( P_5 \). To prove \( r(P_5, K_{1,2,3}) \leq 9 \) and \( r(P_5, K_{2,2,2} - e) \leq 10 \) assume we have a \((P_5, K_{1,2,3})\)-coloring of \( K_9 \) or a \((P_5, K_{2,2,2} - e)\)-coloring of \( K_{10} \). Let \( V \) denote the vertex sets of the complete graphs. Since \( P_4 = B_{1,3} \), by the above result on brooms we already know that \( r(P_4, K_{1,2,3}) \leq 7 \) and \( r(P_4, K_{2,2,2} - e) \leq 8 \). Thus, a red \( P_4 = u_1u_2u_3u_4 \) must occur
in $[V]$, and $P_5 \not\subseteq [V]$ forces all edges between $\{u_1, u_4\}$ and the vertices in $W = V \setminus \{u_1, u_2, u_3, u_4\}$ to be green. In $K_6$ we obtain $|W| = 5$, and $r(P_5, S_4) = 5$ (cf. [5]) guarantees a green $S_4$ in $[W]$ with centre $w_0$ and vertices $w_1, w_2, w_3$ of degree 1 yielding the green $K_{1,2,3} = \{w_0\} + \{u_1, u_4\} + \{w_1, w_2, w_3\}$, a contradiction. In $K_{10}$ we have $|W| = 6$, and $r(P_5, P_4) = 6$ (see Theorem 2.5) forces a green $P_4$ in $[W]$. But then $\{u_1, u_4\} + P_4$ is a green $K_{2,2,2} - e$, a contradiction.

Now let $n \geq 6$. By the induction hypothesis, $r(T_k, K_{1,2,3}) \leq 2k - 1$ and $r(T_k, K_{2,2,2} - e) \leq 2k$ for every tree $T_k \not\subseteq \{S_k, B_{k-3,3}\}$ with $4 \leq k < n$. Suppose we have a $(T_n, K_{1,2,3})$-coloring of $K_{2n-1}$ or a $(T_n, K_{2,2,2} - e)$-coloring of $K_{2n}$ for some tree $T_n$ where $T_n \not\subseteq \{S_n, B_{n-3,3}\}$. Again we use $V'$ to denote the vertex sets of the complete graphs. By Lemma 2.2(i), $T_n$ contains two vertices $v_1$ and $v_2$ of degree 1 with distance $d(v_1, v_2) \geq 3$ such that the tree $T^* = T_n - \{v_1, v_2\}$ of order $n - 2$ is not a star. By the induction hypothesis and the above result on brooms, $r(T^*, K_{1,2,3}) \leq 2n - 5$ and $r(T^*, K_{2,2,2} - e) \leq 2n - 4$. Let $a_1$ and $a_2$ be the two vertices in $T^*$ such that $a_i$ is adjacent to $v_i$ in $T_n$, where $1 \leq i \leq 2$. Since $r(T_n, K_4 - e) = 2n - 1$ (see [2]), one of the following two cases must occur.

**Case 1.** $K_4 \subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green $K_4$ with minimal sum $d_r(u_1) + d_r(u_2) + d_r(u_3) + d_r(u_4)$ of $r$-degrees, and let $W = V \setminus U$. Since $|W| = 2n - 5$ in case of $K_{2n-1}$ and $|W| = 2n - 4$ in case of $K_{2n}$, $T^* \subseteq [W]$, we distinguish two subcases depending on $q_r(a_1, U)$.

**Case 1.1.** $q_r(a_1, U) \geq 1$ and $q_r(a_2, U) \geq 1$. Then $T_n \subseteq [V]_r$, except for $q_r(a_1, U) = q_r(a_2, U) = 1$ where $a_1$ and $a_2$ have the same red neighbor in $U$, say $u_1$. But this gives the green $K_{1,2,3} = \{u_2\} + \{u_3, u_4\} + \{u_1, a_1, a_2\}$, a contradiction for $|V| = 2n - 1$. In the remaining case $|V| = 2n$ let $W' = W \setminus V(T^*)$. Note that $|W'| = n - 2$. If $a_1$ or $a_2$ has a red neighbor in $W'$, then again a red $T_n$ occurs. Otherwise all $n + 1$ vertices in $W' \cup \{u_2, u_3, u_4\}$ are common green neighbors of $a_1$ and $a_2$, and Theorem 2.5 guarantees a green $P_4$ in $[W' \cup \{u_2, u_3, u_4\}]$. But this forces $\{a_1, a_2\} + P_4$ to be a green $K_{2,2,2} - e$, a contradiction.

**Case 1.2.** $q_r(a_1, U) = 0$ or $q_r(a_2, U) = 0$, say $q_r(a_1, U) = 0$. Now let $U' = U \cup \{a_1\}$ and $W' = V \setminus U'$. Note that $|U'|$ is a green $K_3$ and that $|W' \cap V(T^*)| = n - 3$. If $q_r(w, U') \leq 2$ for some $w \in W'$, then we find a green $K_{1,2,3}$ and a green $K_{2,2,2} - e$ in $[U' \cup \{w\}]$, a contradiction. Thus $q_r(w, U') \geq 3$ for every $w \in W'$ yielding $q_r(W', U') \geq 3|W'| \geq 3(2n - 6)$. This implies $q_r(u, U') = d_r(u) \geq n - 2$ for some $u \in U'$. In case of $d_r(a_1) \leq n - 3$ we may assume that $d_r(u_4) \geq n - 2$. But then the green $K_4 = \{u_1, u_2, u_3\}$ would have a smaller sum of $r$-degrees than the green $K_4 = \{u_1, u_2, u_3, u_4\}$. It remains $d_r(a_1) \geq n - 2$. This forces $q_r(a_1, W') \geq n - 2$ and we find a red neighbor $w'$ of $a_1$ in $W' \setminus V(T^*)$ since $|W' \cap V(T^*)| = n - 3$. Moreover, $q_r(w, U') \geq 3$ for every $w \in W'$ yields a red neighbor $w$ of $a_2$ in $U$. But then $T^*$ together with $w$ and $w$ produce a red $T_n$, a contradiction.
Case 2. $K_4 - e \subseteq [V]_g$ and $K_4 \not\subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green $K_4 - e$ where $u_1 u_4$ is red, and let $W = V \setminus U$. Since $K_4 \not\subseteq [V]_g$, $q_r(w, U) \geq 1$ for every $w \in W$. As in Case 1, $T^* \subseteq [W]_r$, and $T_n \not\subseteq [V]_r$ forces $q_r(a_1, U) = q_r(a_2, U) = 1$. Moreover, $a_1$ and $a_2$ must have the same red neighbor in $U$, and $K_4 \not\subseteq [V]_g$ implies that $u_2$ or $u_3$, say $u_2$, is the common red neighbor. But then we obtain the green $K_{1,2,3} = \{u_3\} + \{u_1, u_4\} + \{w_2, a_1, a_2\}$, a contradiction for $|V| = 2n - 1$. In the remaining case $|V| = 2n$ let $W' = W \setminus V(T^*)$. Note that $|W'| = n - 2$. If $a_1$ or $a_2$ has a red neighbor in $W'$, then a red $T_n$ occurs. Otherwise, the $n + 1$ vertices in $W' \cup \{u_1, u_3, u_4\}$ are common green neighbors of $a_1$ and $a_2$, and Theorem 2.5 guarantees a green $P_4$ in $[W' \cup \{u_1, u_3, u_4\}]$. But this gives a green $K_{2,2,2} - e$ and we are done.

The two graphs $G$ of order six with $\chi(G) = 3$ not considered in Theorem 2.1 are $G = K_{1,1,4}$ and $G = K_{2,2,2}$. The values of $r(T_n, K_{1,1,4})$ for $n \geq 9$ follow from a more general result due to Erdős, Faudree, Rousseau and Schelp [6] who investigated $r(T_n, B_m)$ for any tree $T_n$ and the book-graph $B_m = K_{1,1,m}$.

**Theorem 2.8** (Erdős, Faudree, Rousseau and Schelp [6]). If $n \geq 3m - 3$, then

$$r(T_n, B_m) = 2n - 1.$$ 

Applying Theorem 2.8 for $B_4 = K_{1,1,4}$ we obtain $r(T_n, K_{1,1,4}) = 2n - 1$ for any tree $T_n$ with $n \geq 9$. A result due to Rousseau and Sheehan [18] implies $r(P_n, K_{1,1,4}) = 10$ for $4 \leq n \leq 5$ and $r(P_n, K_{1,1,4}) = 2n - 1$ for $n \geq 6$. Moreover, in [13] we determined the missing values of $r(T_n, K_{1,1,4})$ for $n \leq 8$. This proves that any non-star tree $T_n$ with $n \geq 6$ is $K_{1,1,4}$-good.

**Theorem 2.9.** Let $n \geq 4$ and $T_n \neq S_n$. Then

$$r(T_n, K_{1,1,4}) = \begin{cases} 
10 & \text{if } 4 \leq n \leq 5, \\
2n - 1 & \text{if } n \geq 6.
\end{cases}$$

For the remaining graph $G = K_{2,2,2}$ the situation is much more complicated. From (1) we obtain $r(T_n, K_{2,2,2}) \geq 2n$. On the other hand, for $n$ sufficiently large we know that $r(S_n, K_{2,2,2}) > 2n - 2 + \lfloor \sqrt{n - 1} - 6(n - 1)^{11/40} \rfloor$ (see [12], note that $K_{2,2,2} = K_6 - 3K_2$) forcing $r(T_n, K_{2,2,2}) > 2n$ also for non-star trees $T_n$ with maximum degree close to $n - 1$ if $n$ is sufficiently large. Nevertheless, there are non-star trees with small maximum degree where the lower bound $2n$ is attained. For $T_n = P_n$ this follows from a more general result due to Gould and Jacobson [8] who proved that any path $P_n$ with $n \geq 3$ is $(K_{2m} - mK_2)$-good.

**Theorem 2.10** (Gould and Jacobson [8]). If $n \geq 3$ and $m \geq 2$, then

$$r(P_n, K_{2m} - mK_2) = (n - 1)(m - 1) + 2.$$
The following theorem shows that \( r(T_n, K_{2,2,2}) = 2n \) also holds for a special class of trees \( T_n \) with \( \Delta(T_n) = 3 \).

**Theorem 2.11.** Let \( T_n \) be a tree of order \( n \geq 5 \) with \( \Delta(T_n) = 3 \) containing a path \( P_{n-1} \). Then

\[
r(T_n, K_{2,2,2}) = 2n.
\]

To prove Theorem 2.11 we use a result due to Burr, Erdős, Faudree, Rousseau and Schelp [1] who obtained a formula to determine \( r(T_n, C_4) \) depending on \( r(S_{m+1}, C_4) \) where \( m = \Delta(T_n) \).

**Theorem 2.12** (Burr, Erdős, Faudree, Rousseau and Schelp [1]). If \( T_n \) is a tree with \( \Delta(T_n) = m \), then \( r(T_n, C_4) = \max \{ 4, n+1, r(S_{m+1}, C_4) \} \).

Thus, \( r(T_n, C_4) \) is easily evaluated if \( r(S_{m+1}, C_4) \) is known, but \( r(S_{m+1}, C_4) \) has not yet been completely determined (see Parsons [15] and Wu, Sun, Zhang and Radziszowski [19]).

**Proof of Theorem 2.11.** It suffices to prove that \( r(T_n, K_{2,2,2}) \leq 2n \). Let \( T_n \) be a tree with \( \Delta(T_n) = 3 \) containing a path \( P_{n-1} \) and suppose we have a \( (T_n, K_{2,2,2}) \)-coloring of \( K_{2n} \) with vertex set \( V \).

**Claim 2.13.** \( |N_g(v_1) \cap N_g(v_2)| \leq n \) for any two vertices \( v_1 \) and \( v_2 \).

**Proof.** Assume that there are vertices \( v_1 \) and \( v_2 \) with \( |N_g(v_1) \cap N_g(v_2)| \geq n+1 \). Since \( r(S_4, C_4) = 6 \) (cf. [4]), Theorem 2.12 states \( r(T_n, C_4) = n+1 \). Thus, \( T_n \not\subseteq [V]_r \) forces a green \( C_4 = (w_1w_2w_3w_4) \) in \( [N_g(v_1) \cap N_g(v_2)] \). But this gives the green \( K_{2,2,2} = \{v_1, v_2\} + \{w_1, w_3\} + \{w_2, w_4\} \), a contradiction.

By Theorem 2.10 and \( K_{2,2,2} = K_6 - 3K_2 \), a red \( P_{n-1} = u_1u_2 \cdots u_{n-1} \) must occur. First let \( n \) be odd or, in case of \( n \) even, let \( T_n \) not be isomorphic to the tree obtained from \( u_1u_2 \cdots u_{n-1} \) by joining a vertex \( w \in W = V \setminus \{u_1, \ldots, u_{n-1}\} \) to \( u_{n/2} \). Then \( T_n \not\subseteq [V]_r \), implies that there is some \( i \) with \( 1 \leq i \leq \lfloor (n-1)/2 \rfloor - 1 \) such that \( u_{i+1} \) and \( u_{n-1-i} \) are joined green to all \( n+1 \) vertices in \( W \), a contradiction to Claim 2.13. Consider now the remaining case for \( n \) even. Since \( T_n \not\subseteq [V]_r \), all edges from \( u_{n/2} \) to \( W \) have to be green, and then Claim 2.13 forces at least one red edge from every \( u_i \) with \( i \neq n/2 \) to \( W \). Moreover, two independent red edges between \( \{u_1, u_{n/2-1}\} \) and \( W \) would yield a red \( T_n \). Thus we may assume that \( u_1 \) and \( u_{n/2-1} \) have a common red neighbor \( w^* \in W \) and that all edges between \( \{u_1, u_{n/2-1}\} \) and \( W \setminus \{w^*\} \) are green. Then \( T_n \not\subseteq [V]_r \) forces \( u_1u_{n-1} \) to be green. Furthermore, by Claim 2.13, the edges \( u_{n/2-1}u_{n-1} \) and \( u_{n/2}u_{n-1} \) have to be red. Remind that a red edge \( u_{n-1}w \) with \( w \in W \) must occur. But then the red path \( P_{n-1} = u_1 \cdots u_{n/2-1}u_{n-1}u_{n/2} \cdots u_n \) together with the red edge \( u_{n-1}w \) yields the forbidden red \( T_n \), a contradiction, and we are done.
3. Trees $T_n \in \{P_n, B_{2,n-2}\}$ and the Graphs $G$ with $\chi(G) = 2$

It seems to be out of reach to determine the exact value of $r(T_n, G)$ for every tree $T_n$ and all connected bipartite graphs $G$ of order six, i.e., the star $S_6 = K_{1,5}$ and the connected spanning subgraphs of $K_{2,4}$ and $K_{3,3}$. Burr, Erdős, Faudree, Rousseau and Schelp [1] derived upper bounds for $r(T_n, K_{2,4})$ and $r(T_n, K_{3,3})$. They proved that for all sufficiently large $n$,

$$r(T_n, K_{2,4}) < n + 3n^{1/2}.$$ 

Moreover they showed that there exists a constant $c$ such that for every tree $T_n$ with maximum degree $\Delta(T_n) = m$,

$$r(T_n, K_{3,3}) \leq \max \left\{ n + \left\lceil cn^{1/3} \right\rceil, r(S_{m+1}, K_{3,3}) \right\}$$

and

$$r(S_{m+1}, K_{3,3}) < m + 3m^{2/3}$$

for $m$ sufficiently large. Lower bounds can be obtained from $r(T_n, C_4)$ since $C_4 \subseteq K_{2,4}$ and $C_4 \subseteq K_{3,3}$. In [1] it was proved that for all sufficiently large $n$,

$$r(S_{m+1}, C_4) > m + \left\lceil m^{1/2} - 6m^{11/40} \right\rceil.$$ 

This together with Theorem 2.12 implies that $r(T_n, K_{2,4})$ and $r(T_n, K_{3,3})$ differ considerably from the lower bound (1) if $n$ is sufficiently large and $\Delta(T_n) = m$ is close to $n - 1$. Clearly, the same holds for $r(T_n, G)$ if $G$ is any bipartite graph with $C_4 \subseteq G$. Here we restrict ourselves to study $r(T_n, G)$ for two trees with small maximum degree, namely $T_n \in \{P_n, B_{2,n-2}\}$. The choice of these two trees is essentially due to our project to determine $r(T_n, G)$ for every connected graph of order six and all trees of order at most five — the only non-star trees on at most five vertices are the paths $P_4$ and $P_5$ and the broom $B_{2,3}$. Our results show that, except for some small $n$, the trees $T_n \in \{P_n, B_{2,n-2}\}$ are $G$-good for any connected bipartite graph $G$ of order $p(G) = 6$, i.e., $r(T_n, G)$ attains the general lower bound from (1). Instead of $r(T_n, S_6)$ here we consider the more general case $r(T_n, S_m)$. We start by improving the lower bound (1) for $T_n \in \{P_n, B_{2,n-2}\}$ and any connected bipartite graph $G$ in case of small $n$.

**Lemma 3.1.** Let $G \subseteq K_{m_1,m_2}$ be a connected graph of order $m = m_1 + m_2$ where $1 \leq m_1 \leq m_2$. Then $r(P_n, G) \geq m - 1 + \lceil n/2 \rceil$ for $n \geq 2$ and $r(B_{2,n-2}, G) \geq m - 1 + \lceil (n-1)/2 \rceil$ for $n \geq 5$.

**Proof.** From (1) it follows that $r(G, T_n) \geq m - 1 + s(T_n)$. Due to $r(F, G) = r(G, F)$, $s(P_n) = \lceil n/2 \rceil$ for $n \geq 2$ and $s(B_{2,n-2}) = \lceil (n-1)/2 \rceil$ for $n \geq 5$ we obtain the desired results. \[ \blacksquare \]
If $G$ is a connected spanning subgraph of $K_{m_1,m_2}$ with $1 \leq m_1 \leq m_2$, then $s(G) = m_1$, and the general lower bound (1) implies $r(T_n, G) \geq n + m_1 - 1$ for any tree $T_n$. Hence the general lower bound is improved by the lower bounds from Lemma 3.1 for $T_n = P_n$ if $n \leq 2m_2 - 2$ and for $T_n = B_{2,n-2}$ if $n \leq 2m_2 - 3$. The following lemma shows that in case of $T_n = B_{2,n-2}$ the general lower bound can also be improved for $n = 2m_2 - 2$ or $n = 2m_2$ and certain graphs $G \subseteq K_{m_1,m_2}$.

**Lemma 3.2.** Let $n \geq 6$ be even and let $m_1 \leq m_2$. Then $r(B_{2,n-2}, G) \geq n + m_1$ if $m_1 \geq 1$, $n = 2m_2$ and $G = K_{m_1,m_2}$ or if $m_1 \geq 2$, $n = 2m_2 - 2$ and $G \in \{K_{m_1,m_2} - e, K_{m_1,m_2} - 2K_2\}$. Moreover, $r(B_{2,3}, K_{m_1,m_2}) \geq m_1 + m_2 + 2$.

**Proof.** For $n = 2m_2$, the coloring of $K_{n+m_1-1}$ with $[V]_g = 2K_{m_2} + \overline{K_{m_1-1}}$ contains no red $B_{2,n-2}$ and no green $K_{m_1,m_2}$. For $n = 2m_2 - 2$, the coloring of $K_{n+m_1-1}$ with $[V]_g = 2K_{m_2-1} + \overline{K_{m_1-1}}$ contains no red $B_{2,n-2}$ and no green $K_{m_1,m_2} - 2K_2$. Moreover, the coloring of $K_{m_1,m_2+1}$ with $[V]_r = C_{m_1,m_2+1}$ contains no red $B_{2,3}$ and no green $K_{m_1,m_2}$.

Now we consider $r(T_n, S_m)$. Parsons [14] has already determined the exact value of $r(P_n, S_m)$ by explicit formulas and a recurrence.

**Theorem 3.3** (Parsons [14]). Let $n \geq 4$ and $m \geq 4$. Then

$$r(P_n, S_m) = \begin{cases} 2m - 3 & \text{if } m - 1 \leq n < 2m - 3, \\ n & \text{if } n \geq 2m - 3, \end{cases}$$

and $r(P_n, S_m) = \max\{r(P_{n-1}, S_m), r(P_n, S_{m-(n-1)}{2m-3}) + n - 1\}$ if $n < m - 1$.

**Remark.** For $n \geq 4$ and $m = 5$ only $r(P_4, S_5)$ is not explicitly given by Theorem 3.3. Applying the recurrence and Theorem 2.3 we derive $r(P_4, S_5) = 7$.

We use the result of Parsons to completely determine the exact values of $r(B_{2,n-2}, S_m)$ if $n \geq m - 1$.

**Theorem 3.4.** Let $n \geq 5$ and $m \geq 4$. Then $r(B_{2,3}, S_4) = 6$ and

$$r(B_{2,n-2}, S_m) = \begin{cases} 2m - 3 & \text{if } m - 1 \leq n \leq 2m - 3 \text{ and } m \geq 5, \\ n + 1 & \text{if } n = 2m - 2, \\ n & \text{if } n \geq 2m - 1. \end{cases}$$

To prove Theorem 3.4 the straightforward statements of the following lemma will be used.

**Lemma 3.5.** Let $n \geq 5$ and let $\chi$ be a coloring of a complete graph with vertex set $V$ and $P_n = u_1 \cdots u_n \subseteq [V]_r$, but $B_{2,n-2} \not\subseteq [V]_r$. Then $u_iu_3$, $u_1u_{n-1}$, $u_2u_n$ and $u_{n-2}u_n$ have to be green. Furthermore, if $n \geq 7$ and $u_iu_i$ is red for some $i$ with $5 \leq i \leq n - 2$, then $u_{i-2}u_n$ has to be green.
Proof of Theorem 3.4. In [5] it was already shown that \( r(B_{2,3},S_4) = 6 \). From (1) we obtain \( r(B_{2,n-2},S_m) \geq n \). Lemma 3.2 yields \( r(B_{2,n-2},S_m) \geq n + 1 \) if \( n = 2m - 2 \). Moreover, the coloring of \( K_{2m-2} \) with \([V]_r = 2K_{m-2}\) shows \( r(B_{2,n-2},S_m) \geq 2m - 3 \) for \( n \geq m - 1 \). Thus, to establish the results from Theorem 3.4 it suffices to prove that \( r(B_{2,n-2},S_m) \geq n \) for \( n \geq 2m - 1 \) with \( m \geq 4 \) and for \( n = 2m - 3 \) with \( m \geq 5 \) by using the monotonicity property \( r(B_{2,n-2},S_m) \leq r(B_{2,n'},S_m) \) for \( n < n' \). To obtain the desired upper bounds suppose that we have a \( (B_{2,n-2},S_m) \)-coloring of \( K_n \) with vertex set \( V \) where \( n \geq 2m - 1 \) with \( m \geq 4 \) or \( n = 2m - 3 \) with \( m \geq 5 \). Then, by Theorem 3.3, a red \( P_n = u_1u_2\cdots u_n \) occurs. Note that \( S_m \not\subseteq [V]_g \) forces \( \Delta_g \leq m - 2 \). By Lemma 3.5, \( u_1u_3, u_1u_{n-1}, u_2u_n \) and \( u_{n-2}u_n \) have to be green.

Case 1. \( n \geq 2m - 1 \) where \( m \geq 4 \). Since \( d_g(u_1) \leq m - 2 \), \( q_g(u_1,\{u_5,\ldots, u_{n-2}\}) \leq m - 4 \). Thus, \( q_r(u_1,\{u_5,\ldots, u_{n-2}\}) \geq n - 6 - (m - 4) \geq 2m - 1 - m - 2 = m - 3 \), and Lemma 3.5 implies \( q_g(u_n,\{u_3,\ldots, u_{n-4}\}) \geq m - 3 \). But this yields \( d_g(u_n) \geq m - 1 \), a contradiction.

Case 2. \( n = 2m - 3 \) where \( m \geq 5 \). We distinguish two subcases depending on the color of \( u_1u_n \).

Case 2.1. \( u_1u_n \) is green. Then \( q_g(u_1,\{u_5,\ldots, u_{n-2}\}) \leq m - 5 \) because \( d_g(u_1) \leq m - 2 \). This forces \( q_r(u_1,\{u_5,\ldots, u_{n-2}\}) \geq n - 6 - (m - 5) = 2m - 3 - m - 1 = m - 4 \), and Lemma 3.5 yields \( q_g(u_n,\{u_3,\ldots, u_{n-4}\}) \geq m - 4 \). Again we obtain \( d_g(u_n) \geq m - 1 \), a contradiction.

Case 2.2. \( u_1u_n \) is red, i.e., \( C_n = (u_1u_2\cdots u_n) \) is a red cycle. The remaining edges are the diagonals \( u_iu_{i+\ell} \) of length \( \ell \) with \( \ell = 2,\ldots, m - 2 \) and \( i = 1,\ldots, n \), where the indices should be read modulo \( n \). To finish Case 2.2 we use the following properties of the diagonals of \( C_n \).

Claim 3.6. If a diagonal \( u_iu_{i+\ell} \) of length \( \ell \) with \( 2 \leq \ell \leq m - 2 \) and \( 1 \leq i \leq n \) is red, then also \( u_{i+1}u_{i+\ell+1} \) has to be red.

Proof. If \( u_iu_{i+\ell} \) is red and \( u_{i+1}u_{i+\ell+1} \) is green, then the end-vertices of the red \( P_n = u_{i+\ell+1}u_{i+\ell+2}\cdots u_{n+1}\cdots u_{i+\ell}u_{i+\ell-1}\cdots u_{i+1} \) are joined green, a situation already considered in Case 2.1.

Claim 3.7. For \( 2 \leq \ell \leq m - 2 \), all diagonals of length \( \ell \) must have the same color.

Proof. This is an immediate consequence of Claim 3.6.

Claim 3.8. If the diagonals of length \( \ell \) with \( 2 \leq \ell \leq m - 3 \) are red, then the diagonals of length \( \ell + 1 \) have to be green.
Proof. Assume that the diagonals of length \(\ell + 1\) are also red. Using the diagonal \(u_1u_{\ell+1}\) of length \(\ell\), the diagonal \(u_2u_{\ell+3}\) of length \(\ell + 1\) and edges from the red \(C_n\) we obtain the red \(B_{2,n-2}\) with bristles \(u_1u_{\ell+1}, u_{\ell+1}u_{\ell+2}\) and handle \(u_{\ell+1}u_\ell \cdots u_2u_{\ell+3} \cdots u_n\), a contradiction. \(\Box\)

Claim 3.9. The diagonals of length \(\ell\) with \(2 \leq \ell \leq 3\) and, for \(m \geq 6\), also the diagonals of length \(\ell = 4\) have to be green.

Proof. Assume that for some \(\ell\) with \(2 \leq \ell \leq 4\) the diagonals of length \(\ell\) are red. If \(\ell = 2\), then \(u_1u_3\) together with edges of the red \(C_n\) give the red \(B_{2,n-2}\) with bristles \(u_1u_3, u_2u_3\) and handle \(u_3u_4 \cdots u_n\), a contradiction. If \(\ell = 3\), then the red \(B_{2,n-2}\) with bristles \(u_1u_2, u_4u_n\) and handle \(u_1u_4u_3u_6 \cdots u_{n-4}u_{n-1}u_{n-2}\) would occur. If \(\ell = 4\) and \(m \geq 6\), then the diagonals \(u_1u_5\) and \(u_3u_7\) together with edges from the red \(C_n\) would yield the red \(B_{2,n-2}\) with bristles \(u_2u_3, u_3u_4\) and handle \(u_3u_7u_6u_5u_1u_nu_{n-1} \cdots u_8\). \(\Box\)

Now we finish Case 2.2 by deriving a contradiction to \(\Delta_g \leq m - 2\). Note that for \(2 \leq \ell \leq m - 2\) every \(u_i\) is incident to two diagonals of length \(\ell\). Thus, Claim 3.9 yields the desired contradiction for \(5 \leq \ell \leq 7\). In the remaining case \(m \geq 8\) we additionally have to consider the diagonals of length \(\ell \geq 5\). There are \(m - 6\) different diagonal lengths \(\ell\) with \(5 \leq \ell \leq m - 2\) and Claim 3.8 implies that at least \([\frac{(m-6)}{2}]\) of them belong to green diagonals. Hence \(d_g(u_i) \geq 6 + 2[(m-6)/2] \geq m - 1\), a contradiction, and we are done. \(\blacksquare\)

In the following two theorems \(r(P_n, G)\) and \(r(B_{2,n-2}, G)\) are determined for any connected spanning subgraph \(G\) of \(K_{2,4}\).

Theorem 3.10. Let \(n \geq 4\) and let \(G\) be a connected graph of order six where \(G \subseteq K_{2,4}\). Then

\[
    r(P_n, G) = \begin{cases} 
        7 & \text{if } 4 \leq n \leq 5, \\
        8 & \text{if } n = 6, \\
        n + 1 & \text{otherwise.}
    \end{cases}
\]

Proof. From (1) we obtain \(r(P_n, G) \geq n + 1\). Moreover, Lemma 3.1 implies \(r(P_n, G) \geq 7\) for \(4 \leq n \leq 5\) and \(r(P_6, G) \geq 8\). To establish equality it suffices to show \(r(P_5, K_{2,4}) \leq 7\) and \(r(P_n, K_{2,4}) \leq n + 1\) for \(n \geq 7\). Consider any coloring of \(K_7\) not containing a red \(P_5\) and any coloring of \(K_{n+1}, n \geq 7\), not containing a red \(P_n\). We have to prove that a green \(K_{2,4}\) occurs. Let \(P_k = u_1 \cdots u_k\) be a red path of maximum length, \(U = \{u_1, \ldots, u_k\}\) and \(W = V \setminus U\) where \(V\) denotes the vertex sets of the complete graphs. If \(k = 1\), then only green edges occur and we find a green \(K_{2,4}\). Now let \(k \geq 2\). The maximality of \(k\) forces that \(u_1\) and \(u_k\) are joined green to all vertices in \(W\). This yields a green \(K_{2,4}\) if \(|W| \geq 4\). It remains \(|W| = 3\) in case of \(K_7\) and \(2 \leq |W| \leq 3\) in case of \(K_{n+1}, n \geq 7\).
Case 1. $|W| = 3$. Then $k = n - 1 = 4$ in case of $K_7$ and $k = n - 2 \geq 5$ in case of $K_{n+1}$, $n \geq 7$. Let $W = \{w_1, w_2, w_3\}$. Only green edges between $W$ and $\{u_2, u_{k-1}\}$ imply a green $K_{2,4}$. Otherwise we may assume that $u_3w_1$ is red. Since $P_{k+1} \not\subseteq [V]_r$, $w_1$ has to be joined green to $w_2$, $w_3$ and $u_3$. Furthermore, $u_1u_3$ and $u_1u_k$ have to be green, and we obtain the green $K_{2,4} = \{u_1, w_1\} + \{w_2, w_3, u_3, u_k\}$.

Case 2. $|W| = 2$ in case of $K_{n+1}$, $n \geq 7$. This implies $k = n - 1$. Let $W = \{w_1, w_2\}$. If $K_{2,4} \not\subseteq [V]_9$, then at most one vertex from $\{u_2, \ldots, u_{n-2}\}$ is joined green to $w_1$ and to $w_2$. Therefore we may assume that every vertex in $\{u_2, \ldots, u_{(n-1)/2}\}$ is joined red to $w_1$ or to $w_2$. Note that $[(n-1)/2] \geq 3$. Since $P_{k+1} \not\subseteq [V]_r$, a common red neighbor of $u_2$ and $u_3$ in $W$ is forbidden. Thus, we may assume that $u_2w_1$ and $u_3w_2$ are red. Then $P_{k+1} \not\subseteq [V]_r$ forces $w_1w_2$, $w_1u_3$, $w_1u_4$, $u_1u_3$, $u_1u_4$ and $u_1u_{n-1}$ to be green, and this yields the green $K_{2,4} = \{u_1, w_1\} + \{w_2, u_3, u_4, u_{n-1}\}$. ☐

Theorem 3.11. Let $n \geq 5$ and let $G$ be a connected graph of order six where $G \subseteq K_{2,4}$. Then, if $G \not\subseteq K_{2,4}$,

$$r(B_{2,n-2}, G) = \begin{cases} 7 & \text{if } n = 5, \\ 8 & \text{if } n = 6 \text{ and } K_{2,4} - 2K_2 \subseteq G, \\ n + 1 & \text{otherwise,} \end{cases}$$

and

$$r(B_{2,n-2}, K_{2,4}) = \begin{cases} 8 & \text{if } n \leq 7, \\ 10 & \text{if } n = 8, \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof. From (1) we obtain $r(B_{2,n-2}, G) \geq n+1$, and Lemma 3.1 yields $r(B_{2,3}, G) \geq 7$. Lemma 3.2 gives $r(B_{2,3}, K_{2,4}) \geq 8$, $r(B_{2,4}, K_{2,4} - 2K_2) \geq 8$ and $r(B_{2,6}, K_{2,4}) \geq 10$. To establish equality it suffices to show that $r(B_{2,3}, K_{2,4} - e) \leq 7$, $r(B_{2,4}, G^*) \leq 7$ for $G^*$ obtained from $K_{2,4}$ by deleting two edges incident to the same vertex of degree 4 and $r(B_{2,n-2}, G) \leq n + 1$ for $G = K_{2,4}$ if $n = 7$ or $n \geq 9$ and for $G = K_{2,4} - e$ if $n = 8$.

To verify that $r(B_{2,3}, K_{2,4} - e) \leq 7$ and $r(B_{2,4}, G^*) \leq 7$ consider any coloring of $K_7$ with vertex set $V$. If a green $K_{2,4}$ occurs, then we are done. Otherwise, by Theorem 3.10, a red $P_5 = u_1 \cdots u_5$ must occur. Let $U = \{u_1, \ldots, u_5\}$ and let $W = V \setminus U = \{w_1, w_2\}$. Assume first that $B_{2,3} \not\subseteq [V]_r$. Then all edges between $W$ and $\{u_2, u_3, u_4\}$ have to be green. Moreover, at least one edge from $u_1$ to $W$ must be green yielding a green $K_{2,4} - e$. Suppose now that $B_{2,4} \not\subseteq [V]_r$. Then all edges between $W$ and $\{w_2, u_4\}$ have to be green. If $w_1$ or $w_2$ is joined green to both $u_1$ and $u_5$, then a green $G^*$ occurs. Neither $u_1$ nor $u_5$ can be joined red to $w_1$ and to $w_2$ since $B_{2,4} \not\subseteq [V]_r$. Thus we may assume that $u_1w_1$ and $u_5w_2$ are
green and that \( u_1w_2 \) and \( u_5w_1 \) are red. But then \( B_{2,4} \not\subseteq [V] \), forces \( u_3w_1 \) to be green, and we obtain a green \( G^* \).

To prove that \( r(B_{2,n-2}, G) \leq n + 1 \) for \( G = K_{2,4} \) if \( n = 7 \) or \( n \geq 9 \) and for \( G = K_{2,4} - e \) if \( n = 8 \) consider any coloring of \( K_{n+1} \), \( n \geq 7 \), not containing a red \( B_{2,n-2} \). Let \( V = V(K_{n+1}) \). We have to show that a green \( K_{2,4} - e \) occurs in case of \( n = 8 \) and a green \( K_{2,4} \) otherwise.

**Case 1.** There is a red cycle \( C_k = (u_1 \cdots u_k) \) of length \( k = n \) or \( k = n + 1 \). Let \( U = \{u_1, \ldots, u_k\} \). We consider two subcases depending on \( k \).

**Case 1.1.** \( k = n \). Then \( B_{2,n-2} \not\subseteq [V] \), implies that all edges between \( U \) and the vertex \( w \in V \setminus U \) are green. By Theorem 3.4, \( r(B_{2,n-2}, S_5) = n \) if \( n = 7 \) or \( n \geq 9 \), and \( r(B_{2,n-2}, S_4) = n \) if \( n = 8 \). This yields a green \( S_5 \) in \( [U] \) for \( n = 7 \) and for \( n \geq 9 \) and a green \( S_4 \) in \( [U] \) for \( n = 8 \). Together with the green edges incident to \( w \) we obtain a green \( K_{2,4} \) and a green \( K_{2,4} - e \), respectively.

**Case 1.2.** \( k = n + 1 \). Then \( B_{2,n-2} \not\subseteq [V] \) forces all diagonals of length \( \ell \leq 3 \) to be green. If, in addition, all diagonals of length \( \ell = 4 \) are green, then \( \{u_1, u_2\} + \{u_4, u_5, u_{n-1}, u_n\} \) is a green \( K_{2,4} \). The remaining case is that at least one diagonal of length 4, say \( u_1u_5 \), is red. Any red diagonal of length \( \ell \geq 4 \) incident to \( u_3 \) yields a red \( B_{2,n-2} \) with bristles \( u_2u_3 \) and \( u_3u_4 \), a contradiction. Otherwise all diagonals of length \( \ell \geq 2 \) incident to \( u_3 \) are green. Thus, \( u_1, u_6 \) and \( u_7 \) are common green neighbors of \( u_3 \) and \( u_4 \). If \( u_4u_{n+1} \) is also green, then \( \{u_3, u_4\} + \{u_1, u_6, u_7, u_{n+1}\} \) is a green \( K_{2,4} \). On the other hand, if \( u_4u_{n+1} \) is red, then all diagonals incident to \( u_2 \) have to be green since \( B_{2,n-2} \not\subseteq [V] \). But then \( u_2 \) and \( u_3 \) have at least four common green neighbors and again a green \( K_{2,4} \) occurs.

**Case 2.** Every red cycle has length at most \( n - 1 \). If \( K_{2,4} \not\subseteq [V] \), then we are done. Otherwise, by Theorem 3.10, a red \( P_n = u_1 \cdots u_n \) occurs. Let \( U = \{u_1, \ldots, u_n\} \) and let \( w \) be the vertex in \( V \setminus U \). Since \( B_{2,n-2} \not\subseteq [V] \), the edges \( wu_2, wu_3, wu_{n-2} \) and \( wu_{n-1} \) have to be green. By Lemma 3.5, the edges \( u_1u_3, u_1u_{n-1}, u_2u_n \) and \( u_{n-2}u_n \) are green. Moreover, \( C_n \not\subseteq [V] \) forces \( u_1u_n \) to be green, and \( C_{n+1} \not\subseteq [V] \) implies that at least one of the edges \( wu_1 \) and \( wu_n \), say \( wu_n \), is green. To avoid a green \( K_{2,4} = \{u_1, w\} + \{u_2, u_{n-2}, u_{n-1}, u_n\} \), \( u_1u_{n-2} \) has to be red. Then, by Lemma 3.5, \( u_{n-4}u_n \) must be green. Furthermore, \( C_n \not\subseteq [V] \) implies that \( u_{n-3}u_{n-1} \) is green, and \( B_{2,n-2} \not\subseteq [V] \) forces \( u_{n-3}u_n \) to be green. If \( n = 7 \), then \( wu_1, wu_4 \) and \( u_3u_6 \) have to be red as otherwise \( \{w, u_7\} + \{u_1, u_2, u_3, u_4, u_5\} \) contains a green \( K_{2,4} \) or \( \{u_6, u_7\} + \{u_1, u_3, u_4, w\} \) is a green \( K_{2,4} \). But this yields a red \( C_n \), a contradiction. If \( n \geq 8 \), then \( wu_{n-4} \) has to be green if no red \( B_{2,n-2} \) with bristles \( u_{n-4}u_{n-3} \) and \( u_{n-4}w \) shall occur. Hence \( \{w, u_n\} + \{u_1, u_2, u_3, u_{n-4}, u_{n-2}\} \) contains a green \( K_{2,4} \) or \( u_3u_n \) and \( wu_4 \) are red. But then we obtain a red \( B_{2,n-2} \) with bristles \( u_1u_2 \) and \( u_1w \), a contradiction.
Finally we determine $r(P_n,G)$ and $r(B_{2,n−2},G)$ for all connected spanning subgraphs $G$ of $K_{3,3}$.

**Theorem 3.12.** Let $n \geq 4$ and let $G$ be a connected graph of order six where $G \subseteq K_{3,3}$. Then

$$r(P_n,G) = \begin{cases} 7 & \text{if } n = 4, \\ n + 2 & \text{otherwise.} \end{cases}$$

**Proof.** By (1), $r(P_n,G) \geq n + 2$. Moreover, Lemma 3.1 yields $r(P_4,G) \geq 7$. To establish equality it suffices to prove that $r(P_n,K_{3,3}) \leq n + 2$ for $n \geq 5$. Consider any coloring of $K_{n+2}$, $n \geq 5$, not containing a red $P_n$. We have to show that a green $K_{3,3}$ occurs. Let $P_k = u_1 \cdots u_k$ be a red path of maximum length, $U = \{u_1, \ldots, u_k\}$ and $W = V \setminus U = \{w_1, w_2, \ldots, w_{n+2−k}\}$. In case of $k \leq 2$ either at most one red edge occurs or any two red edges are independent.

This yields a green $K_3 - 2K_2$ containing a green $K_{3,3}$. Now let $k \geq 3$. All edges between $\{u_1, u_k\}$ and $W$ have to be green. Since $k \leq n−1$, $|W| = n + 2 − k \geq 3$.

If $q_g(u_i, W) \geq 3$ for some $i$ with $2 \leq i \leq k − 1$, then a green $K_{3,3}$ occurs.

Otherwise $q_r(u_i, W) \geq 1$ for every $i$ with $2 \leq i \leq k − 1$, and we may assume that $u_2w_1$ is red. Since $P_{k+1} \not\subseteq [V]_r$, all edges incident to $w_1$ in $[W]$ have to be green. This produces a green $K_{3,3}$ if $|W| \geq 4$. The remaining case is $|W| = 3$ which implies $k = n − 1 \geq 4$. Again we apply $P_{k+1} \not\subseteq [V]_r$. Thus, $u_1u_k$, $u_1u_3$ and $u_3w_1$ must be green. Moreover we may assume that $u_3w_2$ is red, and this forces $u_2w_2$ and $u_2u_3$ to be green. If $u_2w_3$ is green, then we obtain the green $K_{3,3} = \{u_1, u_2, w_1\} + \{u_k, w_2, w_3\}$. If $u_2w_3$ is red, then $w_2w_3$ and $w_3w_4$ have to be green yielding the green $K_{3,3} = \{u_1, w_1, w_3\} + \{u_3, u_k, w_2\}$, and the proof is complete.

**Theorem 3.13.** Let $n \geq 5$ and let $G$ be a connected graph of order six where $G \subseteq K_{3,3}$. Then

$$r(B_{2,n−2},G) = \begin{cases} n + 3 & \text{if } G = K_{3,3} \text{ and } 5 \leq n \leq 6, \\ n + 2 & \text{otherwise.} \end{cases}$$

**Proof.** By (1), $r(B_{2,n−2},G) \geq n + 2$. Moreover, Lemma 3.2 yields $r(B_{2,3},K_{3,3}) \geq 8$ and $r(B_{2,4},K_{3,3}) \geq 9$. To establish equality we prove $r(B_{2,n−2},K_{3,3}) \leq n + 3$ as well as $r(B_{2,n−2},K_{3,3}−e) \leq n + 2$ for $n \geq 5$ and $r(B_{2,n−2},K_{3,3}) \leq n + 2$ for $n \geq 7$.

Consider any coloring of $K_m$ with $n + 2 \leq m \leq n + 3$ and $n \geq 5$ not containing a red $B_{2,n−2}$. Let $V = V(K_m)$. If a green $K_{3,3}$ occurs, then we are done. Otherwise Theorem 3.12 guarantees a red $P_n = u_1 \cdots u_n$. Let $U = \{u_1, \ldots, u_n\}$. $B_{2,n−2} \not\subseteq [V]_r$ forces only green edges between $\{u_2, u_3, u_{n−2}, u_{n−1}\}$ and $W = V \setminus U$. Hence $K_{3,3} \subseteq [V]_g$ in case of $m = n + 3$, i.e., $|W| = 3$, a contradiction. It remains $m = n + 2$. Let $W = \{w_1, w_2\}$. By Lemma 3.5, $u_1u_3$, $u_1u_{n−1}$, $u_2u_n$, and $u_{n−2}u_n$
have to be green. Thus we find a green $K_{3,3} - e$ in $\{u_1, w_1, w_2\} + \{u_2, u_3, u_{n-1}\}$ proving that $r(B_{2,n-2}, K_{3,3} - e) \leq n + 2$ if $n \geq 5$. Now let $n \geq 7$. To avoid that $\{u_1, w_1, w_2\} + \{u_3, u_{n-2}, u_{n-1}\}$ or $\{w_1, w_2, u_n\} + \{u_2, u_3, u_{n-2}\}$ is a green $K_{3,3}$, $u_1u_{n-2}$ and $u_3u_n$ have to be red, and then $B_{2,n-2} \not\subseteq [V]_r$ implies that $u_1w_1$ and $u_1w_2$ are green. This forces $u_1u_n$ to be red as otherwise $\{u_1, u_2, u_{n-2}\} + \{w_1, w_2, u_n\}$ is a green $K_{3,3}$. Consequently, since $B_{2,n-2} \not\subseteq [V]_r$, all edges between $U$ and $W$ have to be green. By Theorem 3.4, $r(B_{2,n-2}, S_4) = n$ for $n \geq 7$. But this implies a green $S_4$ in $[U]$ yielding a green $K_{3,3}$ together with $w_1$ and $w_2$, a contradiction, and we are done.

4. Concluding Remarks

Summarizing Theorems 2.1, 2.9, the results from [11] concerning non-bipartite graphs $G$ and the results from [13] for $r(S_n, K_{1,1,4})$, we see that $r(T_n, G)$ has been determined for any tree $T_n$ and all connected graphs $G \not\subseteq K_{2,2,2}$ of order six with $\chi(G) \geq 3$, except for $T_n = S_n$ in case of some small $n$ and some $G$ where $\chi(G) = 3$. The exact values of $r(S_n, G)$ are still missing in the following cases (the numbering of $G$ corresponds to the numbering of the 112 connected graphs of order six used in [11]): $G = G_{100} = K_{1,2,3}$ with $n \in \{7, 9, 11\}$, $G = G_{94} = E_2 + (E_1 \cup P_3)$ with $n = 7$, $G = G_{92} = K_{3,3} + e$ with $6 \leq n \leq 12$, $G = G_{78} = E_2 + (E_2 \cup K_2)$ with $6 \leq n \leq 8$, $G = G_{60}$ and $G = G_{79}$ (the two graphs obtained from $K_{1,1,3}$ by joining an additional vertex to one or two of the three vertices of degree 2) with $n = 6$. In all these cases we know that the value of $r(S_n, G)$ differs by at most 2 from the lower bound given in (1). By a detailed case analysis, perhaps assisted by computer algorithms, it should be possible to determine the missing exact values.

To achieve significant progress in evaluating $r(T_n, K_{2,2,2})$ seems to be difficult, especially for trees $T_n$ with maximum degree $\Delta(T_n)$ close to $n - 1$, where we know that, for $n$ sufficiently large, $r(T_n, K_{2,2,2})$ differs considerably from the lower bound $2n$ obtained from (1). In contrast, for some trees with small maximum degree as $P_n$ and a special class of trees with $\Delta(T_n) = 3$, $r(T_n, K_{2,2,2})$ attains the bound $2n$ (see Theorems 2.10 and 2.11). It seems to be promising to study $r(T_n, K_{2,2,2})$ for further trees with small maximum degree, in particular it would be desirable to obtain a characterization of all $K_{2,2,2}$-good trees $T_n$.

As already explained, it seems to be extremely difficult to evaluate $r(T_n, G)$ for trees $T_n$ with maximum degree $\Delta(T_n)$ close to $n - 1$ and all connected bipartite graphs $G$ of order six, i.e., all connected spanning subgraphs of $K_{m_1,m_2}$ with $1 \leq m_1 \leq m_2$ and $m_1 + m_2 = 6$. If $\Delta(T_n)$ is small, then the situation is entirely different. For $T_n \in \{P_n, B_{2,n-2}\}$ we have shown that, except for small $n$, $T_n$ is $G$-good for any connected bipartite graph $G$ of order six, and there might be
other trees $T_n$ with small maximum degree where the general lower bound (1) is attained. Especially, by Theorems 3.3, 3.10 and 3.12, $P_n$ is $G$-good if and only if $n \geq 2m_2 - 1$. This improves in a very special case a general result due to Pokrovskiy and Sudakov [16] who recently have shown that $P_n$ is $G$-good for any graph $G$ on $p(G)$ vertices if $n \geq 4p(G)$.

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