ON SEMISYMMETRIC CUBIC GRAPHS
OF ORDER $20p^2$, $p$ PRIME

MOHSEN SHAHSAVARAN

AND

MOHAMMAD REZA DARAFSHEH

School of Mathematics, Statistics, and Computer Science
College of Science, University of Tehran, Tehran, Iran

e-mail: m.shahsavaran@ut.ac.ir
darafsheh@ut.ac.ir

Abstract

A simple graph is called semisymmetric if it is regular and edge-transitive
but not vertex-transitive. Let $p$ be an arbitrary prime. Folkman proved [Reg-
is no semisymmetric graph of order $2p$ or $2p^2$. In this paper an extension of
his result in the case of cubic graphs of order $20p^2$ is given. We prove that
there is no connected cubic semisymmetric graph of order $20p^2$ or, equiv-
ally, that every connected cubic edge-transitive graph of order $20p^2$ is
necessarily symmetric.

Keywords: edge-transitive graph, vertex-transitive graph, semisymmetric
graph, order of a graph, classification of cubic semisymmetric graphs.

2010 Mathematics Subject Classification: 05E18, 20D60, 05C25, 20B25.

1. Introduction

In this paper all graphs are finite, undirected and simple, i.e., without loops or
multiple edges. A graph is called semisymmetric if it is regular and edge-transitive
but not vertex-transitive.

The class of semisymmetric graphs was first studied by Folkman [9], who
found several infinite families of such graphs and posed eight open problems.

An interesting research problem is to classify connected cubic semisymmetric
graphs of various types of orders. Among these, graphs of orders $kp^i$ for prime $p$
and small $k$ and $i$ have been a target of much research. In [9], Folkman proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for any prime $p$. In [2] it is proved that there is no connected cubic semisymmetric graph of order $4p^2$ and also in [1] the authors proved that there is no connected cubic semisymmetric graph of order $8p^2$ for any prime $p$.

For prime $p$, cubic semisymmetric graphs of order $2p^3$ were investigated in [17], in which they proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray graph.

Connected cubic semisymmetric graphs of orders $4p^3$, $6p^2$, $6p^3$, $8p^3$, $10p^3$, $18p^n$ have been classified in [3, 8, 11, 13] and [22].

In this paper we investigate connected cubic semisymmetric graphs of order $20p^2$ for all primes $p$. During our trial for classifying such graphs, we managed to prove that no such graph can really exist. In fact we first assume the existence of such a graph and obtain a very useful structure for its group of automorphisms. Then we use this structure to prove that the existence of the graph will actually result in a contradiction. By [23] this result is equivalent to saying that any connected cubic edge-transitive graph of order $20p^2$ is symmetric.

2. Preliminaries

In this paper, the cardinality of a finite set $A$ is denoted by $|A|$. A function $f$ acts on its argument from the left, i.e., we write $f(x)$. The composition, $fg$, of two functions $f$ and $g$, is defined as $(fg)(x) = f(g(x))$.

The symmetric and alternating groups of degree $n$, the dihedral group of order $2n$ and the cyclic group of order $n$ are respectively denoted by $S_n$, $A_n$, $D_{2n}$, $Z_n$. If $G$ is a group and $H \leq G$, then $Aut(G)$, $G'$, $Z(G)$, $C_G(H)$ and $N_G(H)$ denote respectively the group of automorphisms of $G$, the commutator subgroup of $G$, the center of $G$, the centralizer and the normalizer of $H$ in $G$. We also write $H \trianglelefteq G$ to denote $H$ is a characteristic subgroup of $G$. If $H \trianglelefteq K \trianglelefteq G$, then $H \leq G$. For a prime $p$ dividing the order of finite $G$, $O_p(G)$ will denote the largest normal $p$-subgroup of $G$. It is easy to verify that $O_p(G) \trianglelefteq G$.

For a group $G$ and a nonempty set $\Omega$, an action of $G$ on $\Omega$ is a function $(g, \omega) \to g.\omega$ from $G \times \Omega$ to $\Omega$, where $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$, for every $g, h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$, if there is no fear of ambiguity. For $\omega \in \Omega$, the stabilizer of $\omega$ in $G$ is defined as $G_\omega = \{g \in G : g\omega = \omega\}$. The action is called semiregular if the stabilizer of each element in $\Omega$ is trivial; it is called regular if it is semiregular and transitive.

For any two groups $G$ and $H$ and any homomorphism $\varphi : H \to Aut(G)$ the external semidirect product $G \rtimes_{\varphi} H$ is defined as the group whose underlying
set is the Cartesian product $G \times H$ and whose binary operation is defined as $(g_1, h_1)(g_2, h_2) = (g_1\varphi(h_1)g_2, h_1h_2)$. If $\varphi(h) = 1$ for each $h \in H$, then the semidirect product will coincide with the usual direct product. If $G = NK$ where $N \unlhd G, K \leq G$ and $N \cap K = 1$, then $G$ is said to be the internal semidirect product of $N$ and $K$. These two concepts are in fact equivalent in the sense that there is some homomorphism $\varphi : K \to \text{Aut}(N)$ where $G \cong N \rtimes_{\varphi} K$.

The dihedral group $D_{2n}$ is defined as

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$ 

So $D_{2n} = \{a^i \mid i = 0, \ldots, n - 1\} \cup \{ba^i \mid i = 0, \ldots, n - 1\}$. All the elements of the form $ba^i$ are of order 2.

Consider the dihedral group of order 12. According to Sylow theorems, the number of Sylow 2-subgroups of $D_{12} = \{a^i \mid i = 0, \ldots, 5\} \cup \{ba^i \mid i = 0, \ldots, 5\}$ divides 3. Also the following three subgroups, all isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, are Sylow 2-subgroups of $D_{12}$:

$$P_1 = \{1, a^3, b, ba^3\}, P_2 = \{1, a^3, ba, ba^4\}, P_3 = \{1, a^3, ba^2, ba^5\}.$$ 

Therefore $D_{12}$ has exactly three Sylow 2-subgroups which are $P_1, P_2$ and $P_3$.

Let $\Gamma$ be a graph. For two vertices $u$ and $v$, we write $u \sim v$ to denote $u$ is adjacent to $v$. If $u \sim v$, then each of the ordered pairs $(u, v)$ and $(v, u)$ is called an arc. The set of all vertices adjacent to a vertex $u$ is denoted by $\Gamma(u)$. The degree or valency of $u$ is $|\Gamma(u)|$. We call $\Gamma$ regular if all of its vertices have the same valency. The vertex set, the edge set, the arc set and the set of all automorphisms of $\Gamma$ are respectively denoted by $V(\Gamma), E(\Gamma), \text{Arc}(\Gamma)$ and $\text{Aut}(\Gamma)$. If $\Gamma$ is a graph and $N \leq \text{Aut}(\Gamma)$, then $\Gamma_N$ will denote a simple undirected graph whose vertices are the orbits of $N$ in its action on $V(\Gamma)$, and where two vertices $Nu$ and $Nv$ are adjacent if and only if $u \sim nv$ in $\Gamma$, for some $n \in N$.

Let $\Gamma_c$ and $\Gamma$ be two graphs. Then $\Gamma_c$ is said to be a covering graph for $\Gamma$ if there is a surjection $f : V(\Gamma_c) \to V(\Gamma)$ which preserves adjacency and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. $f$ is called a covering projection. Clearly, if $\Gamma$ is bipartite, then so is $\Gamma_c$. For each $u \in V(\Gamma)$, the fibre on $u$ is defined as $\text{fib}_u = f^{-1}(u)$. The following important set is a subgroup of $\text{Aut}(\Gamma_c)$ and is called the group of covering transformations for $f$:

$$CT(f) = \{\sigma \in \text{Aut}(\Gamma_c) \mid \forall u \in V(\Gamma), \sigma(\text{fib}_u) = \text{fib}_u\}.$$ 

It is known that $K = CT(f)$ acts semiregularly on each fibre [14]. If this action is regular, then $\Gamma_c$ is said to be a regular $K$-cover of $\Gamma$.

Let $X \leq \text{Aut}(\Gamma)$. Then $\Gamma$ is said to be $X$-vertex-transitive, $X$-edge-transitive or $X$-arc-transitive if $X$ acts transitively on $V(\Gamma), E(\Gamma)$ or $\text{Arc}(\Gamma)$ respectively.
The graph $\Gamma$ is called $X$-semisymmetric if it is regular and $X$-edge-transitive but not $X$-vertex-transitive. Also $\Gamma$ is called $X$-symmetric if it is $X$-vertex-transitive and $X$-arc-transitive. For $X = \text{Aut}(\Gamma)$, we omit $X$ and simply talk about $\Gamma$ being edge-transitive, vertex-transitive, symmetric or semisymmetric. An $X$-edge-transitive but not $X$-vertex-transitive graph is necessarily bipartite, where the two partites are the orbits of the action of $X$ on $V(\Gamma)$. If $\Gamma$ is regular, then the two partite sets have equal cardinality. So an $X$-semisymmetric graph is bipartite such that $X$ is transitive on each partite but $X$ carries no vertex from one partite set to the other.

There are only two symmetric cubic graphs of order 20 which are denoted by $F_{20A}$ and $F_{20B}$. Only $F_{20B}$ is bipartite [7].

Any minimal normal subgroup of a finite group, is the internal direct product of isomorphic copies of a simple group.

A finite simple group $G$ is called a $K_n$-group if its order has exactly $n$ distinct prime divisors, where $n \in \mathbb{N}$. The following two results determine all simple $K_3$-groups and $K_4$-groups [4, 12, 20, 25].

**Theorem 2.1.** (i) If $G$ is a simple $K_3$-group, then $G$ is one of the following groups: $A_5$, $A_6$, $L_2(7)$, $L_2(2^3)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

(ii) If $G$ is a simple $K_4$-group, then $G$ is one of the following groups:

1. $A_7$, $A_8$, $A_9$, $A_{10}$, $M_{11}$, $M_{12}$, $J_2$, $L_2(2^4)$, $L_2(5^2)$, $L_2(7^2)$, $L_2(3^4)$, $L_2(97)$, $L_2(3^5)$, $L_2(577)$, $L_3(2^3)$, $L_3(5)$, $L_3(7)$, $L_3(2^3)$, $L_3(17)$, $L_4(3)$, $U_3(2^3)$, $U_3(7)$, $U_3(3^2)$, $U_4(3)$, $U_5(2)$, $S_4(2^3)$, $S_4(5)$, $S_4(7)$, $S_4(3^2)$, $S_6(2)$, $O_6^+(2)$, $G_2(3)$, $S_5(2^3)$, $S_7(5^2)$, $3^3D_4(2)$, $2F_4(2)'$;
2. $L_2(r)$ where $r$ is a prime, $r^2 - 1 = 2^a \cdot 3^b \cdot s$, $s > 3$ is a prime, $a, b \in \mathbb{N}$;
3. $L_2(2^m)$ where $m$, $2^m - 1$, $2^{\frac{2^m+1}{2}}$, $2^{\frac{3^m-1}{2}}$ are primes greater than 3;
4. $L_2(3^m)$ where $m$, $\frac{3^m+1}{2}$, $\frac{3^m-1}{2}$ are odd primes.

**Theorem 2.2** [21]. If $H$ is a subgroup of a group $G$, then $C_G(H) \trianglelefteq N_G(H)$ and $\frac{N_G(H)}{C_G(H)}$ is isomorphic to a subgroup of $\text{Aut}(H)$.

**Theorem 2.3** [19]. Let $G$ be a finite group and $p$ a prime. If $G$ has an abelian Sylow $p$-subgroup, then $p$ does not divide $|G' \cap Z(G)|$.

**Theorem 2.4** ([18], Theorem 9.1.2). Let $G$ be a finite group and $N \trianglelefteq G$. If $|N|$ and $|\frac{G}{N}|$ are relatively prime, then $G$ has a subgroup $H$ such that $G = NH$ and $N \cap H = 1$ (therefore $G$ is the internal semidirect product of $N$ and $H$).

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

**Theorem 2.5** [21]. For any two distinct primes $p$ and $q$ and any two nonnegative integers $a$ and $b$, every finite group of order $p^a q^b$ is solvable.
In the following proposition, the inverse of a pair \((a, b)\) is meant to be \((b, a)\).
Also for each \(i\), \(A_i\), \(B_i\), \(C_i\) and \(D_i\) are noncyclic groups of order \(i\) with known structures. We will not need their structures.

**Theorem 2.6** [10]. If \(\Gamma\) is a connected cubic \(X\)-semisymmetric graph, then the order of the stabilizer of any vertex is of the form \(2^r \cdot 3\) for some \(0 \leq r \leq 7\). More precisely, if \(\{u, v\}\) is any edge of \(\Gamma\), then the pair \((X_u, X_v)\) can only be one of the following fifteen pairs or their inverses:
\[
(Z_3, Z_3), (S_3, S_3), (S_3, Z_6), (D_{12}, D_{12}), (D_{12}, A_4), (S_4, D_{24}), (S_4, Z_3 \times D_3), (A_4 \times Z_2, D_{12} \times Z_2), (S_4 \times Z_2, D_8 \times S_3), (S_4, S_4), (S_4 \times Z_2, S_4 \times Z_2), (A_{96}, B_{96}), (A_{192}, B_{192}), (C_{192}, D_{192}), (A_{384}, B_{384}).
\]

**Proposition 2.7** [24]. Let \(\Gamma\) be a connected cubic \(X\)-semisymmetric graph for some \(X \leq \text{Aut}(\Gamma)\) and let \(N \trianglelefteq X\). If \(|\frac{X}{N}|\) is not divisible by 3, then \(\Gamma\) is also \(N\)-semisymmetric.

**Proposition 2.8** [17]. Let \(\Gamma\) be a connected cubic \(X\)-semisymmetric graph for some \(X \leq \text{Aut}(\Gamma)\); then either \(\Gamma \cong K_{3,3}\), the complete bipartite graph on 6 vertices, or \(X\) acts faithfully on each of the bipartition sets of \(\Gamma\).

**Theorem 2.9** [15]. Let \(\Gamma\) be a connected cubic \(X\)-semisymmetric graph. Let \(\{U, W\}\) be a bipartition for \(\Gamma\) and assume \(N \trianglelefteq X\). If the actions of \(N\) on both \(U\) and \(W\) are intransitive, then \(N\) acts semiregularly on both \(U\) and \(W\), \(\Gamma_N\) is \(X\)-\(N\)-semisymmetric, and \(\Gamma\) is a regular \(N\)-covering of \(\Gamma_N\).

This theorem has a nice result. For every normal subgroup \(N \trianglelefteq X\) either \(N\) is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of \(N\) is divisible by \(|U| = |W|\). In the latter case, according to Theorem 2.9, the induced action of \(N\) on both \(U\) and \(W\) is semiregular and hence the order of \(N\) divides \(|U| = |W|\). So we have the following handy corollary.

**Corollary 2.10.** If \(\Gamma\) is a connected cubic \(X\)-semisymmetric graph with \(\{U, W\}\) as a bipartition and \(N \trianglelefteq X\), then either \(|N|\) divides \(|U|\) or \(|U|\) divides \(|N|\).

Following [10] (see also [16]) the coset graph \(C(G; H_0, H_1)\) of a group \(G\) with respect to finite subgroups \(H_0\) and \(H_1\) is a bipartite graph with \(\{H_0g \mid g \in G\}\) and \(\{H_1g \mid g \in G\}\) as its bipartition sets of vertices where \(H_0g\) is adjacent to \(H_1g'\) whenever \(H_0g \cap H_1g' \neq \emptyset\). The following proposition may be extracted from [10].

**Proposition 2.11.** Let \(G\) be a finite group and \(H_0, H_1 \leq G\). The coset graph \(C(G; H_0, H_1)\) has the following properties:

(i) \(C(G; H_0, H_1)\) is regular of valency \(d\) if and only if \(H_0 \cap H_1\) has index \(d\) in both \(H_0\) and \(H_1\).
(ii) \(C(G; H_0, H_1)\) is connected if and only if \(G = \langle H_0, H_1 \rangle\).

(iii) \(G\) acts on \(C(G; H_0, H_1)\) by right multiplication. Moreover, this action is faithful if and only if \(\text{Core}_G(H_0 \cap H_1) = 1\).

(iv) In the case when the action of \(G\) is faithful, the coset graph \(C(G; H_0, H_1)\) is \(G\)-semisymmetric.

**Proposition 2.12** [16]. Let \(\Gamma\) be a regular graph and \(G \leq \text{Aut}(\Gamma)\). If \(\Gamma\) is \(G\)-semisymmetric, then \(\Gamma\) is isomorphic to the coset graph \(C(G; G_u, G_v)\) where \(u\) and \(v\) are adjacent vertices.

### 3. Main Result

In this section, our goal is to prove the following important result.

**Theorem 3.1.** There is no connected cubic semisymmetric graph of order \(20p^2\) for any prime \(p\).

In the remainder we first state and prove some lemmas and then prove the main theorem.

**Lemma 3.2.** Let \(G\) be a finite group and \(H \trianglelefteq G\) such that \(G/H\) is nonabelian simple. If \(H \cong \mathbb{Z}_p\) or \(\mathbb{Z}_{2p}\) for an odd prime \(p\), then \(G = \mathbb{Z}(G)\).

**Proof.** Since \(G/H\) is nonabelian simple, \(G\) is nonabelian and \(H\) is a maximal normal subgroup of \(G\). Also \(\text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}\) and \(\text{Aut}(\mathbb{Z}_{2p}) \cong \text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_p) \cong \text{Aut}(\mathbb{Z}_2) \times \text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}\). Because \(H\) is abelian, we have \(H \leq C_G(H) \leq N_G(H) = G\). So either \(C_G(H) = H\) or \(C_G(H) = G\). In the former case, according to Theorem 2.2, we should have \(G \cong \text{Aut}(H) \cong \mathbb{Z}_{p-1}\) which is impossible. On the other hand, \(C_G(H) = G\) implies \(H \leq Z(G)\) which again results in \(Z(G) = H\) or \(Z(G) = G\) since \(H\) is a maximal normal subgroup of \(G\). As \(G\) is not abelian, the latter is impossible and so \(Z(G) = H\). \(\blacksquare\)

**Lemma 3.3.** (i) For any odd prime number \(p\), there are only two groups of order \(2p\) : \(\mathbb{Z}_{2p}\), \(\mathbb{D}_{2p}\). The cardinalities of \(Z(\mathbb{D}_{2p})\) and \(\text{Aut}(\mathbb{D}_{2p})\) are 2 and \(p(p-1)\) respectively.

(ii) There are only four groups of order 70 : \(\mathbb{Z}_{70}\), \(\mathbb{D}_{70}\), \(X_{70}\) and \(Y_{70}\), where

\[X_{70} = \langle a, b \mid a^{35} = b^2 = 1, b^{-1}ab = a^6 \rangle\]
\[Y_{70} = \langle a, b \mid a^{35} = b^2 = 1, b^{-1}ab = a^{-6} \rangle.\]

Moreover,

\[Z(\mathbb{D}_{70}) \cong \mathbb{Z}_2, Z(X_{70}) \cong \mathbb{Z}_5, Z(Y_{70}) \cong \mathbb{Z}_7,\]
|\text{Aut}(\mathbb{D}_{70})| = 24 \cdot 35, \ |\text{Aut}(X_{70})| = 24 \cdot 7 \text{ and } |\text{Aut}(Y_{70})| = 24 \cdot 5.

\textbf{Proof.} We prove part (ii). The proof for part (i) is similar. Let \( G \) be a group of order 70. Using the Sylow theorems, it is easily verified that the number of Sylow 5-subgroups as well as the number of Sylow 7-subgroups of \( G \) is 1. So if \( P \) and \( Q \) are the Sylow 5-subgroup and the Sylow 7-subgroup of \( G \) respectively, then \( P, Q \leq G \) and hence \( N = PQ \leq G \). Now \( N \cong P \times Q \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35} \) is cyclic. Let \( N = \langle a \rangle \) and take \( b \in G \) to be an element of order 2. There is some \( 1 \leq i < 35 \) for which \( b^{-1}ab = a^i \). Therefore \( a^i = b^{-1}(b^{-1}ab)b = a \) and so \( i^2 \equiv 1 \pmod{35} \). This congruence has only four solutions \( i = 1, 34, 6, 29 \) which respectively correspond to \( \mathbb{Z}_{35}, \mathbb{D}_{70}, X_{70} \) and \( Y_{70} \).

Now consider \( X_{70} \). Each element equals \( a^i \) or \( a^ib \) for some \( i \). It can be easily verified that no element of the form \( a^ib \) belongs to the center, and that \( a^i \in Z(X_{70}) \) if and only if \( i \equiv 0 \pmod{7} \). So \( Z(X_{70}) = \langle a^7 \rangle \).

Every automorphism \( f \) of \( X_{70} \) is uniquely characterized by the two values \( f(a) \) and \( f(b) \). By an order argument, we find out that a group of order 70 has only one subgroup of order 35. So \( f \) takes \( \langle a \rangle \) to \( \langle a \rangle \) and hence \( f(a) = a^i \) for some \( i \) coprime to 35. Therefore there are \( \varphi(35) \) possibilities for \( f(a) \), where \( \varphi \) is the Euler function. Also \( f(b) \) must be an element of order 2 and has 7 possibilities, since the elements of order 2 in \( X_{70} \) are of the form \( a^ib \) where \( i \equiv 0 \pmod{5} \). We conclude that \( |\text{Aut}(X_{70})| = \varphi(35) \times 7 = 24 \times 7 \). The corresponding results for \( \mathbb{D}_{70} \) and \( Y_{70} \) follow quite similarly.

\textbf{Lemma 3.4.} There are only three simple \( K_4 \)-groups whose orders are of the form \( 2^i \cdot 3 \cdot 5 \cdot p \) for some prime \( p > 5 \) and some \( 1 \leq i \leq 8 : L_2(2^4), L_2(11) \) and \( L_2(31) \).

\textbf{Proof.} Considering the powers of primes, there is no possibility for such a group in sub-item (4) of item (ii) of Theorem 2.1. By inspecting orders of groups in sub-item (1), the only group of the desired form is \( L_2(2^4) \). As for sub-item (3), let \( L_2(2^m) \) be a group of order \( 2^i \cdot 3 \cdot 5 \cdot p \); then

\[
2^m \cdot 3 \cdot (2^m - 1) \cdot \left( \frac{2^m + 1}{3} \right) = 2^i \cdot 3 \cdot 5 \cdot p,
\]

where \( m, 2^m - 1 \) and \( \frac{2^m + 1}{3} \) are all primes according to Theorem 2.1. This equation has no answer as neither \( 2^m - 1 \) nor \( \frac{2^m + 1}{3} \) could be equal to 5. Finally, consider groups \( L_2(r) \) in sub-item (2). If for odd prime \( r \) and for prime \( s > 3 \) we have \( r^2 - 1 = 2^a \cdot 3^b \cdot s \) and

\[
2^a - 1 \cdot 3^b \cdot s \cdot r = 2^i \cdot 3 \cdot 5 \cdot p,
\]

then \( b = 1, a - 1 = i \) and either \( s = 5 \) or \( r = 5 \). The equality \( r = 5 \) is not possible, since \( L_2(5) \) is not a \( K_4 \)-group. Also if \( s = 5 \), then the equation \( r^2 - 1 = 2^a \cdot 3 \cdot 5 \) gives us only two solutions \( r = 11, 31 \) when \( a \) spans integers 2, 3, \ldots, 9.

\( \blacksquare \)
Lemma 3.5. Suppose \( \Gamma \) is a semisymmetric cubic graph of order \( 20p^2 \) where \( p > 5 \) is a prime. Let \( A = \text{Aut}(\Gamma) \). Then

(i) If \( O_p(A) = 1 \), then \( A \) does not have a normal subgroup of order 10.

(ii) If \( |O_p(A)| = p \), then \( A \) does not have a normal subgroup of order 10p.

**Proof.** Let \( \{U, W\} \) be the bipartition for \( \Gamma \). Then \( |U| = |W| = 10p^2 \). Also if \( u \in U \) is an arbitrary vertex, according to Theorem 2.6, \( |A_u| = 2^r \cdot 3 \) for some \( 0 \leq r \leq 7 \). Due to transitivity of \( \Gamma \) on \( A \), the equality \([A : A_u] = |U|\) holds which yields \(|A| = 2^{r+1} \cdot 3 \cdot 5 \cdot p^2\).

Let \( M \) be a normal subgroup of \( A \) of order 10 or 10p. Then \( M \) is intransitive on the partite sets and according to Theorem 2.9 the quotient graph \( \Gamma_M \) is \( \frac{A}{M} \)-semisymmetric with a bipartition \( \{U_M, W_M\} \). We prove that the combinations \((|O_p(A)|, |M|) = (1, 10)\) or \((p, 10p)\) lead to contradictions.

To prove (i), let \( O_p(A) = 1 \) and \(|M| = 10\). Then \(|U_M| = |W_M| = p^2 \) and \(|\frac{A}{M}| = 2^r \cdot 3 \cdot p^2 \). Let \( \frac{K}{M} \) be a minimal normal subgroup of \( \frac{A}{M} \). If \( \frac{K}{M} \) is unsolvable, it must be a simple group of order \( 2^i \cdot 3 \cdot p^2 \) for some \( i \) and \( p > 5 \). So \( \frac{K}{M} \cong L_2(7) \) and \( p = 7 \). Since 3 does not divide the order of \( \frac{A}{M} \), we conclude that \( \Gamma \) is K-semisymmetric according to Proposition 2.7.

Because \( \frac{K}{M} \) is nonabelian simple, \( M \) is a maximal normal subgroup of \( K \). Also note that \( C_K(M) \leq N_K(M) = K \). So \( M \leq MC_K(M) \leq K \) and hence either \( MC_K(M) = M \) or \( MC_K(M) = K \). If \( MC_K(M) = M \), then \( C_K(M) \leq M \) and so \( C_K(M) = C_K(M) \cap M = Z(M) \). Now according to Theorem 2.2 the order of \( K \) must divide \(|Aut(M)|\). The order of \( M \) is 70 and so according to Lemma 3.3, \( M \cong Z_{70}, D_{70}, X_{70} \) and \( Y_{70} \). We have \(|K| = |L_2(7)| \times |M| = 2^4 \cdot 3 \cdot 5 \cdot 7^2 \). If \( M \cong Z_{70} \), then \(|\frac{K}{Z(M)}| = 2^3 \cdot 3 \cdot 7 \) does not divide \(|Aut(M)| = \varphi(70) = 24 \). Also for the three remaining cases of \( M \), the orders of \( Z(M) \) and \( Aut(M) \) are known according to Lemma 3.3, and in each case one can make sure that \(|K| \) does not divide \(|Aut(M)|\). So the equality \( MC_K(M) = M \) could not be possible.

On the other hand if \( MC_K(M) = K \), then \(|K| = |\frac{M}{C_K(M)}| = \frac{|M||C_K(M)|}{|Z(M)|} \). From this equation, in each of the four possibilities for \( M \) one can obtain \(|C_K(M)|\), since the order of \( Z(M) \) is known in each case. As \( \Gamma \) is K-semisymmetric and \( C_K(M) \leq K \), according to Corollary 2.10 either \(|U| = 10 \cdot 7^2 \) divides \(|C_K(M)|\) or
\(|C_K(M)|\) divides \(10 \cdot 7^2\). In none of the four cases, these are possible. For example if \(M \cong D_{70}\), then from \(|K| = \frac{|M||C_K(M)|}{|Z(M)|}\) we obtain \(|C_K(M)| = 2^4 \cdot 3 \cdot 7\) which is neither divisible by \(10 \cdot 7^2\) nor it divides \(10 \cdot 7^2\). So \(K = M\) cannot be unsolvable.

If \(\frac{K}{M}\) is solvable, it is elementary abelian. Whether it is intransitive or transitive on the partite sets, its order must equal \(p\). Therefore \(|K| = 10p^2\). The Sylow \(p\)-subgroup of \(K\) is normal in \(K\). So it is characteristic in \(K\) and hence normal in \(A\), contradicting the assumption that \(|O_p(A)| = p\).

\[\text{Lemma 3.6. If } p > 5 \text{ is a prime and } \Gamma \text{ is a connected cubic semisymmetric graph of order } 20p^2, \text{ then } \text{Aut}(\Gamma) \text{ has a normal Sylow } p\text{-subgroup.}\]

\[\textbf{Proof.}\] Take \(\{U, W\}\) to be a bipartition for \(\Gamma\) and let \(A = \text{Aut}(\Gamma)\). Then \(|U| = |W| = 10p^2\) and \(|A| = 2^r \cdot 3 \cdot 5 \cdot p^2\) for some \(0 \leq r \leq 7\). Let \(N \cong T^k\) be a minimal normal subgroup of \(A\), where \(T\) is simple.

If \(T\) is nonabelian, then \(k = 1\) and \(N = T\) since the powers of 3 and 5 in \(|A|\) equal 1. According to Corollary 2.10 either \(|N|\) divides \(|U| = 10p^2\) or \(10p^2\) divides \(|N|\). If \(|N|\) divides \(10p^2\), then since \(|N|\) is divisible by at least three distinct primes (Theorem 2.5), we must have \(|N| = 2 \cdot 5 \cdot p^i\) for \(i = 1\) or \(2\), and so \(N\) is a simple \(K_3\)-group. But the order of every simple \(K_3\)-group, listed in Theorem 2.1, is divisible by 3, a contradiction. Therefore \(|N|\) is divisible by \(10p^2\).

Again since the order of every simple \(K_3\)-group is divisible by 3, \(N\) must be a simple \(K_4\)-group whose order is of the form \(2^i \cdot 3 \cdot 5 \cdot p^2\). But no such simple \(K_4\)-group exists.

We conclude that \(N\) is elementary abelian and hence it follows from Corollary 2.10 that \(|N|\) divides \(10p^2\). Therefore \(N \cong Z_2, Z_5, Z_2^5\) or \(Z_3 p\). In each case \(\Gamma_N\) would itself be a connected cubic \(\frac{A}{N}\)-semisymmetric graph of order \(\frac{20p^2}{|N|}\). We claim \(|O_p(A)| = p^2\) by showing that \(|O_p(A)| < p^2\) will result in a contradiction as follows.

\[\text{Case 1. } O_p(A) = 1.\] If this case happens, then the minimal normal subgroup of \(A\) is \(N \cong Z_2\) or \(Z_5\) and \(\Gamma_N\) is \(\frac{A}{N}\)-semisymmetric of order \(10p^2\) or \(4p^2\) respectively. Take \(\{U_N, W_N\}\) to be the bipartition for \(\Gamma_N\). Also let \(\frac{M}{N}\) be a minimal normal subgroup of \(\frac{A}{N}\).

If \(N \cong Z_2\), then \(\frac{A}{N} = 2^r \cdot 3 \cdot 5 \cdot p^2\) and \(|U_N| = |W_N| = 5p^2\). If \(\frac{M}{N}\) is unsolvable, then it must be a simple \(K_4\)-group whose order is of the form \(2^i \cdot 3 \cdot 5 \cdot p^2\). But there is no such simple \(K_4\)-group. If \(\frac{M}{N}\) is solvable, it is elementary abelian and hence according to Corollary 2.10 its order divides \(5p^2\) which yields \(\frac{M}{N} \cong Z_5\) or \(Z_3^5\) for \(i = 1\) or \(2\). It follows from part (i) of Lemma 3.5 that \(\frac{M}{N}\) cannot be isomorphic to \(Z_5\). Accordingly, \(\frac{M}{N} \cong Z_3^5\) and so \(|M| = 2p^i\) for \(i = 1\) or \(2\). If \(P\) is a Sylow \(p\)-subgroup of \(M\), then \(P \leq c M \leq A\) which implies \(P \leq A\), contradicting our assumption.
If \( N \cong \mathbb{Z}_5 \), then \(|A_N| = 2^{r+1} \cdot 3 \cdot p^2 \) and \(|U_N| = |W_N| = 2p^2\). Now if \( \frac{A}{N} \) is unsolvable, then it must be a simple group whose order is of the form \( 2^i \cdot 3 \cdot p^2 \) for some \( i \geq 1 \). But there is no such simple \( K_3 \)-group. If \( \frac{A}{N} \) is solvable, it is elementary abelian and according to Corollary 2.10 its order divides \(|U_N| = 2p^2\) which yields \( \frac{A}{N} \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_p^i \) for \( i = 1 \) or \( 2 \). Again it follows from Lemma 3.5 that \( \frac{A}{N} \) is not isomorphic to \( \mathbb{Z}_2 \) and so \( \frac{A}{N} \cong \mathbb{Z}_p^i \) which results in \(|M| = 5p^i \) for \( i = 1 \) or \( 2 \). Again a Sylow \( p \)-subgroup of \( M \) is normal in \( A \), contradicting our assumption on \( O_p(A) \).

**Case 2.** \(|O_p(A)| = p\). Let \( M = O_p(A) \). According to Theorem 2.9, \( \Gamma_M \) is connected cubic \( \frac{A}{M} \)-semisymmetric with the bipartition \( \{U_M, W_M\} \), where \(|U_M| = |W_M| = 10p\). We have \(|A_M| = 2^{r+1} \cdot 3 \cdot 5 \cdot p \). Let \( \frac{A}{M} \) be a minimal normal subgroup of \( \frac{A}{N} \) and consider the following two subcases based on solvability or unsolvability of \( \frac{A}{M} \).

(a) \( \frac{A}{M} \) is unsolvable. In this case \( \frac{A}{M} \) should be of an order divisible by \(|U_M| = 10p\) and so it is a simple \( K_3 \)-group whose order equals \( 2^i \cdot 3 \cdot 5 \cdot p \) for some \( 1 \leq i \leq 8 \). According to Lemma 3.4, \( \frac{A}{M} \) is isomorphic to either \( L_2(2^i) \) of order \( 2^4 \cdot 3 \cdot 5 \cdot 17 \), \( L_2(11) \) of order \( 2^2 \cdot 3 \cdot 5 \cdot 11 \) or \( L_2(31) \) of order \( 2^5 \cdot 3 \cdot 5 \cdot 31 \). These three groups correspond to \( p = 17, 11 \) and \( 31 \) respectively. In each case the order of \( \frac{A}{N} \cong \frac{A}{M} \) is not divisible by \( 3 \). So \( \Gamma \) would also be \( L \)-semisymmetric according to Proposition 2.7. Also since in all the three cases \( \frac{A}{M} \) is nonabelian simple and \( M \cong \mathbb{Z}_p \), according to Lemma 3.2 \( Z(L) = M \). Now \( L' \cap Z(L) \leq Z(L) \) and \( p \) does not divide the order of \( L' \cap Z(L) \) according to Theorem 2.3. Therefore \( L' \cap Z(L) = 1 \). The relations \( Z(L) \leq L'Z(L) \leq L \) imply \( L'Z(L) = Z(L) \) or \( L' \leq Z(L) \). If \( L'Z(L) = Z(L) \), then \( L' \leq Z(L) \) and so \( L' = L' \cap Z(L) = 1 \) which is not possible as \( L \) is not abelian. On the other hand if \( L'Z(L) = L \), then \(|L| = |L|Z(L)| \) and so \(|L'| = 2^i \cdot 3 \cdot 5 \cdot p \), where \( i \) depends on \( p \) (e.g. \( i = 2 \) for \( p = 11 \)). Since \( \Gamma \) is \( L \)-semisymmetric, according to Corollary 2.10, \( |L'| \) either divides \(|U| = 10p^2 \) or is divisible by \( 10p^2 \). With the order that we just obtained for \( L' \), none of these divisibilities hold.

(b) \( \frac{A}{M} \) is solvable. In this case, \( \frac{A}{M} \) is elementary abelian and hence intransitive on both \( U_M \) and \( W_M \). So \( \frac{A}{M} \cong \mathbb{Z}_2, \mathbb{Z}_5 \) or \( \mathbb{Z}_p \). The isomorphism \( \frac{A}{M} \cong \mathbb{Z}_p \) could not hold as it would lead to \(|L| = p^2 \) which contradicts the assumption that \(|O_p(A)| = p \). In the following we discuss the two remaining cases.

(b1) \( \frac{A}{M} \cong \mathbb{Z}_2 \). This yields \(|L| = 2p \). Consider the graph \( \Gamma_L \) which is connected cubic \( \frac{A}{L} \)-semisymmetric (Theorem 2.9) with the bipartition \( \{U_L, W_L\} \), where \(|U_L| = |W_L| = 5p \). Let \( \frac{A}{L} \) be a minimal normal subgroup of \( \frac{A}{N} \). It is either solvable or unsolvable. In the following we discuss that both cases lead to contradictions.

Suppose \( \frac{A}{M} \) is solvable. Then \( \frac{A}{M} \cong \mathbb{Z}_5 \) or \( \mathbb{Z}_p \). The case \( \frac{A}{M} \cong \mathbb{Z}_5 \) is not possible...
since according to Lemma 3.5, \( A \) does not have any normal subgroup of order \( 10p \). Also the case \( \frac{T}{L} \cong \mathbb{Z}_p \) yields \( |T| = 2p^2 \) and a Sylow \( p \)-subgroup of \( T \) would be normal in \( A \), contradicting our assumption that \( |O_p(A)| = p \).

Now suppose \( \frac{T}{L} \) is unsolvable. So it should be a simple group of order \( 2^i \cdot 3 \cdot 5 \cdot p \) for some \( 1 \leq i \leq 7 \). Again according to Lemma 3.4, \( \frac{T}{L} \cong L_2(2^i) \), \( L_2(11) \) or \( L_2(31) \) which respectively correspond to \( p = 17, 11, 31 \). In each case \( \Gamma \) is \( T \)-semisymmetric according to Proposition 2.7, since the order of \( \frac{T}{L} \cong \frac{A}{T} \) is not divisible by 3. As \( |L| = 2p \) and \( p \) is odd prime, according to Lemma 3.3 we have \( L \cong \mathbb{Z}_{2p} \) or \( D_{2p} \).

First suppose \( L \cong \mathbb{Z}_{2p} \). Then according to Lemma 3.2, \( Z(T) = L \). Now \( T' \cap Z(T) \leq Z(T) \cong \mathbb{Z}_{2p} \) and \( p \) does not divide the order of \( T' \cap Z(T) \) according to Theorem 2.3. Therefore \( T' \cap Z(T) = 1 \) or 2. Since \( \frac{T}{L} \) is nonabelian simple, \( \frac{T}{L} = (\frac{T}{L})' = \frac{T}{L} \) and so \( T' \cap Z(T) = T \). So \( |T| = |T' \cdot |L| \). Because \( |T| = 2^i \cdot 3 \cdot 5 \cdot p^2 \) for some \( i > 1 \) and \( \frac{|L|}{T} = p \) or \( 2p \), we obtain \( |T| = 2^i \cdot 3 \cdot 5 \cdot p \) for some \( j \). The graph \( \Gamma \) is \( T \)-semisymmetric and so according to Corollary 2.10 either \( |T'| \) divides \( |U| = 10p^2 \) or \( |T'| \) is divisible by \( 10p^2 \). Both these cases are inconsistent with the order that we just obtained for \( T' \).

Next assume \( L \cong D_{2p} \). The relations \( L \leq LC_T(L) \leq T \) imply \( LC_T(L) = L \) or \( LC_T(L) = T \). If \( LC_T(L) = T \), then \( |T| = \frac{|L||C_T(L)|}{|L \cap C_T(L)|} \). Since \( |L \cap C_T(L)| = |Z(L)| = |Z(D_{2p})| = 2 \), we have \( |T| = \frac{|L||C_T(L)|}{2} \) and so \( |C_T(L)| = 2^j \cdot 3 \cdot 5 \cdot p \) for some \( j \). With this order, the normal subgroup \( C_T(L) \cong L \) does not satisfy Corollary 2.10. If \( LC_T(L) = L \), then \( C_T(L) \leq L \) and hence \( C_T(L) = C_T(L) \cap \Gamma = Z(L) \cong Z(D_{2p}) \cong \mathbb{Z}_2 \). According to Theorem 2.2, \( \frac{T}{C_T(L)} \leq Aut(L) \cong Aut(D_{2p}) \) and so \( |\frac{T}{C_T(L)}| = 2^2 \cdot 3 \cdot 5 \cdot p^2 \) divides \( |Aut(D_{2p})| = p(p-1) \) which is impossible.

(b2) \( \frac{A}{T} \cong \mathbb{Z}_5 \). In this case \( |L| = 5p \). The graph \( \Gamma_L \) is connected cubic \( \frac{A}{T} \)-semisymmetric (Theorem 2.9) with the bipartition \( \{U_L, W_L\} \), where \( |U_L| = |W_L| = 2p \). Let \( \frac{T}{L} \) be a minimal normal subgroup of \( \frac{A}{T} \).

If \( \frac{T}{L} \) is solvable, then \( \frac{T}{L} \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_p \). The case \( \frac{T}{L} \cong \mathbb{Z}_2 \) is ruled out, as a result of part (ii) of Lemma 3.5. Also the case \( \frac{T}{L} \cong \mathbb{Z}_p \) yields \( |T| = 5p^2 \) and a Sylow \( p \)-subgroup of \( T \) would be normal in \( A \), contradicting our assumption that \( |O_p(A)| = p \).

If \( \frac{T}{L} \) is unsolvable, then it is a simple group of order \( 2^i \cdot 3 \cdot p \) for some \( 1 \leq i \leq 8 \). So \( \frac{T}{L} \cong A_5 \), \( L_2(7) \) according to Theorem 2.1. But since \( p > 5 \), we may only have \( \frac{T}{L} \cong L_2(7) \) and \( p = 7 \). The order of \( \frac{A}{T} \) is not divisible by 3. So \( \Gamma_L \) is also \( G \)-semisymmetric where \( G = \frac{T}{L} \). Therefore \( G \) is transitive on both \( U_L \) and \( W_L \), each with \( 5p = 14 \) points. So for any pair of vertices \( u \in U_L \) and \( w \in W_L \), the stabilizers \( G_u \) and \( G_w \) are of order 12. For any prime power \( q \) all subgroups of the group \( L_2(q) \) have been characterized (see [21], Chapter 3). The only subgroup of
$L_2(7)$ of order 12 is $\mathbb{A}_4$. So $G_u \cong G_w \cong \mathbb{A}_4$. But the pair $(G_u, G_w) = (\mathbb{A}_4, \mathbb{A}_4)$ is not possible for an edge $\{u, w\}$ of a cubic $G$-semisymmetric graph according to Theorem 2.6.

**Lemma 3.7.** For any prime $p$, the group $GL_2(p)$ does not have a subgroup isomorphic to $\mathbb{A}_5$.

**Proof.** For $p = 2, 3$ the claim follows by comparing the orders. Let $p > 3$ and suppose on the contrary that $\mathbb{A}_5 \cong H \leq GL_2(p)$. Then $H = H' \leq (GL_2(p))' = SL_2(p)$. It is a well-known fact that $SL_2(p)$ has only one involution for odd prime $p$, whereas $\mathbb{A}_5$ has more than one element of order 2, a contradiction.

**Lemma 3.8.** Suppose $p > 5$ is a prime and $\Gamma$ is a connected cubic semisymmetric graph of order $20p^2$. Let $A = Aut(\Gamma)$ and take $M$ to be the (normal) Sylow $p$-subgroup of $A$. For $G = \mathbb{A}_5$ or $S_5$ if $\frac{A}{M} \cong G$, then

1. For each vertex $u$ the stabilizer $A_u$ is isomorphic to a subgroup of $G$.
2. $A \cong M \rtimes_\varphi G$ for some homomorphism $\varphi : G \rightarrow Aut(M)$.

**Proof.** For each vertex $u$ of $\Gamma$, $MA_u \leq A$. Therefore $A_u \cong \frac{A_u}{M \cap A_u} \cong \frac{MA_u}{M} \leq \frac{A}{M}$. This proves (1). Also it follows from Theorem 2.4 that $A = MH$ for some subgroup $H \leq A$ where $M \cap H = 1$. So $A$ is the internal semidirect product of $M$ and $H$ and hence it is isomorphic to the external semidirect product of $M$ and $H$; i.e., $A \cong M \rtimes_\psi H$ for some $\psi : H \rightarrow Aut(M)$. Since $G \cong \frac{A}{M} = \frac{MH}{M} \cong \frac{H}{M \cap H} \cong H$, we can write $A \cong M \rtimes_\varphi G$ for some $\varphi : G \rightarrow Aut(M)$.

**Lemma 3.9.** Suppose $p > 5$ is a prime and $\Gamma$ is a connected cubic semisymmetric graph of order $20p^2$. Let $A = Aut(\Gamma)$ and take $M$ to be the Sylow $p$-subgroup of $A$. Then $\frac{A}{M}$ cannot be isomorphic to $\mathbb{A}_5$.

**Proof.** Suppose on the contrary, that $\frac{A}{M} \cong \mathbb{A}_5$. Then for any vertex $u$ from the equality $[A : A_u] = 10p^2$ we obtain $[A_u] = 6$ and hence $A_u \cong Z_6$ or $S_3$. By Lemma 3.8 $A_u \leq \mathbb{A}_5$. Since $\mathbb{A}_5$ does not have elements of order 6, we conclude that $A_u \cong S_3$. Also according to Lemma 3.8, $A \cong M \rtimes_\varphi \mathbb{A}_5$. There are only two possibilities for the kernel of $\varphi : \mathbb{A}_5 \rightarrow Aut(M)$.

(a) If $ker(\varphi) = 1$, then $\mathbb{A}_5$ is isomorphic to a subgroup of $Aut(M)$. Since $M$ of order $p^2$ is abelian, either $M \cong Z_{p^2}$ or $M \cong Z_p \times Z_p$. In the first case $Aut(M)$ is abelian and does not have a subgroup isomorphic to $\mathbb{A}_5$ and in the second case $Aut(M) \cong GL_2(p)$ which again does not have a subgroup isomorphic to $\mathbb{A}_5$ according to Lemma 3.7.

(b) If $ker(\varphi) = \mathbb{A}_5$, then $\varphi$ is the trivial homomorphism and so $A \cong M \times \mathbb{A}_5$. Since $\Gamma$ is semisymmetric, according to Proposition 2.12 $\Gamma$ is isomorphic to $C(A; A_u, A_v)$ where $u$ and $v$ are two adjacent vertices in $\Gamma$. As $\Gamma$ is connected, according to Proposition 2.11 we must have $A = (A_u, A_v)$. In view of $A_u \cong A_v \cong \mathbb{A}_4$ and $A_u \cong \mathbb{A}_4$. So $G_u \cong G_w \cong \mathbb{A}_4$. But the pair $(G_u, G_w) = (\mathbb{A}_4, \mathbb{A}_4)$ is not possible for an edge $\{u, w\}$ of a cubic $G$-semisymmetric graph according to Theorem 2.6. □
S₃, this means that $M \times A_5$ is generated by two of its subgroups, say $H$ and $K$, both of them isomorphic to $S_3$. Now for each element $(m, a) \in H$ we have $(m, a)^6 = 1$ which means $m^6 = 1$ in $M$. As $|M| = p^2$ and $p > 5$, this results in $m = 1$. Therefore the first component of each element of $H$ (and similarly for $K$) equals 1. Consequently the first component of each element in $M \times A_5 = \langle H, K \rangle$ equals 1 which is a contradiction. 

**Lemma 3.10.** Suppose $p > 5$ is a prime and $\Gamma$ is a connected cubic semisymmetric graph of order $20p^2$. Let $A = \text{Aut}(\Gamma)$ and take $M$ to be the Sylow $p$-subgroup of $A$. Then $\frac{A}{M}$ cannot be isomorphic to $S_5$.

**Proof.** If $\frac{A}{M} \cong S_5$, then according to Lemma 3.8, $A \cong M \times_\varphi S_5$ for some homomorphism $\varphi : S_5 \to \text{Aut}(M)$. Moreover, according to the same Lemma for each vertex $u$, $A_u$ is isomorphic to a subgroup of $S_5$. From $[A : A_u] = 10p^2$ we have $|A_u| = 12$. Subgroups of $S_5$ of order 12 are $A_4$ and $D_12$. So $A_u \cong D_{12}$ or $A_4$ for any vertex $u$. Now there are three possibilities for the kernel of $\varphi$.

(a) If $\ker(\varphi) = 1$, then $S_5$ and hence $A_5$ is isomorphic to a subgroup of $\text{Aut}(M)$. But as we showed in part (a) of the proof of Lemma 3.9, $A_5$ cannot be a subgroup of $\text{Aut}(M)$. So this case cannot happen.

(b) If $\ker(\varphi) = S_5$, then $\varphi$ is the trivial homomorphism and so $A \cong M \times S_5$. As in part (b) of the proof of Lemma 3.9 we conclude that $A = \langle A_u, A_v \rangle$ where $u$ and $v$ are two adjacent vertices. Since $|A_u| = |A_v| = 12$, this means that $M \times S_5$ is generated by two of its subgroups, say $H$ and $K$, each of order 12. For each element $(m, a) \in H$ we have $(m, a)^{12} = 1$ which means $m^{12} = 1$. As $|M| = p^2$ and $p > 5$, it follows that $m = 1$. Therefore the first component of each element of $H$ (and similarly for $K$) equals 1. Consequently the first component of each element in $M \times S_5 = \langle H, K \rangle$ equals 1 which is a contradiction.

(c) If $\ker(\varphi) = A_5$, then the image of $\varphi$ is isomorphic to $\frac{S_5}{A_5} \cong \mathbb{Z}_2$. So there is some $T \in \text{Aut}(M)$ of order 2 for which $\varphi(x) = 1$ for all $x \in A_5$ and $\varphi(x) = T$ for any $x \notin A_5$. For any two elements $(x, g)$ and $(y, h)$ from $M \times_\varphi S_5$ the multiplication $(x, g)(y, h)$ equals $(xy, gh)$ if $g \in A_5$ and equals $(xT(y), gh)$ if $g \notin A_5$. Now it is easy to see that in the group $M \times_\varphi S_5$ for any positive integer $n$ if $g \in A_5$, then $(x, g)^n = (x^n, g^n)$ for all $x \in M$, and if $g \notin A_5$, then $(x, g)^{2n} = (x^nT(x^n), g^{2n})$ and $(x, g)^{2n+1} = (x^{n+1}T(x^n), g^{2n+1})$ for all $x \in M$. Take $\{u, v\}$ to be a fixed edge in $\Gamma$. Then according to Proposition 2.12, $\Gamma \cong C(A; A_u, A_v)$. Now according to part (ii) of Proposition 2.11, $A = \langle A_u, A_v \rangle$ and also according to part (i) of Proposition 2.11, $|A_u \cap A_v| = 4$, i.e., $A_u \cap A_v$ is a common Sylow 2-subgroup of both $A_u$ and $A_v$. Since $A \cong M \times_\varphi S_5$, a similar result holds for $M \times_\varphi S_5$; i.e., there are two subgroups $U, V \leq M \times_\varphi S_5$ where

1. $U, V \in \{D_{12}, A_4\}$; and
2. $M \times_\varphi S_5 = \langle U, V \rangle$; and
3. $U \cap V$ is a common Sylow 2-subgroup of both $U$ and $V$. 

We will show however, that the existence of $U$ and $V$ with the above specifications will lead to a contradiction.

Let $H \leq M \cong S_5$ and $H \cong \mathbb{D}_{12}$ or $A_4$. We say $H$ is of type 1 if all elements of $H$ have their second component in $A_5$. We also call $H$ of type 2 if there is at least one element in $H$ whose second component is not in $A_5$.

Define $K := \{(x, g) \in H \mid g \in A_5\}$. Then $K \leq H$. Define $f : K \to M$ by $f(x, g) = x$. Then $f$ is a homomorphism and hence $K/\ker(f) \cong \text{Im}(f)$. Consequently $|K/\ker(f)|$ divides both 12 and $p^2$ and hence $K = \ker(f)$. This means that if $(x, g) \in H$ and $g \in A_5$, then $x = 1$.

Suppose $H$ is of type 1. Then the first component of each element of $H$ equals 1 and so $H$ is isomorphic to a subgroup of $A_5$. Since $A_5$ does not have any element of order 6, $H$ cannot be isomorphic to $\mathbb{D}_{12}$. Hence $H \cong A_4$.

Now suppose $H$ is of type 2. If $(x, g) \in H$ is an arbitrary element with $g \notin A_5$, then $(xT(x), g^2) = (x, g)^2 \in H$. Since $g^2 \in A_5$, we conclude $xT(x) = 1$ and so $T(x) = x^{-1}$. Also if $(y, h) \in H$ is another element with $h \notin A_5$, then $(yT(x), hg) = (y, h)(x, g) \in H$. Again $hg \in A_5$ implies that $yT(x) = 1$ or $T(x) = y^{-1}$. Therefore $x^{-1} = y^{-1}$ and so $x = y$. In other words, for any pair of elements $(x, g) \in H$ and $(y, h) \in H$ with $g \notin A_5$ and $h \notin A_5$ we must have $x = y$. Certainly there are always elements in $H$ whose second component lies in $A_5$ (and hence their first component is 1).

Therefore we can write

$$H = \{(x, g_1), (x, g_2), \ldots, (x, g_n), (1, h_1), \ldots, (1, h_m)\},$$

where $n + m = 12$ and where $g_1, \ldots, g_n \notin A_5$ and $h_1, \ldots, h_m \in A_5$. If $\overline{H} = \{(1, h_1), \ldots, (1, h_m)\}$ then $\overline{H} \leq H$, and if $H_1 = \{h_1, \ldots, h_m\}$ then $H_1 \leq A_5$ and $H_1 \cong \overline{H}$. Multiplying all the elements of $H$ by $(x, g_t)$, $t$ arbitrary, we again obtain $H$.

Therefore

$$H = \{(1, g_t h_1), (1, g_t h_2), \ldots, (1, g_t h_n), (x, g_t h_1), \ldots, (x, g_t h_m)\}.$$

Comparing the two equalities for $H$ and by taking into account that $g_t g_i \in A_5$ for $i = 1, \ldots, n$ and $g_t h_j \notin A_5$ for $j = 1, \ldots, m$, we obtain

$$\{g_t g_1, g_t g_2, \ldots, g_t g_n\} = \{h_1, \ldots, h_m\}$$

and

$$\{g_t h_1, \ldots, g_t h_m\} = \{g_1, \ldots, g_n\}.$$

This results in $m = n = 6$. So $|\overline{H}| = 6$. Since $A_4$ does not have a subgroup of order 6, the first conclusion is that $H$ cannot be isomorphic to $A_4$ and hence $H \cong A_4$. The group $\mathbb{D}_{12}$ has only two subgroups of order 6, namely $Z_6$ and $S_3$. If $\overline{H} \cong Z_6$, then $Z_6 \cong H_1 \leq A_5$. But $A_5$ does not have elements of order 6. Therefore $\overline{H}$ cannot be isomorphic to $Z_6$ and hence $\overline{H} \cong S_3$. Since $H \cong \mathbb{D}_{12}$,
we can write $H = \{a^i \mid i = 0, \ldots, 5\} \cup \{ba^i \mid i = 0, \ldots, 5\}$. As $\overline{H} \cong S_3$ does not have any element of order 6, $a$ must be of the form $(x, g)$ for some $g \notin A_5$. As for $b$, there are two possible cases: either $b = (1, h) \in \overline{H}$ in which case we call $H$ of (sub)type 2.1 or $b = (x, g') \in H - \overline{H}$ in which case we call $H$ of (sub)type 2.2.

If $H$ is of type 2.1 and $b = (1, h), h \in A_5$, then

$$H^1(x, g, h) = \{(x, g)^i \mid i = 0, \ldots, 5\} \cup \{(1, h)(x, g)^i \mid i = 0, \ldots, 5\}$$

$$= \{(1, 1), (x, g), (1, g^2), (x, g^3), (1, g^4), (x, g^5),$$

$$\quad (1, h), (x, hg), (1, hg^2), (x, hg^3), (1, hg^4), (x, hg^5)\}.$$

In this case all the Sylow 2-subgroups of $H$ are as follows:

$$HP_1^1(x, g, h) = \{(1, 1), (x, g^3), (1, h), (x, hg^3)\},$$

$$HP_2^1(x, g, h) = \{(1, 1), (x, g^3), (x, hg), (1, hg^4)\},$$

$$HP_3^1(x, g, h) = \{(1, 1), (x, g^3), (1, hg^2), (x, hg^5)\}.$$

Also if $H$ is of type 2.2 and $b = (x, g'), g' \notin A_5$, then

$$H^2(x, g, g') = \{(1, 1), (x, g), (1, g^2), (x, g^3), (1, g^4), (x, g^5),$$

$$\quad (x, g'), (1, g'g), (x, g'g^2), (1, g'g^3), (x, g'g^4), (1, g'g^5)\}$$

and all the Sylow 2-subgroups of $H$ are as follows:

$$HP_1^2(x, g, g') = \{(1, 1), (x, g^3), (x, g'), (1, g'g^3)\},$$

$$HP_2^2(x, g, g') = \{(1, 1), (x, g^3), (1, g'g), (x, g'g^4)\},$$

$$HP_3^2(x, g, g') = \{(1, 1), (x, g^3), (x, g'g^2), (1, g'g^5)\}.$$

In the above notations, the superindex $j = 1, 2$ refers to type $2j$ of $H$.

Now we go back and consider $U$ and $V$. There are only three possible cases for their types.

**Case c1.** Both $U$ and $V$ are of type 1. This means $(A_u, A_v) = (A_4, A_4)$ which is not possible according to Theorem 2.6.

**Case c2.** One of $U$ and $V$ is of type 1 and the other has type 2. Without loss of generality assume $U$ has type 1 and $V$ is of type 2. So every element of $U$ has its first component equal to 1 and so will be every element of $U \cap V$ which is a Sylow 2-subgroup of both $U$ and $V$. No matter what the subtype of $V$ is, this says that the first component of every element of $V$ should be 1. For example, if

$$U \cap V = VP_2^2(x, g, g') = \{(1, 1), (x, g^3), (1, g'g), (x, g'g^4)\}$$
for some \( x \in M, g, g' \notin A_5 \), then \( x = 1 \) and hence the first component of every element of \( V = V^2(x, g, g') \) is 1. Now \( U \) and \( V \) cannot both generate \( M \rtimes \varphi S_5 \) since every element of \( \langle U, V \rangle \) is an alternating product of elements from \( U \) and \( V \) which will have its first component equal to 1 because \((1, t)(1, s) = (1, ts)\) for any \( t, s \in S_5 \).

Case c3. Both \( U \) and \( V \) are of type 2. In this case, there are four possible cases for the subtypes of \( U \) and \( V \). Suppose \( U = U^j(x, a, b) \) and \( V = V^j(y, \alpha, \beta) \) for some \( x, y \in M \) and some suitable \( a, b, \alpha, \beta \in S_5 \). Since \( U \cap V \) is a Sylow 2-subgroup of both \( U \) and \( V \), we must have \( UP^j(x, a, b) = VP^j(y, \alpha, \beta) \) for some \( i', j' \in \{1, 2, 3\} \). We compare the first components of non-identity elements from the two sides of this equality. If \( x = 1 \), then inevitably \( y = 1 \). If \( x \neq 1 \), then \( x = y \). Therefore \( x = y \) in any case.

Define \( M_x = (\{1\} \times \mathbb{A}_5) \cup (\{x\} \times (S_5 - \mathbb{A}_5)) \). It is easy to check that \( M_x \leq M \rtimes \varphi S_5 \). Obviously \( U \cup V \subset M_x \), and so \( \langle U, V \rangle \leq M_x \). Therefore \( \langle U, V \rangle \neq M \rtimes \varphi S_5 \).  

**Proof of Theorem 3.1.** According to [6] there is no connected cubic semisymmetric graph of order 20p^2 for \( p = 2, 3 \) and 5. So let \( p > 5 \) and suppose on the contrary that \( \Gamma \) is a connected cubic semisymmetric graph of order 20p^2. Let \( \{U, W\} \) be a bipartition for \( \Gamma \) and \( A = \text{Aut}(\Gamma) \). Then \( |U| = |W| = 10p^2 \) and \( |A| = 2^{r+1} \cdot 3 \cdot 5 \cdot p^2 \) for some \( 0 \leq r \leq 7 \). Also assume \( M \) is a Sylow p-subgroup of \( A \). According to Lemma 3.6, \( M \trianglelefteq A \). Due to its order, \( M \) is intransitive on both \( U \) and \( W \) and so according to Theorem 2.9, \( \Gamma_M \) is a connected cubic \( G \)-semisymmetric graph of order 20 with the bipartition \( \{U_M, W_M\} \), where \( G = \frac{A}{M} \) and \( |U_M| = |W_M| = 10 \).

Therefore \( \Gamma_M \) is \( G \)-edge-transitive and hence edge-transitive. Now if \( \Gamma_M \) is not vertex-transitive, then it must be semisymmetric, but there is no semisymmetric cubic graph of order 20 according to [6]. So \( \Gamma_M \) should be vertex-transitive and hence symmetric since according to [23] a cubic vertex and edge-transitive graph is necessarily symmetric. According to [7] there are only two symmetric cubic graphs of order 20: F20A and F20B. Since \( \Gamma_M \) is bipartite and F20A is not bipartite, we should have \( \Gamma_M \cong F20B \).

The automorphism group of F20B has 240 elements [7] and \( G = \frac{A}{M} \) is isomorphic to a subgroup of \( \text{Aut}(F20B) \) of order \( |G| = 2^{r+1} \cdot 3 \cdot 5 \). The equality is not possible since \( G \) is not transitive on \( V(F20B) \) whereas \( \text{Aut}(F20B) \) is. So \( |G| < 240 \) and hence \( 1 \leq r + 1 < 3 \). Also \( G \) is transitive on both \( U_M \) and \( W_M \) and according to Proposition 2.8 the action of \( G \) on each of \( U_M \) and \( W_M \) is faithful. Therefore \( G \) is a transitive permutation group of degree 10. Transitive permutation groups of degree 10 have been completely classified in [5]. There are 45 such groups up to isomorphism which are denoted by \( T1, T2, \ldots, T45 \) in [5]. Among these, the only groups whose orders are of the form \( 2^i \cdot 3 \cdot 5 \) for \( 1 \leq i \leq 3 \), are \( T7 \cong \mathbb{A}_5 \) of order 60, and \( T11, T12 \) and \( T13 \cong S_5 \) of order 120. We first argue
that $G$ could not be isomorphic to $T_{11}$ or $T_{12}$.

In [5] all the transitive groups of degree 10 have been defined with a set of generating permutations on ten points. If

$$a = (1, 2, 3, 4, 5), \quad b = (6, 7, 8, 9, 10),$$
$$e = (1, 5)(2, 3), \quad f = (6, 10)(7, 8), \quad g = (1, 2), \quad h = (6, 7) \text{ and}$$
$$i = (1, 6)(2, 7)(3, 8)(4, 9)(5, 10),$$

then

$$T_{11} = \langle ab, ef, i \rangle, \quad T_{12} = \langle ab, ef, ghi \rangle.$$ 

Using the GAP software it is easy to verify that $H = \langle i \rangle \trianglelefteq T_{11}$. Obviously $|H| = 2$.

If $G \cong T_{11}$, then according to Theorem 2.9 the quotient graph of $\Gamma_M$ with respect to $H$ which we denote by $(\Gamma_M)_H$, would be $R$-semisymmetric of order 10, where $R = \frac{T_{11}}{H}$. This implies that $R$ is transitive on each partite set and by Proposition 2.8, $R$ would be a transitive permutation group of degree 5. Again according to [5] the only transitive permutation group of degree 5 and of order 60 is $A_5$. Now the stabilizer of any vertex of $(\Gamma_M)_H$ under the action of $R$ has $|R| = 12$ points and the only subgroup of $A_5$ of order 12 is isomorphic to $A_4$. So for an edge $\{u, w\}$ of the cubic $R$-semisymmetric graph $(\Gamma_M)_H$, we have $(R_u, R_w) = (A_4, A_4)$ which is not possible according to Theorem 2.6. Therefore the assumption that $G \cong T_{11}$, leads to a contradiction.

Now suppose $G \cong T_{12}$. Since $G$ is transitive on either of the two partite sets, the stabilizers of any vertex in one partite set are isomorphic. Consider one vertex, e.g. 1. Using GAP software, we easily obtain the stabilizer of 1 under $T_{12}$ as

$$\langle (2, 4)(3, 5)(7, 9)(8, 10), (3, 5, 4)(8, 10, 9) \rangle.$$ 

Using GAP one finds out that this group is nonabelian of order 12 which has the following group of order 4 as a normal subgroup:

$$\langle (2, 3)(4, 5)(7, 8)(9, 10), (2, 4)(3, 5)(7, 9)(8, 10) \rangle.$$ 

There are only 3 nonabelian groups of order 12 up to isomorphism: $A_4$, $D_{12}$ and the dicyclic group of order 12. Among these, the only one having a normal subgroup of order 4, is $A_4$. So the stabilizer of any vertex in $\Gamma_M$ would be isomorphic to $A_4$ which is not possible according to Theorem 2.6.

Finally $\frac{A_4}{H} \cong T_7 \cong A_5$ is not possible according to Lemma 3.9 and $\frac{A_7}{H} \cong T_{13} \cong S_5$ is not possible according to Lemma 3.10.

Since every case for $\frac{A_4}{H}$ is contradictory, we conclude that there is no connected cubic semisymmetric graph of order $20p^2$. \[ \blacksquare \]
Acknowledgement

The authors would like to thank the financial support of INSF (Iran National Science Foundation). We also thank the anonymous referees for their helpful suggestions.

References


On Semisymmetric Cubic Graphs of Order $20p^2$, $p$ Prime


Received 8 October 2018
Revised 26 February 2019
Accepted 5 March 2019