THE HILTON-SPENCER CYCLE THEOREMS VIA KATONA’S SHADOW INTERSECTION THEOREM

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Abstract

A family $\mathcal{A}$ of sets is said to be intersecting if every two sets in $\mathcal{A}$ intersect. An intersecting family is said to be trivial if its sets have a common element. A graph $G$ is said to be $r$-EKR if at least one of the largest intersecting families of independent $r$-element sets of $G$ is trivial. Let $\alpha(G)$ and $\omega(G)$ denote the independence number and the clique number of $G$, respectively. Hilton and Spencer recently showed that if $G$ is the vertex-disjoint union of a cycle $C$ raised to the power $k$ and $s$ cycles $1C, \ldots, sC$ raised to the powers $k_1, \ldots, k_s$, respectively, $1 \leq r \leq \alpha(G)$, and

$$\min(\omega(1C^{k_1}), \ldots, \omega(sC^{k_s})) \geq \omega(C^k),$$

then $G$ is $r$-EKR. They had shown that the same holds if $C$ is replaced by a path $P$ and the condition on the clique numbers is relaxed to

$$\min(\omega(1C^{k_1}), \ldots, \omega(sC^{k_s})) \geq \omega(P^k).$$

We use the classical Shadow Intersection Theorem of Katona to obtain a significantly shorter proof of each result for the case where the inequality for the minimum clique number is strict.

Keywords: cycle, independent set, intersecting family, Erdős-Ko-Rado theorem, Hilton-Spencer theorem, Katona’s shadow intersection theorem.

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1. Introduction

Unless stated otherwise, we shall use small letters such as $x$ to denote non-negative integers or elements of a set, capital letters such as $X$ to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote families (sets whose members are sets themselves). The set of positive integers is denoted by $\mathbb{N}$. The set $\{i \in \mathbb{N}: m \leq i \leq n\}$ is denoted by $[m, n]$, $[1, n]$ is abbreviated to $[n]$, and $[0]$ is taken to be the empty set $\emptyset$. For a set $X$, the power set of $X$ (that is, $\{A: A \subseteq X\}$) is denoted by $2^X$. The family of $r$-element subsets of $X$ is denoted by $\binom{X}{r}$. The family of $r$-element sets in a family $\mathcal{F}$ is denoted by $\mathcal{F}^r$. If $\mathcal{F} \subseteq 2^X$ and $x \in X$, then the family $\{A \in \mathcal{F}: x \in A\}$ is denoted by $\mathcal{F}(x)$ and called the star of $\mathcal{F}$ with centre $x$.

A family $\mathcal{A}$ is said to be intersecting if for every $A, B \in \mathcal{A}$, $A$ and $B$ intersect (that is, $A \cap B \neq \emptyset$). The stars of a family $\mathcal{F}$ are among the simplest intersecting subfamilies of $\mathcal{F}$. We say that $\mathcal{F}$ has the star property if at least one of the largest intersecting subfamilies of $\mathcal{F}$ is a star of $\mathcal{F}$.

Determining the size of a largest intersecting subfamily of a given family $\mathcal{F}$ is one of the most popular endeavours in extremal set theory. This started in [11], which features the classical result known as the Erdős-Ko-Rado (EKR) Theorem. The EKR Theorem states that if $r \leq n/2$ and $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Thus, $\binom{[n]}{r}$ has the star property for $r \leq n/2$ (clearly, for $n/2 < r \leq n$, $\binom{[n]}{r}$ itself is intersecting). There are various proofs of the EKR Theorem (see [9,16,24,25,27]), two of which are particularly short and beautiful: Katona’s [25], which introduced the elegant cycle method, and Daykin’s [9], using the fundamental Kruskal-Katona Theorem [26,28]. The EKR Theorem gave rise to some of the highlights in extremal set theory [1,14,27,30] and inspired many variants and generalizations; see [4,10,13,15,17,21,22].

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We may represent an edge $\{v, w\}$ by $vw$. A subset $I$ of $V(G)$ is an independent set of $G$ if $vw \notin E(G)$ for every $v, w \in I$. Let $\mathcal{I}_G$ denote the family of independent sets of $G$. An independent set $J$ of $G$ is maximal if $J \not\subseteq I$ for each independent set $I$ of $G$ such that $I \neq J$. The size of a smallest maximal independent set of $G$ is denoted by $\mu(G)$. The size of a largest independent set of $G$ is denoted by $\alpha(G)$. A subset $X$ of $V(G)$ is a clique of $G$ if $vw \in E(G)$ for every $v, w \in X$ with $v \neq w$. The size of a largest clique of $G$ is called the clique number of $G$ and denoted by $\omega(G)$.

Holroyd and Talbot introduced the problem of determining whether $\mathcal{I}_G^{(r)}$ has the star property for a given graph $G$ and an integer $r \geq 1$. Following their terminology, a graph $G$ is said to be $r$-EKR if $\mathcal{I}_G^{(r)}$ has the star property. The Holroyd-Talbot (HT) Conjecture [22, Conjecture 7] claims that $G$ is $r$-EKR if $\mu(G) \geq 2r$. This was verified by Borg [2] for $\mu(G)$ sufficiently large depending on $r$ (see also [6, Lemma 4.4 and Theorem 1.4]). By the EKR Theorem, the
conjecture is true if \( G \) has no edges. The HT Conjecture has been verified for several classes of graphs \([2, 3, 7, 8, 12, 18–23, 29, 31]\). As demonstrated in \([8]\), for \( r > \mu(G)/2 \), whether \( G \) is \( r \)-EKR or not depends on \( G \) and \( r \) (both cases are possible). Naturally, graphs \( G \) of particular interest are those that are \( r \)-EKR for all \( r \leq \alpha(G) \).

For \( n \geq 1 \), the graphs \([n], \binom{[n]}{2}\) and \([n], \{\{i, i + 1\}: i \in [n - 1]\}\) are denoted by \( K_n \) and \( P_n \), respectively. For \( n \geq 3 \), \(([n], E(P_n) \cup \{\{n, 1\}\})\) is denoted by \( C_n \). A copy of \( K_n \) and \( P_n \) is called a complete graph. A copy \( P_n \) is called an \( n \)-path or simply a path, and a vertex of \( P \) is called an end-vertex if it is not adjacent to more than one vertex. A copy of \( C_n \) is called an \( n \)-cycle or simply a cycle (normally, this terminology is used for \( n \geq 3 \), but we may include the case \( n = 2 \)). If \( H \) is a subgraph of a graph \( G \) (that is, \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \)), then we say that \( G \) contains \( H \). For \( v, w \in V(G) \), the distance \( d_G(v, w) \) is \( \min\{k: v, w \in V(P) \text{ for some } (k + 1) \text{-path } P \text{ contained by } G\} \). The \( k \)-th power of \( G \), denoted by \( G^k \), is the graph with vertex set \( V(G) \) and edge set \( \{vw: v, w \in V(G), 1 \leq d_G(v, w) \leq k\} \); \( G^k \) is also referred to as \( G \) raised to the power \( k \).

Note that \( P_n^k = K_n \) for \( n \leq k + 1 \), and \( C_n^k = K_n \) for \( n \leq 2k + 1 \). Also note that \( \omega(P_n^k) = k + 1 \) if \( n \geq k + 1 \), \( \omega(P_n^k) = n \) if \( n \leq k \); \( \omega(C_n^k) = k + 1 \) if \( n \geq 2k + 2 \), \( \omega(C_n^k) = n \) if \( n \leq 2k + 1 \); \( \alpha(P_n^k) = \lfloor n/(k + 1) \rfloor \), \( \alpha(C_n^k) = \lfloor n/(k + 1) \rfloor \) if \( n \geq k + 1 \), and \( \omega(C_n^k) = 1 \) if \( 2 \leq n \leq k + 1 \).

The following remarkable analogue of the EKR theorem was obtained by Talbot \([29]\).

**Theorem 1** \([29]\). For \( 1 \leq r \leq \alpha(C_n^k) \), \( C_n^k \) is \( r \)-EKR.

Talbot introduced a compression technique to prove Theorem 1. In vague terms, his compression technique rotates anticlockwise the elements of the independent sets of the intersecting family which are distinct from a specified vertex (see Section 2).

If \( G, G_1, \ldots, G_k \) are graphs such that the vertex sets of \( G_1, \ldots, G_k \) are pairwise disjoint and \( G = \left( \bigcup_{i=1}^{k} V(G_i), \bigcup_{i=1}^{k} E(G_i) \right) \), then \( G \) is said to be the disjoint union of \( G_1, \ldots, G_k \), and \( G_1, \ldots, G_k \) are said to be vertex-disjoint.

Inspired by the work of Talbot, Hilton and Spencer \([19]\) went on to prove the following result, which is stated with notation used in \([19, 20]\).

**Theorem 2** \([19]\). If \( G \) is the disjoint union of a path \( P \) raised to the powers \( k_1, \ldots, k_s \) and \( s \) cycles \( C_i \), \( i = 1, \ldots, s \), raised to the powers \( k_i \), respectively, \( 1 \leq r \leq \alpha(G) \), and

\[
\min (\omega(C_1^{k_1}), \ldots, \omega(C_s^{k_s})) \geq \omega(P^k),
\]

then \( G \) is \( r \)-EKR. Moreover, for any end-vertex \( x \) of \( P \), \( I_{G^r}(x) \) is a largest intersecting subfamily of \( I_{G^r} \).
However, it was desired to obtain a generalization of Theorem 1, and this was eventually achieved by Hilton and Spencer [20] with the following theorem.

**Theorem 3** [20]. If $G$ is the disjoint union of $s + 1$ cycles $C, 1C, \ldots, sC$ raised to the powers $k, k_1, \ldots, k_s$, respectively, $1 \leq r \leq \alpha(G)$, and

$$\min (\omega(1C^{k_1}), \ldots, \omega(sC^{k_s})) \geq \omega(C^k),$$

then $G$ is $r$-EKR. Moreover, for any $x \in V(C)$, $I_G^{(r)}(x)$ is a largest intersecting subfamily of $I_G^{(r)}$.

Hilton and Spencer [20] conjectured that every disjoint union of powers of cycles is $r$-EKR.

The proof of Theorem 3 is also inspired by Talbot’s proof of Theorem 1. In particular, an essential ingredient in the proof of Theorem 3 is the use of Theorem 2 for the special case where $P^k$ is a complete graph as the base case of an induction argument.

In this paper, we give a significantly shorter and simpler proof of Theorem 2 and of Theorem 3, except for the cases of equality in conditions (1) and (2), respectively. In other words, we prove the following two results.

**Theorem 4.** Theorem 2 is true if the inequality in (1) is strict.

**Theorem 5.** Theorem 3 is true if the inequality in (2) is strict.

Our argument is based on the Shadow Intersection Theorem of Katona [27], hence demonstrating yet another application of this classical and useful result in extremal set theory.

## 2. The New Proof

Let $P, 1C, \ldots, sC$ be as in Theorem 2. Let $p = |V(P)|$ and $c_i = |V(iC)|$. For $1 \leq i \leq s$, we label the vertices of $iC$ by $v_{1,i}, v_{2,i}, \ldots, v_{c_i,i}$, where $E(iC) = \{v_{j,i}v_{j+1,i} : j \in [c_i - 1]\} \cup \{v_{c_i,i}v_{1,i}\}$. We may assume that $P = P_p$, that is, $V(P) = [p]$ and $E(P) = \{\{i, i+1\} : i \in [p-1]\}$. Let $H$ be the union of $1C^{k_1}, \ldots, sC^{k_s}$, and let $f : V(H) \to V(H)$ be the bijection given by

$$f(v_{c_i,i}) = v_{1,i} \quad \text{and} \quad f(v_{j,i}) = v_{j+1,i} \quad \text{for} \quad 1 \leq i \leq s \quad \text{and} \quad 1 \leq j \leq c_i - 1.$$

Let $f^1 = f$, and for any integer $t \geq 2$, let $f^t = f \circ f^{t-1}$ and $f^{-t} = f^{-1} \circ f^{-(t-1)}$. Note that for $t \geq 1$, one can think of $f^t$ as $t$ clockwise rotations, and of $f^{-t}$ as $t$ anticlockwise rotations. For $I \in \mathcal{I}_H$, we denote the set $\{f^t(x) : x \in I\}$ by $f^t(I)$, and for $A \subseteq \mathcal{I}_H$, we denote the family $\{f^t(A) : A \in A\}$ by $f^*(A)$. The notation $f^{-t}(I)$ and $f^{-t}(A)$ is defined similarly.

The new argument presented in this paper lies entirely in the proof of the following important case, which both Theorem 4 and Theorem 5 pivot on.
Lemma 6. Theorem 2 is true if $P^k$ is a complete graph and the inequality in (1) is strict.

As shown in this section, Theorem 4 follows from Lemma 6 by applying the compression method in [21], and Theorem 5 follows from Lemma 6 by applying the same compression method of Talbot in [29].

We now start working towards the proof of Lemma 6.

Let $A$ be a family of $r$-element sets. The shadow of $A$, denoted by $\partial A$, is the family $\bigcup_{A \in A} (\partial A)$. A special case of Katona’s Shadow Intersection Theorem [27] is that

\begin{equation}
|A| \leq |\partial A| \quad \text{if } A \text{ is intersecting.}
\end{equation}

**Proof of Lemma 6.** Suppose that $P^k$ is a complete graph. Then, $\omega(P^k) = p$. Suppose $\min(\omega(1C^{k_1}), \ldots, \omega(1C^{k_s})) > p$. Note that this implies that for every $i \in [s]$, $c_i \geq p + 1$ and, for every $v_{h,i}, v_{j,i} \in V(1C^{k_i})$,

\begin{equation}
(4) \quad \text{if } v_{h,i} \in \{ f^{-q}(v_{h,i}) : q \in [p] \} \cup \{ f^q(v_{h,i}) : q \in [p] \}, \text{ then } v_{h,i}v_{j,i} \in E(C^{k_i}).
\end{equation}

It is worth pointing out that the strict inequality is only used for (4), from which we obtain Claim 7.

Let $A$ be an intersecting subfamily of $\mathcal{I}_G^{(r)}$. Recall that $V(P^k) = [p]$. Let $A_0 = \{ A \in A : A \cap [p] = \emptyset \}$ and $A_i = \{ A \in A : A \cap [p] = \{i\} \}$ for $1 \leq i \leq p$. Since $P^k$ is a complete graph, the families $A_0, A_1, \ldots, A_p$ partition $A$. Let $A_0' = A_0$ and $A_i' = \{ A \setminus \{i\} : A \in A_i \}$ for $1 \leq i \leq p$. Since $A$ is intersecting,

\begin{equation}
(5) \quad \text{for any } i, j \in \{0\} \cup [p] \text{ with } i \neq j, \text{ each set in } A_i' \text{ intersects each set in } A_j'.
\end{equation}

**Claim 7.** The families $\partial A_0, f^1(A_1', f^2(A_2'), \ldots, f^p(A_p')$ are pairwise disjoint.

**Proof.** Suppose $B \in f^i(A_1') \cap f^j(A_j')$ for some $i, j \in [p]$ with $i < j$. Then, $B = f^i(A_1') = f^j(A_j')$ for some $A_1 \in A_1'$ and $A_j \in A_j'$. Thus, $A_1 = f^{j-i}(A_j)$. Since $1 \leq j - i < p$, (4) gives us $A_1 \cap A_j = \emptyset$, but this contradicts (5). Therefore, $f^1(A_1'), f^2(A_2'), \ldots, f^p(A_p')$ are pairwise disjoint.

Suppose $B \in \partial A_0 \cap f^i(A_i')$ for some $i \in [p]$. Then, $C \setminus \{x\} = B = f^i(A_i)$ for some $C \in A_0, x \in C$, and $A_i \in A_i'$. Since $1 \leq i \leq p$, (4) gives us $C \cap A_i = \emptyset$, but this contradicts (5). The claim follows.

Let $A_0^* = \{ A \cup \{1\} : A \in \partial A_0 \}$ and $A_i^* = \{ A \cup \{1\} : A \in f^i(A_i') \}$ for $1 \leq i \leq p$. For $0 \leq i \leq p$, $A_i^* \subseteq \mathcal{I}_G^{(r)}(1)$. By Claim 7, $\sum_{i=0}^{p} |A_i^*| = |\bigcup_{i=0}^{p} A_i^*| \leq |\mathcal{I}_G^{(r)}(1)|$. By (3), $|A_0| \leq |\partial(A_0)| = |A_0^*|$. We have

\begin{equation}
|A| = \sum_{i=0}^{p} |A_i| = |A_0| + \sum_{i=1}^{p} |A_i^*| \leq \sum_{i=0}^{p} |A_i^*| \leq |\mathcal{I}_G^{(r)}(1)|,
\end{equation}

and the lemma is proved.
The full Theorem 4 is now obtained by the line of argument laid out in [21], hence making use of established facts regarding compressions on independent sets.

For any edge uv of a graph G, let \( \delta_{u,v} : \mathcal{I}_G \rightarrow \mathcal{I}_G \) be defined by

\[
\delta_{u,v}(A) = \begin{cases} 
(A\backslash \{v\}) \cup \{u\} & \text{if } v \in A, u \notin A, \text{ and } (A\backslash \{v\}) \cup \{u\} \in \mathcal{I}_G; \\
A & \text{otherwise},
\end{cases}
\]

and let \( \Delta_{u,v} : 2^{\mathcal{I}_G} \rightarrow 2^{\mathcal{I}_G} \) be the compression operation (also called a shifting operation) defined by

\[
\Delta_{u,v}(A) = \{ \delta_{u,v}(A) : A \in \mathcal{A} \} \cup \{ A \in \mathcal{A} : \delta_{u,v}(A) \in \mathcal{A} \}.
\]

It is well known, and easy to see, that

\[
|\Delta_{u,v}(A)| = |A|
\]

(see [11, 15]). For any \( x \in V(G) \), let \( N_G(x) \) denote the set \( \{ y \in V(G) : xy \in E(G) \} \). The following is given by [8, Lemma 2.1] (which is actually stated for \( \mathcal{I}_G^{(r)} \) but proved for \( \mathcal{I}_G \)) and essentially originated in [21]. We omit the proof.

**Lemma 8** [8, 21]. If \( G \) is a graph, \( uv \in E(G) \), \( A \) is an intersecting subfamily of \( \mathcal{I}_G \), \( \mathcal{B} = \Delta_{u,v}(A) \), \( \mathcal{B}_0 = \{B \in \mathcal{B} : v \notin B\} \), \( \mathcal{B}_1 = \{B \in \mathcal{B} : v \in B\} \), and \( \mathcal{B}'_1 = \{B\backslash \{v\} : B \in \mathcal{B}_1\} \), then

(i) \( \mathcal{B}_0 \) is intersecting;
(ii) if \( |N_G(u)\backslash (\{v\} \cup N_G(v))| \leq 1 \), then \( \mathcal{B}'_1 \) is intersecting;
(iii) if \( |N_G(u)\backslash (\{v\} \cup N_G(v))| = 0 \), then \( \mathcal{B}_0 \cup \mathcal{B}'_1 \) is intersecting.

For a vertex \( v \) of a graph \( G \), let \( G - v \) denote the graph obtained by deleting \( v \) (that is, \( G - v = (V(G)\backslash \{v\}, \{xy \in E(G) : x, y \notin \{v\}\}) \)), and let \( G \downarrow v \) be the graph obtained by deleting \( v \) and the vertices adjacent to \( v \) (that is, \( G \downarrow v = (V(G)\backslash \{v\} \cup N_G(v), \{xy \in E(G) : x, y \notin \{v\} \cup N_G(v)\}) \)).

**Proof of Theorem 4.** We use induction on \( |V(P)| \). If \( P^k \) is a complete graph, then the result is given by Lemma 6. Note that this captures the base case \( |V(P)| = 1 \). Now suppose that \( P^k \) is not a complete graph. Then, \( |V(P)| \geq k+2 \). If \( r = 1 \), then the result is trivial. Suppose \( r > 1 \). Let \( \mathcal{A} \) be an intersecting subfamily of \( \mathcal{I}_G^{(r)} \). Let \( u = p-1 \) and \( v = p \). Let \( \mathcal{B} = \Delta_{u,v}(A) \), \( \mathcal{B}_0 = \{B \in \mathcal{B} : v \notin B\} \), \( \mathcal{B}_1 = \{B \in \mathcal{B} : v \in B\} \), and \( \mathcal{B}'_1 = \{B\backslash \{v\} : B \in \mathcal{B}_1\} \). By Lemma 8(i), \( \mathcal{B}_0 \) is intersecting. We have \( N_G(u)\backslash (\{v\} \cup N_G(v)) = \{p-k-1\} \), so, by Lemma 8(ii), \( \mathcal{B}'_1 \) is intersecting. Let \( H_0 = P_{p-1} \) and \( H_1 = P_{p-k-1} \). Clearly, \( \mathcal{B}_0 \subseteq \mathcal{I}_{G-u}^{(r)} \), \( \mathcal{B}'_1 \subseteq \mathcal{I}_{G-v}^{(r)} \), \( G-v \) is the union of \( H_0^k \) and \( 1C_{k_1} \ldots C_{k_s} \), and \( G \downarrow v \) is the union of \( H_1^k \) and \( 1C_{k_1} \ldots C_{k_s} \). The condition \( \min(\omega(1C_{k_1}), \ldots, \omega(sC_{k_s})) > \omega(P^k) \).
in the theorem gives us \( \min(\omega(C_1^{k_1}), \ldots, \omega(C_{s}^{k_s})) \geq \omega(H_0^k) \). By the induction hypothesis, \( |B_0| \leq |I_{G-v}^{(r)}(1)| \) and \( |B'_1| \leq |I_{G_{v-v}}^{(r-1)}(1)| \). We have

\[
|A| = |B| = |B_0| + |B'_1| \leq |I_{G-v}^{(r)}(1)| + |I_{G_{v}}^{(r-1)}(1)|
\]

\[
= |\{ A \in I_G^{(r)} : 1 \in A, v \notin A \}| + |\{ A \in I_G^{(r)} : 1 \in A, v \in A \}| = |I_{G}^{(r)}(1)|,
\]

as required.

**Proof of Theorem 5.** We use induction on \( c = |V(C)| \). We may assume that \( C = C_c \). If \( C^k \) is a complete graph, then the result is given by Lemma 6. Note that this captures the base case \( c = 2 \). Now suppose that \( C^k \) is not a complete graph. Then, \( c \geq 2k + 2 \). If \( r = 1 \), then the result is trivial. Suppose \( r > 1 \). Let \( A \) be an intersecting subfamily of \( I_G^{(r)} \).

Let \( g : V(G) \to V(G) \) be the Talbot compression [20,29] given by

\[
g(v) = v \quad \text{for } v \in V(G) \setminus V(C),
\]

\[
g(1) = 1, \quad \text{and}
\]

\[
g(1 + j) = 1 + j - 1 \quad \text{for } 1 \leq j \leq c - 1.
\]

For \( X \in I_G \) and \( X \subseteq I_G \), we use the notation \( g^t(X) \) and \( g^t(X) \) similarly to the way it is used above for \( f \). Let \( F \) be the union of \( C_{c-1}^k \) and \( 1C_{k_1}, \ldots, sC_{k_s} \). Let \( K \) be the union of \( C_{c-k-1}^k \) and \( 1C_{k_1}, \ldots, sC_{k_s} \). Let

\[
B = \{ A \in A : 1 \notin A, g(A) \in I_F^{(r)} \},
\]

\[
C = \{ A \in A : 1 \in A, g(A) \in I_F^{(r)} \},
\]

\[
D_0 = \{ A \in A : 1 \in A, k + 2 \in A \},
\]

\[
D_i = \{ A \in A : 1 + c - i, k + 2 - i \in A \} \quad \text{for } 1 \leq i \leq k.
\]

Note that these families partition \( A \). Let

\[
F = \left( g^{k-1}(E) - \{1\} \right) \cup \bigcup_{i=0}^{k} \left( g^k(D_i) - \{1\} \right),
\]

where \( E = g(B) \cap g(C) \) and, for any family \( G, G - \{1\} = \{ G \setminus \{1\} : G \in \mathcal{G} \} \).

**Claim 9 (See [20,29]).** The following hold

(i) \( |A| = |g(B \cup C)| + |F| \);

(ii) \( g(B \cup C) \) is an intersecting subfamily of \( I_F^{(r)} \);

(iii) \( g(F) \) is an intersecting subfamily of \( I_K^{(r-1)} \) of size \( |F| \);

(iv) \( |I_G^{(r)}(1)| = |I_F^{(r)}(1)| + |I_K^{(r-1)}(1)| \).
By the induction hypothesis and Claim 9(ii)–(iii), $|g(B \cup C)| \leq |I_F^{(r)}(1)|$ and $|F| = |g(F)| \leq |I_K^{(r-1)}(1)|$. Thus, by Claim 9(i) and Claim 9(iv), we have

$$|A| = |g(B \cup C)| + |F| \leq |I_F^{(r)}(1)| + |I_K^{(r-1)}(1)| = |I_G^{(r)}(1)|,$$

and the theorem is proved.

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