SOME RESULTS ON PATH-FACTOR CRITICAL AVOIDABLE GRAPHS

SIZHONG ZHOU

School of Science
Jiangsu University of Science and Technology
Zhenjiang, Jiangsu 212003, China

e-mail: zsz_cumt@163.com

Abstract

A path factor is a spanning subgraph $F$ of $G$ such that every component of $F$ is a path with at least two vertices. We write $P_{\geq k} = \{P_i : i \geq k\}$. Then a $P_{\geq k}$-factor of $G$ means a path factor in which every component admits at least $k$ vertices, where $k \geq 2$ is an integer. A graph $G$ is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, $G$ admits a $P_{\geq k}$-factor excluding $e$. A graph $G$ is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $Q \subseteq V(G)$ with $|Q| = n$, $G - Q$ is a $P_{\geq k}$-factor avoidable graph.

Let $G$ be an $(n + 2)$-connected graph. In this paper, we demonstrate that (i) $G$ is a $(P_{\geq 2}, n)$-factor critical avoidable graph if $tough(G) > \frac{n+2}{2}$; (ii) $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph if $tough(G) > \frac{n+1}{2}$; (iii) $G$ is a $(P_{\geq 2}, n)$-factor critical avoidable graph if $I(G) > \frac{n+2}{3}$; (iv) $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph if $I(G) > \frac{n+3}{2}$. Furthermore, we claim that these conditions are sharp.

Keywords: graph, toughness, isolated toughness, $P_{\geq k}$-factor, $(P_{\geq k}, n)$-factor critical avoidable graph.

2010 Mathematics Subject Classification: 05C70, 05C38, 90B10.

1. Introduction

In this paper, we discuss only finite undirected simple graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the vertex set of $G$ and $E(G)$ denotes the edge set of $G$. For $x \in V(G)$, the degree of $x$ in $G$ is denoted by $d_G(x)$. For a set $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of $G$ induced by $X$ and write $G - X$ for $G[V(G) \setminus X]$. We let $i(G)$ and $\omega(G)$ denote the number of isolated vertices and the number of connected components of $G$, respectively. Let $P_n$ and
$K_n$ denote the path and the complete graph of order $n$, respectively. The join $G + H$ denotes the graph with vertex set $V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}.$$ 

Chvátal [2] first introduced the toughness of a graph $G$, denoted by $\text{tough}(G)$, namely,

$$\text{tough}(G) = \min \left\{ \frac{|X|}{\omega(G - X)} : X \subseteq V(G), \omega(G - X) \geq 2 \right\},$$

if $G$ is not complete; otherwise, $\text{tough}(G) = +\infty$.

Yang, Ma and Liu [13] first posed isolated toughness of a graph $G$, denoted by $I(G)$, namely,

$$I(G) = \min \left\{ \frac{|X|}{i(G - X)} : X \subseteq V(G), i(G - X) \geq 2 \right\},$$

if $G$ is not a complete graph; otherwise, $I(G) = +\infty$.

A path factor is a spanning subgraph $F$ of $G$ such that every component of $F$ is a path with at least two vertices. We write $P_{\geq k} = \{P_i : i \geq k\}$. Then a $P_{\geq k}$-factor of $G$ means a path factor in which every component admits at least $k$ vertices, where $k \geq 2$ is an integer. A $\{P_k\}$-factor $F$ of $G$ is simply called a $P_k$-factor if every component of $F$ is isomorphic to $P_k$.

A 1-factor of $G$ is a spanning subgraph $F$ of $G$ such that $d_F(x) = 1$ holds for any $x \in V(G)$. A graph $R$ is a factor-critical graph if for any $x \in V(R)$, $R - \{x\}$ admits a 1-factor. Let $R$ be a factor-critical graph with $V(R) = \{x_1, x_2, \ldots, x_n\}$, $n$ new vertices $y_1, y_2, \ldots, y_n$ together with new edges $x_1y_1, x_2y_2, \ldots, x_ny_n$ are added to $R$. Then the resulting graph is said to be a sun. By Kaneko [7], $K_1$ and $K_2$ are also suns. A big sun is a sun of order at least 6. We use $\text{sun}(G)$ to denote the number of sun components of $G$.


**Theorem 1** [11]. A graph $G$ admits a $P_{\geq 2}$-factor if and only if $i(G-X) \leq 2 |X|$ for every $X \subseteq V(G)$.


**Theorem 2** [7]. A graph $G$ admits a $P_{\geq 3}$-factor if and only if $\text{sun}(G-X) \leq 2 |X|$ for every $X \subseteq V(G)$.

A graph $G$ is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, $G$ admits a $P_{\geq k}$-factor excluding $e$. A graph $G$ is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $Q \subseteq V(G)$ with $|Q| = n$, $G - Q$ is a $P_{\geq k}$-factor
avoidable graph. Obviously, a \((P_{\geq k}, 0)\)-factor critical avoidable graph is simply called a \(P_{\geq k}\)-factor avoidable graph.

Kelmas [10] claimed a result on the existence of path factors in subgraphs.

**Theorem 3** [10]. Let \(G\) be a 3-connected claw-free graph and \(|V(G)| \equiv 1 \pmod{3}\). Then for any \(x \in V(G)\) and any \(e \in E(G)\), \(G - \{x, e\}\) has a \(\{P_3\}\)-factor, namely, \(G - \{x\}\) is a \(\{P_3\}\)-factor avoidable graph.

Motivated by Theorem 3, we consider a more general problem.

**Problem 1.** Find sufficient conditions for a graph to be a \((P_{\geq k}, n)\)-factor critical avoidable graph.

Kano, Lu and Yu [8] verified that a graph \(G\) has a \(\{P_3\}\)-factor if \(i(G - S) \leq \frac{2}{3}|S|\) for every \(S \subset V(G)\). Zhou, Yang and Xu [22] proved that an \(n\)-connected graph \(G\) is \((P_{\geq 3}, n)\)-factor critical if its toughness \(\text{tough}(G) \geq \frac{n + 1}{2}\). Some other results on path factors can be found in [3, 15, 17, 18]. Lots of authors derived some toughness conditions for the existence of graph factors [4, 5, 9, 20]. Some results on the relationships between isolated toughness and graph factors are obtained by Gao, Liang and Chen [6]. For many other results on graph factors, see [1, 12, 14, 16, 19, 21, 23]. In this paper, we study \((P_{\geq k}, n)\)-factor critical avoidable graphs and get some sufficient conditions for graphs to be \((P_{\geq k}, n)\)-factor critical avoidable graphs depending on toughness and isolated toughness, which are given in Sections 2 and 3.

## 2. Toughness and \((P_{\geq k}, n)\)-Factor Critical Avoidable Graphs

In this section, we explore the relationship between toughness and \((P_{\geq k}, n)\)-factor critical avoidable graphs, and derive two toughness conditions for the existence of \((P_{\geq k}, n)\)-factor critical avoidable graphs for \(k = 2, 3\).

**Theorem 4.** Let \(G\) be an \((n + 2)\)-connected graph, where \(n \geq 0\) is an integer. If its toughness \(\text{tough}(G) > \frac{n^2}{4}\), then \(G\) is a \((P_{\geq 2}, n)\)-factor critical avoidable graph.

**Proof.** Theorem 4 obviously holds for a complete graph. Next, we assume that \(G\) is not complete. Let \(Q \subset V(G)\) with \(|Q| = n\), and \(G' = G - Q\), and let \(e \in E(G')\) and \(H = G' - e\). Since \(G\) is \((n + 2)\)-connected, \(H\) is connected. To prove Theorem 4, it suffices to show that \(H\) admits a \(P_{\geq 2}\)-factor. On the contrary, suppose that \(H\) has no \(P_{\geq 2}\)-factor. Then by Theorem 1, there exists a set \(X \subset V(H)\) such that

\[(1) \quad i(H - X) \geq 2|X| + 1.\]
Since $H$ is connected, we have $X \neq \emptyset$. Thus,
\begin{equation}
(2) \quad i(H - X) \geq 2|X| + 1 \geq 3.
\end{equation}

Note that $\omega(G - (Q \cup X)) \geq \omega(G - (Q \cup X) - e) - 1$. Combining this with (2), we derive
\begin{equation}
(3) \quad \omega(G - (Q \cup X)) \geq \omega(G - (Q \cup X) - e) - 1 = \omega(H - X) - 1 \geq i(H - X) - 1 \geq 2.
\end{equation}

**Claim 1.** $|X| \geq 2$.

**Proof.** Assume $|X| = 1$. Since $H = G' - e$, we easily know that $i(H - X) = i(G' - e - X) \leq i(G' - X) + 2$. Then by (2), we derive $i(G' - X) \geq i(H - X) - 2 \geq 1$, which implies that there exists an isolated vertex $u$ in $G' - X$, i.e., $d_{G' - X}(u) = 0$. Thus, we have $d_G(u) \leq d_{G'}(u) + |Q| = d_{G'}(u) + n \leq d_{G' - X}(u) + |X| + n = 0 + 1 + n = n + 1$, contradicting that $G$ is $(n + 2)$-connected. Therefore, $|X| \geq 2$. \qed

According to (1), (2), (3), Claim 1 and the definition of tough$(G)$, we have
\begin{align*}
tough(G) & \leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{|Q| + |X|}{i(H - X) - 1} \\
& = \frac{n + |X|}{i(H - X) - 1} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{4} = \frac{n + 2}{4},
\end{align*}
which contradicts tough$(G) > \frac{n + 2}{4}$. Theorem 4 is verified. \qed

**Remark 5.** Now, we claim that the result in Theorem 4 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Clearly, $G$ is $(n + 2)$-connected and tough$(G) = \frac{n + 2}{4}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with $|Q| = n$ and $e$ be the edge of $K_2$. Then $G - Q - e$ is a graph isomorphic to $K_2 + (5K_1)$, and it obviously has no $P_{2}$-factor. Thus, $G$ is not a $(P_{2}, n)$-factor critical avoidable graph.

**Theorem 6.** Let $G$ be an $(n + 2)$-connected graph, where $n \geq 0$ is an integer. If its toughness tough$(G) > \frac{n + 1}{2}$, then $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph.

**Proof.** Theorem 6 obviously holds for a complete graph. In the following, we assume that $G$ is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e \in E(G')$ and $H = G' - e$. Since $G$ is $(n + 2)$-connected, $H$ is connected.

To prove Theorem 6, it suffices to show that $H$ admits a $P_{3}$-factor. On the contrary, suppose that $H$ has no $P_{3}$-factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that
\begin{equation}
(4) \quad \text{sun}(H - X) \geq 2|X| + 1.
\end{equation}
Some Results on Path-Factor Critical Avoidable Graphs

Claim 1. $X \neq \emptyset$.

Proof. Assume that $X = \emptyset$. Then it follows from (4) that

$\text{sun}(H) \geq 1.$

Since $H$ is connected, we have $\text{sun}(H) = 1$ and $H$ itself is a sun.

Since $G$ is $(n+2)$-connected, $|V(G)| \geq n+3$. Thus, $|V(H)| = |V(G)| - n \geq 3$, which implies that $H$ is a big sun. Hence, $|V(H)| \geq 6$. Let $R$ be the factor-critical graph of $H$. Then $|V(R)| \geq 3$ and there exists $w \in V(R)$ such that $\omega(G' - \{w\}) = \omega(H - \{w\}) = 2$. Thus, we have

$\omega(G - Q - \{w\}) = \omega(G' - \{w\}) = 2.$

In terms of (6) and the definition of $\text{tough}(G)$, we get

$\text{tough}(G) \leq \frac{|Q \cup \{w\}|}{\omega(G - (Q \cup \{w\}))} = \frac{n + 1}{2},$

contradicting to $\text{tough}(G) > \frac{n+1}{2}$. Hence, $X \neq \emptyset$. 

By (4) and Claim 1, we gain $\omega(G - (Q \cup X)) = \omega(G' - X) \geq \omega(G' - X - e) - 1 = \omega(H - X) - 1 \geq \text{sun}(H - X) - 1 \geq 2|X| \geq 2$. Combining this with Claim 1 and the definition of $\text{tough}(G)$, we have

$\text{tough}(G) \leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{2} = \frac{n + 1}{2},$

this contradicts $\text{tough}(G) > \frac{n+1}{2}$. This finishes the proof of Theorem 6.

Remark 7. Now, we show that the conditions in Theorem 6 are best possible, which cannot be replaced by $G$ being $(n + 1)$-connected and $\text{tough}(G) \geq \frac{n+1}{2}$.

Let $G = K_{n+1} + (2K_2)$. We easily see that $G$ is $(n + 1)$-connected and $\text{tough}(G) = \frac{n+1}{2}$. Let $Q \subseteq V(K_{n+1}) \subseteq V(G)$ with $|Q| = n$, and $e$ be an edge of $2K_2$. Then $G - Q - e$ is a graph isomorphic to $K_1 + (2K_1 \cup K_2)$, and it obviously has no $P_{2k}$-factor, and so $G$ is not a $(P_{2k}, n)$-factor critical avoidable graph.

3. Isolated Toughness and $(P_{2k}, n)$-Factor Critical Avoidable Graphs

In this section we give two sufficient conditions using isolated toughness for a graph to be a $(P_{2k}, n)$-factor critical avoidable graph for $k = 2, 3$.

Theorem 8. Let $G$ be an $(n + 2)$-connected graph, where $n \geq 0$ is an integer. If its isolated toughness $I(G) > \frac{n+2}{3}$, then $G$ is a $(P_{2k}, n)$-factor critical avoidable graph.
Proof. Theorem 8 obviously holds for a complete graph. In what follows, we assume that $G$ is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e \in E(G')$ and $H = G' - e$. Since $G$ is $(n + 2)$-connected, $H$ is connected. To prove Theorem 8, it suffices to show that $H$ admits a $P_{\geq 2}$-factor. On the contrary, suppose that $H$ has no $P_{\geq 2}$-factor. Then by Theorem 1, there exists a set $X \subset V(H)$ such that

$$(7) \quad i(H - X) \geq 2|X| + 1.$$ 

Claim 1. $|X| \geq 2$.

Proof. If $X = \emptyset$, then by (7) and $H$ being connected, we obtain

$$1 \leq i(H) = 0,$$

which is a contradiction.

Next, we consider $|X| = 1$. Note that $i(H - X) = i(G' - e - X) \leq i(G' - X) + 2$. Combining this with (7), we derive $i(G' - X) \geq i(H - X) - 2 \geq 2|X| + 1 - 2 = 2|X| - 1 = 1$, which hints that there exists $w \in V(G') \setminus X$ with $d_{G' - X}(w) = 0$. Therefore, we admit $d_G(w) = d_{G' + Q}(w) \leq d_{G'}(w) + |Q| = d_{G'}(w) + n \leq d_{G' - X}(w) + |X| + n = 0 + 1 + n = n + 1$, which contradicts that $G$ is $(n + 2)$-connected. Thus, we derive $|X| \geq 2$. □

According to (7) and Claim 1, we get

$$(8) \quad i(G - (Q \cup X)) \geq i(G - (Q \cup X) - e) - 2 = i(H - X) - 2 \geq 2|X| - 1 \geq 3.$$ 

It follows from (8), Claim 1 and the definition of $I(G)$ that

$$I(G) \leq \frac{|Q \cup X|}{i(G - (Q \cup X))} \leq \frac{|Q| + |X|}{2|X| - 1} \leq \frac{n + \frac{1}{2}}{2|X| - 1} = \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{|X| - \frac{1}{2}}{2(2|X| - 1)}$$

$$= \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{1}{2} \leq \frac{n + \frac{1}{2}}{3} + \frac{1}{2} = \frac{n + 2}{3},$$

which contradicts $I(G) > \frac{n + 2}{3}$. Theorem 8 is proved. □

Remark 9. Now, we explain that the result in Theorem 8 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Obviously, $G$ is $(n + 2)$-connected and $I(G) = \frac{n + 2}{3}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with $|Q| = n$, and $e$ be the edge of $K_2$. Then $G - Q - e$ is a graph isomorphic to $K_2 + (5K_1)$, and it obviously has no $P_{\geq 2}$-factor. Thus, $G$ is not a $(P_{\geq 2}, n)$-factor critical avoidable graph.
Theorem 10. Let $G$ be an $(n+2)$-connected graph, where $n$ is a positive integer. If its isolated toughness $I(G) > \frac{n+3}{2}$, then $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph.

Proof. Theorem 10 obviously holds for a complete graph. Next, we assume that $G$ is not complete. Let $Q \subseteq V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e = xy \in E(G')$ and $H = G' - e$. Since $G$ is $(n+2)$-connected, $H$ is connected. To prove Theorem 10, it suffices to show that $H$ admits a $P_{\geq 3}$-factor. On the contrary, suppose that $H$ has no $P_{\geq 3}$-factor. Then by Theorem 2, there exists a set $X \subseteq V(H)$ such that

$$\text{sun}(H - X) \geq 2|X| + 1.$$ 

Claim 1. $X \neq \emptyset$.

Proof. Assume $X = \emptyset$. Then $\text{sun}(H) \geq 1$. This implies $\text{sun}(H) = 1$ since $H$ is connected.

Note that $G$ is $(n+2)$-connected. Hence, $|V(G)| \geq n + 3$. Thus, $|V(H)| = |V(G)| - n \geq (n+3) - n = 3$, which implies that $H$ is a big sun. Therefore, $|V(H)| \geq 6$. Let $R$ be the factor-critical subgraph of $H$. Then $i(H - V(R)) = |V(R)| \geq 3$. Next, we consider two cases.

Case 1. $x, y \in V(H) \setminus V(R)$. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$.

Thus, we easily see

$$i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\})) = i(G' - ((V(R) \setminus \{z\}) \cup \{y\}))$$

$$= i(G' - ((V(R) \setminus \{z\}) \cup \{y\}) - e)$$

$$= i(H - ((V(R) \setminus \{z\}) \cup \{y\}))$$

$$= |V(R)| \geq 3.$$ 

Combining this with the definition of $I(G)$ and $I(G) > \frac{n+3}{2}$, we admit. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily get

$$\frac{n + 3}{2} < I(G) \leq \frac{|Q \cup (V(R) \setminus \{z\}) \cup \{y\}|}{i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\}))}$$

$$= \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{V(R)} + 1 \leq \frac{n}{3} + 1 = \frac{n + 3}{3},$$ 

which is a contradiction.

Case 2. $x \in V(R)$ or $y \in V(R)$. In this case, $i(G - (Q \cup (V(R)))) = i(G' - V(R)) = i(G' - V(R) - e) = i(H - V(R)) = |V(R)| \geq 3$. Thus, we get

$$I(G) \leq \frac{|Q \cup V(R)|}{i(G - (Q \cup V(R)))} = \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \leq \frac{n}{3} + 1 = \frac{n + 3}{3},$$ 

which contradicts $I(G) > \frac{n+3}{2}$. Hence, $X \neq \emptyset$. 

□
Let \( \text{Sun}(H - X) \) denote the union of sun components of \( H - X \), which consists of \( a \) isolated vertices, \( b \) \( K_2 \)-components and \( c \) big sun components \( S_1, S_2, \ldots, S_c \). Let \( R_i \) be the factor-critical subgraph of \( S_i \) for \( 1 \leq i \leq c \), and write \( Z = \bigcup_{1 \leq i \leq c} V(R_i) \). We select one vertex from every \( K_2 \) component of \( H - X \), and the set of such vertices is denoted by \( Y \). Clearly, \( |Y| = b \). Then \( i(H - (X \cup Y \cup Z)) = a + b + |Z| \) and it follows from (9) and Claim 1 that
\[
(10) \quad \text{sun}(H - X) = a + b + c \geq 2|X| + 1 \geq 3.
\]

**Claim 2.** \( 0 \leq a \leq 1 \).

**Proof.** Assume that \( a \geq 2 \). By (10), \( c \geq 0 \) and \( |V(R_i)| \geq 3 \), we derive
\[
i(G - (Q \cup X \cup Y \cup Z \cup \{x}\))) = i(G' - (X \cup Y \cup Z \cup \{x}\))) = i(H - (X \cup Y \cup Z \cup \{x}\))) \geq i(H - (X \cup Y \cup Z)) - 1 = a + b + |Z| - 1 \geq a + b + 3c - 1 \geq a + b + c - 1 \geq 2.
\]

Combining this with the definition of \( I(G) \) and \( I(G) > \frac{n+3}{2} \), we derive
\[
\frac{n + 3}{2} < I(G) \leq \frac{|Q \cup X \cup Y \cup Z \cup \{x}\)|}{i(G - (Q \cup X \cup Y \cup Z \cup \{x}\)))} \leq \frac{n + |X| + b + |Z| + 1}{a + b + |Z| - 1},
\]

namely,
\[
(11) \quad 0 > \frac{n + 1}{2} (a + b + |Z|) + a - |X| - \frac{3n + 5}{2}.
\]

It follows from (10), (11), \( a \geq 2, c \geq 0, |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \) and Claim 1 that
\[
0 > \frac{n + 1}{2} (a + b + |Z|) + a - |X| - \frac{3n + 5}{2}
\geq \frac{n + 1}{2} (a + b + 3c) + 2 - |X| - \frac{3n + 5}{2}
\geq \frac{n + 1}{2} (a + b + c) - |X| - \frac{3n + 1}{2}
\geq \frac{n + 1}{2} (2|X| + 1) - |X| - \frac{3n + 1}{2}
= n(|X| - 1) \geq 0,
\]

which is a contradiction. Therefore, \( 0 \leq a \leq 1 \). □

We easily see that \( x \notin V(aK_1) \) or \( y \notin V(aK_1) \) since \( 0 \leq a \leq 1 \) (by Claim 2).
Some Results on Path-Factor Critical Avoidable Graphs

Claim 3. \( x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \) or \( y \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \).

**Proof.** Assume that \( x, y \notin V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \). Note that \( x \notin V(aK_1) \) or \( y \notin V(aK_1) \). Hence, there is at least one vertex in \( \{x, y\} \) such that the vertex does not belong \( V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \). Without loss of generality, we let \( x \notin V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \). Then \( x \in V(G) \setminus (Q \cup V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)) \). Thus, we obtain

\[
i(G - (Q \cup X \cup (Y \cup Z) \{x\} \cup \{x\})) = a + b + |Z| \geq a + b + 3c \geq 3
\]

by (10), \( c \geq 0 \) and \( |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \). In terms of the definition of \( I(G) \), we derive

\[
I(G) \leq \frac{|Q \cup X \cup Y \cup Z \{x\}|}{i(G - (Q \cup X \cup Y \cup Z \{x\}))} \leq \frac{n + |X| + b + |Z| + 1}{a + b + |Z|}.
\]

It follows from (10), (12), \( a \geq 0, \ c \geq 0, \ |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \) and \( I(G) > \frac{n+3}{2} \) that

\[
0 \geq (I(G) - 1)(a + b + |Z|) + a - n - |X| - 1
\]
\[
\geq (I(G) - 1)(a + b + 3c) - n - |X| - 1
\]
\[
\geq (I(G) - 1)(a + b + c) - n - |X| - 1
\]
\[
\geq (I(G) - 1)(2|X| + 1) - n - |X| - 1
\]
\[
= I(G)(2|X| + 1) - n - 3|X| - 2,
\]

which implies

\[
I(G) \leq \frac{3|X| + n + \frac{2}{2|X| + 1}}{2|X| + 1}.
\]

From (13), Claim 1 and \( n \geq 1 \), we have

\[
I(G) \leq \frac{3|X| + n + \frac{2}{2|X| + 1}}{2|X| + 1} = \frac{3}{2} + \frac{n + \frac{1}{2}}{2|X| + 1} \leq \frac{3}{2} + \frac{n + \frac{1}{2}}{3} = \frac{n + 3}{2} + \frac{1 - n}{6} \leq \frac{n + 3}{2},
\]

which contradicts \( I(G) > \frac{n+3}{2} \). Claim 3 is verified. \( \square \)

Without loss of generality, we let \( x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \) by Claim 3. Then there exists \( z \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c) \) such that \( xz \in E(G) \) and there is at least one vertex of \( \{x, z\} \) with degree 1 in the subgraph \( (bK_2) \cup S_1 \cup \cdots \cup S_c \). Thus, we obtain

\[
i(G - (Q \cup X \cup ((Y \cup Z) \{z\} \cup \{x\})) = a + b + |Z| \geq a + b + 3c \geq 3
\]
by (10), \( c \geq 0 \) and \( |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \). Combining this with the definition of \( I(G) \) and \( I(G) > \frac{n+3}{2} \), we obtain

\[
n + \frac{3}{2} > I(G) \geq \frac{|Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}|}{i(G - (Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}))} = \frac{n + |X| + b + |Z|}{a + b + |Z|},
\]

that is,

\[
0 > \frac{n+1}{2} (a + b + |Z|) - n - |X| + a.
\]

Combining this with (10), \( a \geq 0 \), \( c \geq 0 \), \( n \geq 1 \), \( |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \) and Claim 1, we derive

\[
0 > \frac{n+1}{2} (a + b + |Z|) - n - |X| + a \geq \frac{n+1}{2} (a + b + c) - n - |X| \geq \frac{n+1}{2} (2|X| + 1) - n - |X| = n|X| + \frac{1}{2} - \frac{n}{2} \geq \frac{n+1}{2} - \frac{n}{2} = \frac{n+1}{2} \geq 1,
\]

which is a contradiction. This finishes the proof of Theorem 10.

**Remark 11.** Next, we elaborate that the conditions in Theorem 10 are best possible, which cannot be replaced by \( G \) being \((n+1)\)-connected and \( I(G) \geq \frac{n+3}{2} \).

Let \( G = K_{n+1} + (2K_2) \). It is clear that \( G \) is \((n+1)\)-connected and \( I(G) = \frac{n+3}{2} \).

Let \( Q \subset V(K_{n+1}) \subseteq V(G) \) with \( |Q| = n \), and \( e \) be an edge of \( 2K_2 \). Then \( G - Q - e \) is a graph isomorphic to \( K_1 + (2K_1 \cup K_2) \), and it obviously has no \( P_{\geq 3} \)-factor. Therefore, \( G \) is not a \((P_{\geq 3}, n)\)-factor critical avoidable graph.

**Acknowledgements**

The authors would like to thank the anonymous referees for their comments on this paper. This work is supported by Six Big Talent Peak of Jiangsu Province (Grant No. JY–022).

**References**


Some Results on Path-Factor Critical Avoidable Graphs 11


Received 9 June 2020
Revised 3 September 2020
Accepted 3 September 2020