EXTENDING POTOČNIK AND ŠAJNA’S CONDITIONS ON THE EXISTENCE OF VERTEX-TRANSITIVE SELF-COMPLEMENTARY $k$-HYPERGRAPHS

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Abstract

Let $\ell$ be a positive integer, $k = 2^\ell$ or $k = 2^\ell + 1$, and let $n$ be a positive integer with $n \equiv 1 \pmod{2^\ell+1}$. For a prime $p$, $n_{(p)}$ denotes the largest integer $i$ such that $p^i$ divides $n$. Potočnik and Šajna showed that if there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$, then for every prime $p$ we have $p^{n_{(p)}} \equiv 1 \pmod{2^\ell+1}$. Here we extend their result to a larger class of integers $k$.

Keywords: vertex-transitive hypergraphs, self-complementary hypergraphs.

2010 Mathematics Subject Classification: 05C65.

1. Introduction

For a prime $p$ and a positive integer $n$, let $n_{(p)}$ denote the largest integer $i$ for which $p^i$ divides $n$. Using this notation, we combine the theorems of Rao and Muzychuk as follows.

**Theorem 1.1** (Rao/Muzychuk). For a positive integer $n$, there exists a vertex-transitive self-complementary graph of order $n$ if and only if $p^{n_{(p)}} \equiv 1 \pmod{4}$ for every prime $p$.

For an interesting discussion of the history of the vertex-transitive self-complementary graph problem, see [1].

For every integer $k \geq 2$, a $k$-uniform hypergraph, or $k$-hypergraph, for short, is a pair $(V; E)$ consisting of a vertex set $V$ and edge set $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all $k$-subsets of $V$. Clearly a 2-hypergraph is just a simple graph. A hypergraph $H$ is called vertex-transitive if for every two vertices $u, v$ of $H$ there is an automorphism $\phi$ of $H$ such that $u = \phi(v)$. A $k$-hypergraph $H = (V; E)$ is called self-complementary if there is a permutation $\sigma$ of the set $V$, called a self-complementing permutation, such that for every $k$-subset $e$ of $V$, $e \subseteq E$ if and only if $\sigma(e) \notin E$. In other words, $H$ is isomorphic to $\overline{H} = (V; \binom{V}{k} \setminus E)$. In 2009, Potočnik and Šajna [5] proposed studying the problem analogous to the previous theorem for $k$-hypergraphs. In particular, they extended Muzychuk’s necessary condition to $k$-hypergraphs when $k = 2^\ell$ or $k = 2^{\ell+1}$ for some positive integer $\ell$. Shortly after, Gosselin [3] established the sufficiency of the Potočnik and Šajna result.

**Theorem 1.2** (Potočnik-Šajna/Gosselin). Let $m$ be a positive integer, $k = 2^m$ or $k = 2^m + 1$, and let $n$ be a positive integer with $n \equiv 1 \pmod{2^{m+1}}$. Then there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$ if and only if for every prime $p$ we have $p^{n_{(p)}} \equiv 1 \pmod{2^{m+1}}$.

In Theorem 1.2, the only considered values of $k$ are of the form $k = 2^m$ or $k = 2^m + 1$, for some positive integer $m$. We now consider any integer $k \geq 2$ and look at the binary expansion of $k$. Then there are positive integers $\ell$ and $m$ such that $k = \sum_{i \leq \ell} k_i 2^i + 2^m$ or $k = 1 + \sum_{i \leq \ell} k_i 2^i + 2^m$, where $k_i \in \{0, 1\}$, for every $i$. In Theorem 1.2, each such $k_i = 0$. Furthermore, in Theorem 1.2, $n \equiv 1 \pmod{2^{m+1}}$. This suggests our next theorem which extends the necessary condition of Potočnik and Šajna for more values of $k$.

**Theorem 1.3.** Let $\ell$, $k$, $n$, and $m$ be positive integers such that $1 < k < n$, $1 \leq \ell \leq m$ and $n \equiv 1 \pmod{2^{\ell+1}}$, $k = \sum_{i \leq \ell} k_j 2^i$ or $k = \sum_{i \leq j \leq m} k_j 2^j + 1$, where $k_j \in \{0, 1\}$ for every $j$, $\ell \leq j \leq m$. If there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$, then for every prime $p$ we have $p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}}$. 
2. Proof of Theorem 1.3

If \( H \) is a self-complementary \( k \)-hypergraph, then the set of all self-complementing permutations of \( H \) will be denoted by \( C(H) \). In [7] the following characterization of self-complementing permutations for \( k \)-hypergraphs was given. Here \( |c| \) denotes the order of a cycle \( c \).

**Theorem 2.1.** Let \( n \) and \( k \) be positive integers, \( 2 \leq k \leq n \). A permutation \( \sigma \) of \([1, n] \) with cycles \( c_1, \ldots, c_\lambda \) is a self-complementing permutation of a \( k \)-hypergraph of order \( n \) if and only if \( \sigma \) satisfies (3). In other words, with exception of a single fixed point, every cycle of \( \sigma \) has order divisible by \( 2^{t+1} \).

**Proof.** Let \( \sigma \) be a self-complementing permutation of a self-complementary \( k \)-hypergraph of order \( n \) with cycles \( c_1, \ldots, c_\lambda \). By Theorem 2.1 there exists a nonnegative integer \( t \) such that

1. \( k = a_t 2^t + s_t \), where \( a_t \) is odd and \( 0 \leq s_t < 2^t \),
2. \( n = b_t 2^{t+1} + r_t \), where \( 0 \leq r_t < 2^t + s_t \), and
3. \( \sum_{|c| \leq t} |c| = r_t \).

First observe that \( t = 0 \) implies \( s_t = 0 \) and \( r_t = 0 \) and \( n \) is even, a contradiction. Thus, \( t \geq 1 \). Since \( a_t \) is odd, it follows that \( t \geq \ell \), and since \( k < 2^{n+1} \), we have \( t \leq m \). Consequently, as \( n \equiv 1 \pmod{2^{m+1}} \), we have that \( n \equiv 1 \pmod{2^{\ell+1}} \) and \( r_t = 1 \). Thus, exactly one cycle \( c_1 \), necessarily of length 1, satisfies (3). In other words, with exception of a single fixed point, every cycle of \( \sigma \) has order divisible by \( 2^{\ell+1} \), and hence by \( 2^{t+1} \).

The proof of Theorem 1.3 uses the technique of Muzychuk [2]. The proof also depends on the first two Sylow theorems (see [4], for example). The following theorem is well-known. We give it however with proof, for completeness.
**Theorem 2.3.** Let \( p \) be a prime and \( G \) a finite group. If \( P \) is a Sylow \( p \)-subgroup of its normalizer in \( G \), then \( P \) is a Sylow \( p \)-subgroup of the group \( G \).

**Proof.** To prove this theorem, we shall use the notion of group action. If we have a group \( G \) acting on a set \( X \), we use symbols \( X_{fix} \), \( G_x \), and \( O_x \) to denote the set of all fixed points of \( X \), the stabilizer of a point \( x \) in \( G \), and the orbit of \( x \), respectively. Recall that for any point \( x \), the Orbit-Stabilizer Theorem (see, for instance, [4] Section 8.3 Lemma 3) asserts that \( |O_x| = |G/G_x| \), and clearly \( O_x = \{x\} \) if and only if \( G_x = G \).

The well-known Orbit Decomposition Theorem (see [4]) states that if a group \( G \) acts on a finite set \( X \neq \emptyset \), and \( x_1, \ldots, x_n \in X \) are representatives of mutually disjoint orbits with at least two elements, then

\[
|X| = |X_{fix}| + \sum_{i=1}^{n} |G/G_{x_i}|.
\]

Thus, the Orbit Decomposition Theorem implies that if \( G \) is a \( p \)-group, then

\[
|X| \equiv |X_{fix}| \pmod{p}.
\]

By \( N_G(H) \) we denote the normalizer of a subgroup \( H \) in \( G \); that is the largest subgroup of \( G \) in which \( H \) is normal, namely \( N_G(H) = \{g \in G : gHg^{-1} = H\} \).

Now we have the following fact.

**Fact.** If \( H \) is a \( p \)-subgroup of \( G \), then \( |N_G(H)/H| \equiv |G/H| \pmod{p} \).

To prove it, we consider the following action of \( H \) on the set \( G/H \) of right cosets: for every \( a \in H \) and every coset \( Hb \), we define \( a(Hb) = Hba^{-1} \). It is straightforward to verify that we are indeed defining a group action. Clearly, for every \( a \in H \), and for every \( b \in G \), \( Hba^{-1} = Hb \) if and only if \( bab^{-1} \in H \), and hence, \( (G/H)_{fix} = N_G(H)/H \). Since \( H \) is a \( p \)-group, \( |G/H| - |N_G(H)/H| = |G/H| - |(G/H)_{fix}| \) is divisible by \( p \).

If \( P \) is a Sylow \( p \)-subgroup of \( N_G(P) \), then \( |N_G(P)/P| \not\equiv 0 \pmod{p} \), and by our Fact, it follows that \( P \) is a Sylow \( p \)-subgroup of \( G \).

**Proof of Theorem 1.3.**

Suppose that \( H = (V; E) \) is a self-complementary vertex-transitive \( k \)-hypergraph of order \( n \), where \( k \) and \( n \) satisfy the conditions of our theorem. Let \( p \) be a prime; if \( n_{(p)} = 0 \), then the result is clear. Thus assume that \( n_{(p)} > 0 \). We shall find a self-complementary vertex-transitive \( k \)-subhypergraph \( H' \) of \( H \) of order \( p^{n_{(p)}} \) such that the cycles of a self-complementing permutation of \( H' \) are cycles of a self-complementing permutation \( \sigma \) of \( H \) and the fixed point of \( \sigma \) is one of the vertices of \( H' \). By Corollary 2.2, all cycles of \( \sigma \) have order divisible by \( 2^{\ell+1} \),
with the exception of a single fixed point. Hence the order of $H'$, that is $p^{r(n)}$, is congruent to 1 modulo $2^{k+1}$, and the statement of Theorem 1.3 follows.

Let $M = \text{Aut}(H)$ be the automorphism group of $H$. For any group $K$, denote the set of the Sylow $p$-subgroups of $K$ by $\text{Syl}_p(K)$.

Note that for every $\sigma \in C(H)$ we have $\sigma^2 \in \text{Aut}(H)$. Moreover a product of a number of automorphisms and self-complementing permutations is an automorphism of $H$ if the number of self-complementing permutations is even; otherwise, the product is a self-complementing permutation of $H$. The set $G = \text{Aut}(H) \cup C(H)$ is a group which is generated by $\text{Aut}(H) \cup \{\sigma\}$, where $\sigma$ is an arbitrary element of $C(H)$.

Define $\mathcal{P}$ to be the set of $p$-subgroups $P$ of $M$ with the property that there exists a vertex $v$ of $H$ and $\tau \in C(H)$ such that

(1) $\tau(v) = v$;
(2) $\tau P \tau^{-1} = P$ ($\tau$ normalizes $P$);
(3) $P_v \in \text{Syl}_p(M_v)$.

We will show that $\mathcal{P}$ is not empty and any maximal element of $\mathcal{P}$ is, in fact, a Sylow $p$-subgroup of $M$.

Since $H$ is self-complementary, $C(H)$ is not empty. Choose any $\sigma \in C(H)$. By Corollary 2.2 there is a fixed point $v$ of $\sigma$. Let $P \in \text{Syl}_p(M_v)$.

Note that if $p$ does not divide $|M_v|$, then $P$ is trivial. Since $P$ is a subgroup of $M_v$, then $P = P_v$, and clearly $\sigma P \sigma^{-1}$ is a subgroup of $M_v$ isomorphic to $P$. By the second Sylow Theorem, there exists $g \in M_v$ such that $\sigma P \sigma^{-1} = g P g^{-1}$. Set $\tau = g^{-1} \sigma$. Then $\tau \in C(H)$, $\tau(v) = v$, $\tau P \tau^{-1} = P$, and $P_v \in \text{Syl}_p(M_v)$. Hence $P \in \mathcal{P}$ and $\mathcal{P} \neq \emptyset$.

From now on we shall assume that

- $P \in \mathcal{P}$ is a maximal element of $\mathcal{P}$,
- $N$ is the normalizer of $P$ in $M$,
- $Q$ is a Sylow $p$-subgroup of $N$ containing $P$ ($Q$ exists by the second Sylow Theorem).

Claim. $P$ is a Sylow $p$-subgroup of $M$.

Proof. To prove this claim, it suffices to show that $Q \in \mathcal{P}$, and hence $Q = P$ by the maximality of $P$. It will then follow that $P$ is a Sylow $p$-subgroup of its own normalizer in $M$, and hence by Theorem 2.3, it is a Sylow $p$-subgroup of $M$.

Since $P \in \mathcal{P}$, there are $\tau \in C(H)$ and a vertex $v$ such that $\tau(v) = v$, $\tau P \tau^{-1} = P$ and $P_v \in \text{Syl}_p(M_v)$. It is straightforward to show that $\tau$ normalizes $N$, that is, $\tau N \tau^{-1} = N$. Thus, $\tau N = N \tau$.

Since $Q$ is a subgroup of $N$ and $\tau N \tau^{-1} = N$, we have that $\tau Q \tau^{-1}$ is a subgroup of $N$ and since $|\tau Q \tau^{-1}| = |Q|$, we conclude that $\tau Q \tau^{-1}$ is a Sylow $p$-subgroup of $N$. 
Recall that \( v \) is a fixed point of \( \tau \), and let \( U = N(v) \), where \( N(v) = \{ h(v) : h \in N \} \). Then we have \( \tau(U) = \tau(N(v)) = (\tau N)(v) = (N \tau)(v) \), since \( \tau N = N \tau \) by our previous argument. This implies that \( \tau(U) = N(\tau(v)) = N(v) = U \).

By Corollary 2.2, every cycle \( c \) of the self-complementing permutation \( \tau \) has length divisible by \( 2^{\ell+1} \), with the exception of one fixed point. Since \( \tau(U) = U \), for every cycle \( c \) of the permutation \( \tau \) we know that either all the vertices of \( c \) are in \( U \) or else, the set of vertices of \( U \) is disjoint with \( U \). Therefore, \( U \) is a set of vertices of a self-complementary vertex-transitive \( k \)-hypergraph \( H' = (U; E \cap \binom{U}{k}) \) with self-complementing permutation \( \tau \) (restricted to \( U \)) and vertex-transitive group of automorphisms containing \( N \). Moreover, vertex \( v \), the fixed point of \( \tau \), is in \( U \). Hence we have

\[
|U| \equiv 1 \left( \mod 2^{\ell+1} \right).
\]

Since \( \tau Q \tau^{-1} \) and \( Q \) are two Sylow \( p \)-subgroups of the group \( N \), by the second Sylow Theorem, there is \( g \in N \) such that \( \tau Q \tau^{-1} = g Q g^{-1} \).

Hence \( (g^{-1} \tau) Q (g^{-1} \tau)^{-1} = Q \).

Write \( \sigma = \tau^{-1} g \). By the definition of \( U \) and since \( g \in N \), we have \( g(U) = U \), and hence, \( \sigma(U) = U \). We have \( \sigma Q \sigma^{-1} = Q \), and the restriction of \( \sigma \in C(H) \) to the set \( U \) is also a self-complementing permutation of \( H' \).

By Corollary 2.2, the permutation \( \sigma \) has a fixed point \( u \), and all remaining cycles are of lengths congruent to \( 1 \) \( \mod 2^{\ell+1} \). Since \( |U| \equiv 1 \left( \mod 2^{\ell+1} \right) \) and the cycles of the restriction of \( \sigma \) to \( U \) are the cycles of \( \sigma \), we have \( u \in U \).

Since the group \( N \) is transitive on the set \( U \), there is \( h \in N \) such that \( h(v) = u \). Thus the subgroups \( M_u \) and \( M_u \) are conjugate, that is,

\[
M_u = h M_u h^{-1}.
\]

Moreover, we also have

\[
P_u = h P_u h^{-1}.
\]

Hence \( |M_u| = |M_u| \) and \( |P_u| = |P_u| \), and therefore \( P_u \) is a Sylow \( p \)-subgroup of \( M_u \). Since \( P_u \leq Q_u \leq M_u \) and \( Q_u \) is a \( p \)-subgroup of \( M_u \), it follows that \( Q_u = P_u \) and \( Q_u \) is a Sylow \( p \)-subgroup of \( M_u \). Finally, we have \( Q \in \mathcal{P} \). This completes the proof of the claim.

Now we shall show that the orbit \( P(v) \) induces a self-complementary vertex-transitive \( k \)-hypergraph of order \( p^\ell \), where \( r = n(p) \). Note first that since \( \tau P = P \tau \) and \( \tau(v) = v \), we have

\[
\tau(P(v)) = P(\tau(v)) = P(v)
\]

and therefore the \( k \)-subhypergraph of \( H \) induced by \( P(v) \) is self-complementary and vertex-transitive.
Write $|M| = p^d q$, where $q$ and $p$ are relatively prime. Then $|P| = p^d$ by the Claim. Since $M$ acts transitively on $V$ we have

$$|M_v| = \frac{|M|}{|M(v)|} = \frac{p^d q}{p^r m} = p^{d-r} s,$$

for some positive integers $m$ and $s$ both relatively prime with $p$.

Since $P_v \in \text{Syl}_p(M_v)$, it follows that $|P_v| = p^{d-r}$. On the other hand, since $P \in \text{Syl}_p(M)$ and $P_v \in \text{Syl}_p(M_v)$ we have

$$p^{d-r} = |P_v| = \frac{|P|}{|P(v)|} = \frac{p^d}{|P(v)|}.$$ 

This implies $|P(v)| = p^r$. Since $\tau$ is a self-complementing permutation of $H$, by Corollary 2.2, the length of every cycle of $\tau$, with exception of a single fixed point, is divisible by $2^{\ell+1}$. Since $\tau(P(v)) = P(v)$, we know that $P(v)$ is the union of orbits of $\tau$, including the fixed point $v$. Hence $p^r \equiv 1 \mod 2^{\ell+1}$ as claimed.

**Acknowledgement**

The authors express their thanks to the anonymous referees for reading carefully the manuscript and for their important comments.

**References**


