EXTENDING POTOČNIK AND ŠAJNA’S CONDITIONS ON THE EXISTENCE OF VERTEX-TRANSITIVE SELF-COMPLEMENTARY $k$-HYPERGRAPHS

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Abstract

Let $\ell$ be a positive integer, $k = 2^\ell$ or $k = 2^\ell + 1$, and let $n$ be a positive integer with $n \equiv 1 \pmod{2^{\ell+1}}$. For a prime $p$, $n_{(p)}$ denotes the largest integer $i$ such that $p^i$ divides $n$. Potočnik and Šajna showed that if there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$, then for every prime $p$ we have $p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}}$. Here we extend their result to a larger class of integers $k$.

Keywords: vertex-transitive hypergraphs, self-complementary hypergraphs.

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1. Introduction

For a prime $p$ and a positive integer $n$, let $n(p)$ denote the largest integer $i$ for which $p^i$ divides $n$. Using this notation, we combine the theorems of Rao and Muzychuk as follows.

**Theorem 1.1 (Rao/Muzychuk).** For a positive integer $n$, there exists a vertex-transitive self-complementary graph of order $n$ if and only if $p^{n(p)} \equiv 1 \pmod{4}$ for every prime $p$.

For an interesting discussion of the history of the vertex-transitive self-complementary graph problem, see [1].

For every integer $k \geq 2$, a $k$-uniform hypergraph, or $k$-hypergraph, for short, is a pair $(V; E)$ consisting of a vertex set $V$ and edge set $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all $k$-subsets of $V$. Clearly a 2-hypergraph is just a simple graph. A hypergraph $H$ is called vertex-transitive if for every two vertices $u, v$ of $H$ there is an automorphism $\phi$ of $H$ for which $u = \phi(v)$. A $k$-hypergraph $H = (V; E)$ is called self-complementary if there is a permutation $\sigma$ of the set $V$, called a self-complementing permutation, such that for every $k$-subset $e$ of $V$, $e \in E$ if and only if $\sigma(e) \notin E$. In other words, $H$ is isomorphic to $\overline{H} = (V; \binom{V}{k} \setminus E)$. In 2009, Potočnik and Šajna [5] proposed studying the problem analogous to the previous theorem for $k$-hypergraphs. In particular, they extended Muzychuk’s necessary condition to $k$-hypergraphs when $k = 2^\ell$ or $k = 2^\ell + 1$ for some positive integer $\ell$. Shortly after, Gosselin [3] established the sufficiency of the Potočnik and Šajna result.

**Theorem 1.2 (Potočnik-Šajna/Gosselin).** Let $m$ be a positive integer, $k = 2^m$ or $k = 2^m + 1$, and let $n$ be a positive integer with $n \equiv 1 \pmod{2^{m+1}}$. Then there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$ if and only if for every prime $p$ we have $p^{n(p)} \equiv 1 \pmod{2^{m+1}}$.

In Theorem 1.2, the only considered values of $k$ are of the form $k = 2^m$ or $k = 2^m + 1$, for some positive integer $m$. We now consider any integer $k \geq 2$ and look at the binary expansion of $k$. Then there are positive integers $\ell$ and $m$ such that $k = \sum_{0 \leq i < m} k_i 2^i + 2^m$ or $k = 1 + \sum_{0 \leq i < m} k_i 2^i + 2^m$, where $k_i \in \{0, 1\}$, for every $i$. In Theorem 1.2, each such $k_i = 0$. Furthermore, in Theorem 1.2, $n \equiv 1 \pmod{2^{m+1}}$. This suggests our next theorem which extends the necessary condition of Potočnik and Šajna for more values of $k$.

**Theorem 1.3.** Let $\ell, k, n$ and $m$ be positive integers such that $1 < k < n$, $1 \leq \ell \leq m$ and $n \equiv 1 \pmod{2^{m+1}}$, $k = \sum_{0 \leq j \leq m} k_j 2^j$ or $k = \sum_{0 \leq j \leq m} k_j 2^j + 1$, where $k_j \in \{0, 1\}$ for every $j$, $\ell \leq j \leq m$. If there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$, then for every prime $p$ we have $p^{n(p)} \equiv 1 \pmod{2^{\ell+1}}$. 
2. Proof of Theorem 1.3

If \( H \) is a self-complementary \( k \)-hypergraph, then the set of all self-complementing permutations of \( H \) will be denoted by \( C(H) \). In [7] the following characterization of self-complementing permutations for \( k \)-hypergraphs was given. Here \(|c|\) denotes the order of a cycle \( c \).

**Theorem 2.1.** Let \( n \) and \( k \) be positive integers, \( 2 \leq k \leq n \). A permutation \( \sigma \) of \([1, n]\) with cycles \( c_1, \ldots, c_\lambda \) is a self-complementing permutation of a \( k \)-hypergraph of order \( n \) if and only if there is a nonnegative integer \( t \) such that the following hold.

(i) \( k = a_t 2^t + s_t \), for some integers \( a_t \) and \( s_t \), where \( a_t \) is odd and \( 0 \leq s_t < 2^t \);

(ii) \( n = b_t 2^{t+1} + r_t \), for some integers \( b_t \) and \( r_t \), where \( 0 \leq r_t < 2^t + s_t \); and

(iii) \( \sum_{c_i \mid c_i \leq t} |c_i| = r_t \).

In [7], the condition (iii) has the form of inequality \( \sum_{c_i \mid c_i \leq t} |c_i| \leq r_t \). However, since \( r_t = \sum_{c_i \mid c_i \leq t} |c_i| \) (mod \( 2^{t+1} \)) and \( r_t < 2^{t+1} \), we have equality (iii).

Theorem 2.1 implies the following corollary.

**Corollary 2.2.** Let \( \ell, k, n \) and \( m \) be positive integers such that \( 1 < k < n \), \( 1 \leq \ell \leq m \) and \( n \equiv 1 \) (mod \( 2^{m+1} \)), \( k = \sum_{\ell \leq j \leq m} k_j 2^j \) or \( k = \sum_{\ell \leq j \leq m} k_j 2^j + 1 \), where \( k_j \in \{0,1\} \) for every \( j, \ell \leq j \leq m \). Then every cycle of order greater than one of any self-complementing permutation of a self-complementary \( k \)-hypergraph of order \( n \) has order divisible by \( 2^{t+1} \).

Note that any such a permutation has exactly one cycle of order one.

**Proof.** Let \( \sigma \) be a self-complementing permutation of a self-complementary \( k \)-hypergraph of order \( n \) with cycles \( c_1, \ldots, c_\lambda \). By Theorem 2.1 there exists a nonnegative integer \( t \) such that

1. \( k = a_t 2^t + s_t \), where \( a_t \) is odd and \( 0 \leq s_t < 2^t \),
2. \( n = b_t 2^{t+1} + r_t \), \( r_t \in \{0, \ldots, 2^t - 1 + s_t\} \), and
3. \( \sum_{c_i \mid c_i \leq t} |c_i| = r_t \).

First observe that \( t = 0 \) implies \( s_t = 0 \), and hence \( r_t = 0 \) and \( n \) is even, a contradiction. Thus, \( t \geq 1 \). Since \( a_t \) is odd, it follows that \( t \geq \ell \), and since \( k < 2^{m+1} \), we have \( t \leq m \). Consequently, as \( n \equiv 1 \) (mod \( 2^{m+1} \)), we have that \( n \equiv 1 \) (mod \( 2^{t+1} \)) and \( r_t = 1 \). Thus, exactly one cycle \( c_1 \), necessarily of length 1, satisfies (3). In other words, with exception of a single fixed point, every cycle of \( \sigma \) has order divisible by \( 2^{t+1} \), and hence by \( 2^{t+1} \).

The proof of Theorem 1.3 uses the technique of Muzychuk [2]. The proof also depends on the first two Sylow theorems (see [4], for example). The following theorem is well-known. We give it however with proof, for completeness.
Theorem 2.3. Let $p$ be a prime and $G$ a finite group. If $P$ is a Sylow $p$-subgroup of its normalizer in $G$, then $P$ is a Sylow $p$-subgroup of the group $G$.

Proof. To prove this theorem, we shall use the notion of group action. If we have a group $G$ acting on a set $X$, we use symbols $X_{fix}$, $G_x$, and $O_x$ to denote the set of all fixed points of $X$, the stabilizer of a point $x$ in $G$, and the orbit of $x$, respectively. Recall that for any point $x$, the Orbit-Stabilizer Theorem (see, for instance, [4] Section 8.3 Lemma 3) asserts that $|O_x| = |G/G_x|$, and clearly $O_x = \{x\}$ if and only if $G_x = G$.

The well-known Orbit Decomposition Theorem (see [4]) states that if a group $G$ acts on a finite set $X \neq \emptyset$, and $x_1, \ldots, x_n \in X$ are representatives of mutually disjoint orbits with at least two elements, then

$$|X| = |X_{fix}| + \sum_{i=1}^n |G/G_{x_i}|.$$ 

Thus, the Orbit Decomposition Theorem implies that if $G$ is a $p$-group, then

$$|X| \equiv |X_{fix}| \pmod{p}.$$ 

By $N_G(H)$ we denote the normalizer of a subgroup $H$ in $G$; that is the largest subgroup of $G$ in which $H$ is normal, namely $N_G(H) = \{g \in G: gHg^{-1} = H\}$.

Now we have the following fact.

Fact. If $H$ is a $p$-subgroup of $G$, then $|N_G(H)/H| \equiv |G/H| \pmod{p}$.

To prove it, we consider the following action of $H$ on the set $G/H$ of right cosets: for every $a \in H$ and every coset $Hb$, we define $a(Hb) = Hba^{-1}$. It is straightforward to verify that we are indeed defining a group action. Clearly, for every $a \in H$, and for every $b \in G$, $Hba^{-1} = Hb$ if and only if $bab^{-1} \in H$, and hence, $(G/H)_{fix} = N_G(H)/H$. Since $H$ is a $p$-group, $|G/H| - |N_G(H)/H| = \frac{|G/H| - |(G/H)_{fix}|}{|N_G(H)/H|}$ is divisible by $p$.

If $P$ is a Sylow $p$-subgroup of $N_G(P)$, then $|N_G(P)/P| \not\equiv 0 \pmod{p}$, and by our Fact, it follows that $P$ is a Sylow $p$-subgroup of $G$. 

Proof of Theorem 1.3.

Suppose that $H = (V; E)$ is a self-complementary vertex-transitive $k$-hypergraph of order $n$, where $k$ and $n$ satisfy the conditions of our theorem. Let $p$ be a prime; if $n_{(p)} = 0$, then the result is clear. Thus assume that $n_{(p)} > 0$. We shall find a self-complementary vertex-transitive $k$-subhypergraph $H'$ of $H$ of order $p^{n_{(p)}}$ such that the cycles of a self-complementing permutation of $H'$ are cycles of a self-complementing permutation $\sigma$ of $H$ and the fixed point of $\sigma$ is one of the vertices of $H'$. By Corollary 2.2, all cycles of $\sigma$ have order divisible by $2^{\ell+1}$,
with the exception of a single fixed point. Hence the order of $H'$, that is $p^{r(n)}$, is congruent to 1 modulo $2^κ+1$, and the statement of Theorem 1.3 follows.

Let $M = \text{Aut}(H)$ be the automorphism group of $H$. For any group $K$, denote the set of the Sylow $p$-subgroups of $K$ by $\text{Syl}_p(K)$.

Note that for every $σ ∈ C(H)$ we have $σ^2 ∈ \text{Aut}(H)$. Moreover a product of a number of automorphisms and self-complementing permutations is an automorphism of $H$ if the number of self-complementing permutations is even; otherwise, the product is a self-complementing permutation of $H$. The set $G = \text{Aut}(H) ∪ C(H)$ is a group which is generated by $\text{Aut}(H) ∪ \{σ\}$, where $σ$ is an arbitrary element of $C(H)$.

Define $\mathcal{P}$ to be the set of $p$-subgroups $P$ of $M$ with the property that there exists a vertex $ν$ of $H$ and $τ ∈ C(H)$ such that

1. $τ(ν) = ν$;
2. $τPτ^{-1} = P$ (τ normalizes $P$);
3. $P_ν ∈ \text{Syl}_p(M_ν)$.

We will show that $\mathcal{P}$ is not empty and any maximal element of $\mathcal{P}$ is, in fact, a Sylow $p$-subgroup of $M$.

Since $H$ is self-complementary, $C(H)$ is not empty. Choose any $σ ∈ C(H)$. By Corollary 2.2 there is a fixed point $ν$ of $σ$. Let $P ∈ \text{Syl}_p(M_ν)$.

Note that if $p$ does not divide $|M_ν|$, then $P$ is trivial. Since $P$ is a subgroup of $M_ν$, then $P = P_ν$, and clearly $σPσ^{-1}$ is a subgroup of $M_ν$ isomorphic to $P$.

By the second Sylow Theorem, there exists $g ∈ M_ν$ such that $σPσ^{-1} = gPg^{-1}$. Set $τ = g^{-1}σ$. Then $τ ∈ C(H)$, $τ(ν) = ν$, $τPτ^{-1} = P$, and $P_ν ∈ \text{Syl}_p(M_ν)$. Hence $P ∈ \mathcal{P}$ and $\mathcal{P} ≠ ∅$.

From now on we shall assume that

- $P ∈ \mathcal{P}$ is a maximal element of $\mathcal{P}$,
- $N$ is the normalizer of $P$ in $M$,
- $Q$ is a Sylow $p$-subgroup of $N$ containing $P$ ($Q$ exists by the second Sylow Theorem).

**Claim.** $P$ is a Sylow $p$-subgroup of $M$.

**Proof.** To prove this claim, it suffices to show that $Q ∈ \mathcal{P}$, and hence $Q = P$ by the maximality of $P$. It will then follow that $P$ is a Sylow $p$-subgroup of its own normalizer in $M$, and hence by Theorem 2.3, it is a Sylow $p$-subgroup of $M$.

Since $P ∈ \mathcal{P}$, there are $τ ∈ C(H)$ and a vertex $ν$ such that $τ(ν) = ν$, $τPτ^{-1} = P$ and $P_ν ∈ \text{Syl}_p(M_ν)$. It is straightforward to show that $τ$ normalizes $N$, that is, $τNτ^{-1} = N$. Thus, $τN = Nτ$.

Since $Q$ is a subgroup of $N$ and $τNτ^{-1} = N$, we have that $τQτ^{-1}$ is a subgroup of $N$ and since $|τQτ^{-1}| = |Q|$, we conclude that $τQτ^{-1}$ is a Sylow $p$-subgroup of $N$. 
Recall that \( v \) is a fixed point of \( \tau \), and let \( U = N(v) \), where \( N(v) = \{ h(v) : h \in N \} \). Then we have \( \tau(U) = \tau(N(v)) = (\tau N)(v) = (N\tau)(v) \), since \( \tau N = N\tau \) by our previous argument. This implies that \( \tau(U) = N(\tau(v)) = N(v) = U \).

By Corollary 2.2, every cycle \( c \) of the self-complementing permutation \( \tau \) has length divisible by \( 2^{\ell+1} \), with the exception of one fixed point. Since \( \tau(U) = U \), for every cycle \( c \) of the permutation \( \tau \) we know that either all the vertices of \( c \) are in \( U \) or else, the set of vertices of \( c \) is disjoint with \( U \). Therefore, \( U \) is a set of vertices of a self-complementary vertex-transitive \( k \)-hypergraph \( H' = (U ; E \cap \binom{U}{k}) \) with self-complementing permutation \( \tau \) (restricted to \( U \)) and vertex-transitive group of automorphisms containing \( N \). Moreover, vertex \( v \), the fixed point of \( \tau \), is in \( U \). Hence we have

\[
|U| \equiv 1 \pmod{2^{\ell+1}}.
\]

Since \( \tau Q \tau^{-1} \) and \( Q \) are two Sylow \( p \)-subgroups of the group \( N \), by the second Sylow Theorem, there is \( g \in N \) such that \( \tau Q \tau^{-1} = g Q g^{-1} \).

Hence \( (g^{-1} \tau) Q (g^{-1} \tau)^{-1} = Q \).

Write \( \sigma = \tau^{-1} g \). By the definition of \( U \) and since \( g \in N \), we have \( g(U) = U \), and hence, \( \sigma(U) = U \). We have \( \sigma Q \sigma^{-1} = Q \), and the restriction of \( \sigma \in C(H) \) to the set \( U \) is also a self-complementing permutation of \( H' \).

By Corollary 2.2, the permutation \( \sigma \) has a fixed point \( u \), and all remaining cycles are of lengths congruent to 1 (mod \( 2^{\ell+1} \)). Since \( |U| \equiv 1 \pmod{2^{\ell+1}} \) and the cycles of the restriction of \( \sigma \) to \( U \) are the cycles of \( \sigma \), we have \( u \in U \).

Since the group \( N \) is transitive on the set \( U \), there is \( h \in N \) such that \( h(v) = u \). Thus the subgroups \( M_u \) and \( M_u \) are conjugate, that is,

- \( M_u = h M_u h^{-1} \).

Moreover, we also have

- \( P_u = h P_u h^{-1} \).

Hence \( |M_u| = |M_u| \) and \( |P_u| = |P_u| \), and therefore \( P_u \) is a Sylow \( p \)-subgroup of \( M_u \). Since \( P_u \leq Q_u \leq M_u \) and \( Q_u \) is a \( p \)-subgroup of \( M_u \), it follows that \( Q_u = P_u \) and \( Q_u \) is a Sylow \( p \)-subgroup of \( M_u \). Finally, we have \( Q \in \mathcal{P} \). This completes the proof of the claim.

Now we shall show that the orbit \( P(v) \) induces a self-complementary vertex-transitive \( k \)-hypergraph of order \( p^r \), where \( r = n(p) \). Note first that since \( \tau P = P \tau \) and \( \tau(v) = v \), we have

\[
\tau(P(v)) = P(\tau(v)) = P(v)
\]

and therefore the \( k \)-subhypergraph of \( H \) induced by \( P(v) \) is self-complementary and vertex-transitive.
Write $|M| = p^d q$, where $q$ and $p$ are relatively prime. Then $|P| = p^d$ by the Claim. Since $M$ acts transitively on $V$ we have

$$|M_v| = \frac{|M|}{(M(v))} = \frac{p^d q}{p^r m} = p^{d-r} s,$$

for some positive integers $m$ and $s$ both relatively prime with $p$.

Since $P_v \in \text{Syl}_p(M_v)$, it follows that $|P_v| = p^{d-r}$. On the other hand, since $P \in \text{Syl}_p(M)$ and $P_v \in \text{Syl}_p(M_v)$ we have

$$p^{d-r} = |P_v| = \frac{|P|}{|P(v)|} = \frac{p^d}{|P(v)|}.$$

This implies $|P(v)| = p^r$. Since $\tau$ is a self-complementing permutation of $H$, by Corollary 2.2, the length of every cycle of $\tau$, with exception of a single fixed point, is divisible by $2^{\ell+1}$. Since $\tau(P(v)) = P(v)$, we know that $P(v)$ is the union of orbits of $\tau$, including the fixed point $v$. Hence $p^r \equiv 1 (\text{mod } 2^{\ell+1})$ as claimed.

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