Abstract

Let $\ell$ be a positive integer, $k = 2^\ell$ or $k = 2^\ell + 1$, and let $n$ be a positive integer with $n \equiv 1 \pmod{2^{\ell+1}}$. For a prime $p$, $n(p)$ denotes the largest integer $i$ such that $p^i$ divides $n$. Potočnik and Šajna showed that if there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$, then for every prime $p$ we have $p^{n(p)} \equiv 1 \pmod{2^{\ell+1}}$. Here we extend their result to a larger class of integers $k$.

Keywords: vertex-transitive hypergraphs, self-complementary hypergraphs.

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1. Introduction

For a prime $p$ and a positive integer $n$, let $n(p)$ denote the largest integer $i$ for which $p^i$ divides $n$. Using this notation, we combine the theorems of Rao and Muzychuk as follows.

**Theorem 1.1** (Rao/Muzychuk). For a positive integer $n$, there exists a vertex-transitive self-complementary graph of order $n$ if and only if $p^{n(p)} \equiv 1 \pmod{4}$ for every prime $p$.

For an interesting discussion of the history of the vertex-transitive self-complementary graph problem, see [1].

For every integer $k \geq 2$, a $k$-uniform hypergraph, or $k$-hypergraph, for short, is a pair $(V; E)$ consisting of a vertex set $V$ and edge set $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all $k$-subsets of $V$. Clearly, a 2-hypergraph is just a simple graph. A hypergraph $H$ is called **vertex-transitive** if for every two vertices $u, v$ of $H$ there is an automorphism $\phi$ of $H$ for which $u = \phi(v)$. A $k$-hypergraph $H = (V; E)$ is called **self-complementary** if there is a permutation $\sigma$ of the set $V$, called a self-complementing permutation, such that for every $k$-subset $e$ of $V$, $e \in E$ if and only if $\sigma(e) \notin E$. In other words, $H$ is isomorphic to $\overline{H} = (V; \binom{V}{k} \setminus E)$. In 2009, Potočnik and Šajna [5] proposed studying the problem analogous to the previous theorem for $k$-hypergraphs. In particular, they extended Muzychuk’s necessary condition to $k$-hypergraphs when $k = 2^\ell$ or $k = 2^\ell + 1$ for some positive integer $\ell$. Shortly after, Gosselin [3] established the sufficiency of the Potočnik and Šajna result.

**Theorem 1.2** (Potočnik-Šajna/Gosselin). Let $m$ be a positive integer, $k = 2^m$ or $k = 2^m + 1$, and let $n$ be a positive integer with $n \equiv 1 \pmod{2^m+1}$. Then there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$ if and only if for every prime $p$ we have $p^{n(p)} \equiv 1 \pmod{2^m+1}$.

In Theorem 1.2, the only considered values of $k$ are of the form $k = 2^m$ or $k = 2^m + 1$, for some positive integer $m$. We now consider any integer $k \geq 2$ and look at the binary expansion of $k$. Then there are positive integers $\ell$ and $m$ such that $k = \sum_{i=1}^{m} k_i 2^i + 2^m$ or $k = 1 + \sum_{i=1}^{m} k_i 2^i + 2^m$, where $k_i \in \{0, 1\}$, for every $i$. In Theorem 1.2, each such $k_i = 0$. Furthermore, in Theorem 1.2, $n \equiv 1 \pmod{2^m+1}$. This suggests our next theorem which extends the necessary condition of Potočnik and Šajna for more values of $k$.

**Theorem 1.3.** Let $\ell, k, n$ and $m$ be positive integers such that $1 < k < n$, $1 \leq \ell \leq m$ and $n \equiv 1 \pmod{2^m+1}$, $k = \sum_{j=1}^{m} k_j 2^j$ or $k = \sum_{j=1}^{m} k_j 2^j + 1$, where $k_j \in \{0, 1\}$ for every $j$, $\ell \leq j \leq m$. If there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$, then for every prime $p$ we have $p^{n(p)} \equiv 1 \pmod{2^\ell+1}$.
2. Proof of Theorem 1.3

If $H$ is a self-complementary $k$-hypergraph, then the set of all self-complementing permutations of $H$ will be denoted by $C(H)$. In [7] the following characterization of self-complementing permutations for $k$-hypergraphs was given. Here $|c|$ denotes the order of a cycle $c$.

**Theorem 2.1.** Let $n$ and $k$ be positive integers, $2 \leq k \leq n$. A permutation $\sigma$ of $[1,n]$ with cycles $c_1, \ldots, c_\lambda$ is a self-complementing permutation of a $k$-hypergraph of order $n$ if and only if there is a nonnegative integer $t$ such that the following hold.

(i) $k = a_t 2^t + s_t$, for some integers $a_t$ and $s_t$, where $a_t$ is odd and $0 \leq s_t < 2^t$;
(ii) $n = b_t 2^{t+1} + r_t$, for some integers $b_t$ and $r_t$, where $0 \leq r_t < 2^t + s_t$; and
(iii) $\sum_{|c_i| \leq t} |c_i| = r_t$.

In [7], the condition (iii) has the form of inequality $\sum_{|c_i| \leq t} |c_i| \leq r_t$. However, since $r_t = \sum_{|c_i| \leq t} |c_i| \mod 2^{t+1}$ and $r_t < 2^{t+1}$, we have equality (iii).

Theorem 2.1 implies the following corollary.

**Corollary 2.2.** Let $\ell, k, n$ and $m$ be positive integers such that $1 < k < n$, $1 \leq \ell \leq m$ and $n \equiv 1 \mod 2^{m+1}$, $k = \sum_{\ell \leq j \leq m} k_j 2^j$ or $k = \sum_{\ell \leq j \leq m} k_j 2^j + 1$, where $k_j \in \{0,1\}$ for every $j$, $\ell \leq j \leq m$. Then every cycle of order greater than one of any self-complementing permutation of a self-complementary $k$-hypergraph of order $n$ has order divisible by $2^{\ell+1}$.

Note that any such a permutation has exactly one cycle of order one.

**Proof.** Let $\sigma$ be a self-complementing permutation of a self-complementary $k$-hypergraph of order $n$ with cycles $c_1, \ldots, c_\lambda$. By Theorem 2.1 there exists a nonnegative integer $t$ such that

1. $k = a_t 2^t + s_t$, where $a_t$ is odd and $0 \leq s_t < 2^t$,
2. $n = b_t 2^{t+1} + r_t$, $r_t \in \{0, \ldots, 2^t - 1 + s_t\}$, and
3. $\sum_{|c_i| \leq t} |c_i| = r_t$.

First observe that $t = 0$ implies $s_t = 0$, and hence $r_t = 0$ and $n$ is even, a contradiction. Thus, $t \geq 1$. Since $a_t$ is odd, it follows that $t \geq \ell$, and since $k < 2^{m+1}$, we have $t \leq m$. Consequently, as $n \equiv 1 \mod 2^{m+1}$, we have that $n \equiv 1 \mod 2^{\ell+1}$ and $r_t = 1$. Thus, exactly one cycle $c_1$, necessarily of length 1, satisfies (3). In other words, with exception of a single fixed point, every cycle of $\sigma$ has order divisible by $2^{\ell+1}$, and hence by $2^{\ell+1}$. 

The proof of Theorem 1.3 uses the technique of Muzychuk [2]. The proof also depends on the first two Sylow theorems (see [4], for example). The following theorem is well-known. We give it however with proof, for completeness.
Theorem 2.3. Let $p$ be a prime and $G$ a finite group. If $P$ is a Sylow $p$-subgroup of its normalizer in $G$, then $P$ is a Sylow $p$-subgroup of the group $G$.

Proof. To prove this theorem, we shall use the notion of group action. If we have a group $G$ acting on a set $X$, we use symbols $X_{fix}$, $G_x$, and $O_x$ to denote the set of all fixed points of $X$, the stabilizer of a point $x$ in $G$, and the orbit of $x$, respectively. Recall that for any point $x$, the Orbit-Stabilizer Theorem (see, for instance, [4] Section 8.3 Lemma 3) asserts that $|O_x| = |G/G_x|$, and clearly $O_x = \{x\}$ if and only if $G_x = G$.

The well-known Orbit Decomposition Theorem (see [4]) states that if a group $G$ acts on a finite set $X \neq \emptyset$, and $x_1, \ldots, x_n \in X$ are representatives of mutually disjoint orbits with at least two elements, then

$$|X| = |X_{fix}| + \sum_{i=1}^{n} |G/G_{x_i}|.$$ 

Thus, the Orbit Decomposition Theorem implies that if $G$ is a $p$-group, then

$$|X| \equiv |X_{fix}| \pmod{p}.$$ 

By $N_G(H)$ we denote the normalizer of a subgroup $H$ in $G$; that is the largest subgroup of $G$ in which $H$ is normal, namely $N_G(H) = \{g \in G: gHg^{-1} = H\}$.

Now we have the following fact.

Fact. If $H$ is a $p$-subgroup of $G$, then $|N_G(H)/H| \equiv |G/H| \pmod{p}$.

To prove it, we consider the following action of $H$ on the set $G/H$ of right cosets: for every $a \in H$ and every coset $Hb$, we define $a(Hb) = Hba^{-1}$. It is straightforward to verify that we are indeed defining a group action. Clearly, for every $a \in H$, and for every $b \in G$, $Hba^{-1} = Hb$ if and only if $bab^{-1} \in H$, and hence, $(G/H)_{fix} = N_G(H)/H$. Since $H$ is a $p$-group, $|G/H| - |N_G(H)/H| = |G/H| - |(G/H)_{fix}|$ is divisible by $p$.

If $P$ is a Sylow $p$-subgroup of $N_G(P)$, then $|N_G(P)/P| \not\equiv 0 \pmod{p}$, and by our Fact, it follows that $P$ is a Sylow $p$-subgroup of $G$. $\blacksquare$

Proof of Theorem 1.3.

Suppose that $H = (V; E)$ is a self-complementary vertex-transitive $k$-hypergraph of order $n$, where $k$ and $n$ satisfy the conditions of our theorem. Let $p$ be a prime; if $n(p) = 0$, then the result is clear. Thus assume that $n(p) > 0$. We shall find a self-complementary vertex-transitive $k$-subhypergraph $H'$ of $H$ of order $p^{n(p)}$ such that the cycles of a self-complementing permutation of $H'$ are cycles of a self-complementing permutation $\sigma$ of $H$ and the fixed point of $\sigma$ is one of the vertices of $H'$. By Corollary 2.2, all cycles of $\sigma$ have order divisible by $2^{\ell+1}$,
with the exception of a single fixed point. Hence the order of $H'$, that is $p^{\ell+1}$, is congruent to 1 modulo $2^k+1$, and the statement of Theorem 1.3 follows.

Let $M = \text{Aut}(H)$ be the automorphism group of $H$. For any group $K$, denote the set of the Sylow $p$-subgroups of $K$ by $\text{Syl}_p(K)$.

Note that for every $\sigma \in C(H)$ we have $\sigma^2 \in \text{Aut}(H)$. Moreover a product of a number of automorphisms and self-complementing permutations is an automorphism of $H$ if the number of self-complementing permutations is even; otherwise, the product is a self-complementing permutation of $H$. The set $G = \text{Aut}(H) \cup C(H)$ is a group which is generated by $\text{Aut}(H) \cup \{\sigma\}$, where $\sigma$ is an arbitrary element of $C(H)$.

Define $\mathcal{P}$ to be the set of $p$-subgroups $P$ of $M$ with the property that there exists a vertex $v$ of $H$ and $\tau \in C(H)$ such that

1. $\tau(v) = v$;
2. $\tau P \tau^{-1} = P$ ($\tau$ normalizes $P$);
3. $P_v \in \text{Syl}_p(M_v)$.

We will show that $\mathcal{P}$ is not empty and any maximal element of $\mathcal{P}$ is, in fact, a Sylow $p$-subgroup of $M$.

Since $H$ is self-complementary, $C(H)$ is not empty. Choose any $\sigma \in C(H)$. By Corollary 2.2 there is a fixed point $v$ of $\sigma$. Let $P \in \text{Syl}_p(M_v)$.

Note that if $p$ does not divide $|M_v|$, then $P$ is trivial. Since $P$ is a subgroup of $M_v$, then $P = P_v$, and clearly $\sigma P \sigma^{-1}$ is a subgroup of $M_v$ isomorphic to $P$. By the second Sylow Theorem, there exists $g \in M_v$ such that $\sigma P \sigma^{-1} = g P g^{-1}$. Set $\tau = g^{-1} \sigma$. Then $\tau \in C(H)$, $\tau(v) = v$, $\tau P \tau^{-1} = P$, and $P_v \in \text{Syl}_p(M_v)$. Hence $P \in \mathcal{P}$ and $\mathcal{P} \neq \emptyset$.

From now on we shall assume that

- $P \in \mathcal{P}$ is a maximal element of $\mathcal{P}$,
- $N$ is the normalizer of $P$ in $M$,
- $Q$ is a Sylow $p$-subgroup of $N$ containing $P$ ($Q$ exists by the second Sylow Theorem).

**Claim.** $P$ is a Sylow $p$-subgroup of $M$.

**Proof.** To prove this claim, it suffices to show that $Q \in \mathcal{P}$, and hence $Q = P$ by the maximality of $P$. It will then follow that $P$ is a Sylow $p$-subgroup of its own normalizer in $M$, and hence by Theorem 2.3, it is a Sylow $p$-subgroup of $M$.

Since $P \in \mathcal{P}$, there are $\tau \in C(H)$ and a vertex $v$ such that $\tau(v) = v$, $\tau P \tau^{-1} = P$ and $P_v \in \text{Syl}_p(M_v)$. It is straightforward to show that $\tau$ normalizes $N$, that is, $\tau N \tau^{-1} = N$. Thus, $\tau N = N \tau$.

Since $Q$ is a subgroup of $N$ and $\tau N \tau^{-1} = N$, we have that $\tau Q \tau^{-1}$ is a subgroup of $N$ and since $|\tau Q \tau^{-1}| = |Q|$, we conclude that $\tau Q \tau^{-1}$ is a Sylow $p$-subgroup of $N$. 

Recall that \( v \) is a fixed point of \( \tau \), and let \( U = N(v) \), where \( N(v) = \{ h(v) : h \in N \} \). Then we have \( \tau(U) = \tau(N(v)) = (\tau N)(v) = (N\tau)(v) \), since \( \tau N = N\tau \) by our previous argument. This implies that \( \tau(U) = N(\tau(v)) = N(v) = U \).

By Corollary 2.2, every cycle \( c \) of the self-complementing permutation \( \tau \) has length divisible by \( 2^{\ell+1} \), with the exception of one fixed point. Since \( \tau(U) = U \), for every cycle \( c \) of the permutation \( \tau \) we know that either all the vertices of \( c \) are in \( U \) or else, the set of vertices of \( c \) restricted to \( U \) is disjoint with \( U \). Therefore, \( U \) is a set of vertices of a self-complementary vertex-transitive \( k \)-hypergraph \( H' = (U; E \cap \left( \binom{U}{k} \right)) \) with self-complementing permutation \( \tau \) (restricted to \( U \)) and vertex-transitive group of automorphisms containing \( N \). Moreover, vertex \( v \), the fixed point of \( \tau \), is in \( U \). Hence we have

\[
|U| \equiv 1 \left( \text{mod } 2^{\ell+1} \right).
\]

Since \( \tau Q \tau^{-1} \) and \( Q \) are two Sylow \( p \)-subgroups of the group \( N \), by the second Sylow Theorem, there is \( g \in N \) such that \( \tau Q \tau^{-1} = g Q g^{-1} \).

Hence \( (g^{-1} \tau) Q (g^{-1} \tau)^{-1} = Q \).

Write \( \sigma = \tau^{-1} g \). By the definition of \( U \) and since \( g \in N \), we have \( g(U) = U \), and hence, \( \sigma(U) = U \). We have \( \sigma Q \sigma^{-1} = Q \), and the restriction of \( \sigma \in C(H) \) to the set \( U \) is also a self-complementing permutation of \( H' \).

By Corollary 2.2, the permutation \( \sigma \) has a fixed point \( u \), and all remaining cycles are of lengths congruent to \( 1 \) (mod \( 2^{\ell+1} \)). Since \( |U| \equiv 1 \) (mod \( 2^{\ell+1} \)) and the cycles of the restriction of \( \sigma \) to \( U \) are the cycles of \( \sigma \), we have \( u \in U \).

Since the group \( N \) is transitive on the set \( U \), there is \( h \in N \) such that \( h(v) = u \). Thus the subgroups \( M_u \) and \( M_u \) are conjugate, that is,

- \( M_u = h M_u h^{-1} \).

Moreover, we also have

- \( P_u = h P_u h^{-1} \).

Hence \( |M_u| = |M_u| \) and \( |P_u| = |P_u| \), and therefore \( P_u \) is a Sylow \( p \)-subgroup of \( M_u \). Since \( P_u \leq Q_u \leq M_u \) and \( Q_u \) is a \( p \)-subgroup of \( M_u \), it follows that \( Q_u = P_u \) and \( Q_u \) is a Sylow \( p \)-subgroup of \( M_u \). Finally, we have \( Q \in \mathcal{P} \). This completes the proof of the claim.

Now we shall show that the orbit \( P(v) \) induces a self-complementary vertex-transitive \( k \)-hypergraph of order \( p^r \), where \( r = \nu(p) \). Note first that since \( \tau P = P \tau \) and \( \tau(v) = v \), we have \( \tau(P(v)) = P(\tau(v)) = P(v) \)

and therefore the \( k \)-subhypergraph of \( H \) induced by \( P(v) \) is self-complementary and vertex-transitive.
Write $|M| = p^d q$, where $q$ and $p$ are relatively prime. Then $|P| = p^d$ by the Claim. Since $M$ acts transitively on $V$ we have

$$|M_v| = \frac{|M|}{|M(v)|} = \frac{p^d q}{p^r m} = p^{d-r} s,$$

for some positive integers $m$ and $s$ both relatively prime with $p$.

Since $P_v \in \text{Syl}_p(M_v)$, it follows that $|P_v| = p^{d-r}$. On the other hand, since $P \in \text{Syl}_p(M)$ and $P_v \in \text{Syl}_p(M_v)$ we have

$$p^{d-r} = |P_v| = \frac{|P|}{|P(v)|} = \frac{p^d}{|P(v)|}.$$ 

This implies $|P(v)| = p^r$. Since $\tau$ is a self-complementing permutation of $H$, by Corollary 2.2, the length of every cycle of $\tau$, with exception of a single fixed point, is divisible by $2^{\ell+1}$. Since $\tau(P(v)) = P(v)$, we know that $P(v)$ is the union of orbits of $\tau$, including the fixed point $v$. Hence $p^r \equiv 1 \pmod{2^{\ell+1}}$ as claimed.

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**References**


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