THE EXISTENCE OF PATH-FACTOR COVERED GRAPHS

GUOWEI DAI

Faculty of Mathematics & Statistics
Central China Normal University
Laoyu Road 152, Wuhan, Hubei 430079, P.R. China

e-mail: daiguowei1990@163.com

Abstract

A spanning subgraph \( H \) of a graph \( G \) is called a \( P_{\geq k} \)-factor of \( G \) if every component of \( H \) is isomorphic to a path of order at least \( k \), where \( k \geq 2 \). A graph \( G \) is called a \( P_{\geq k} \)-factor covered graph if there is a \( P_{\geq k} \)-factor of \( G \) covering \( e \) for any \( e \in E(G) \). In this paper, we obtain two special classes of \( P_{\geq 2} \)-factor covered graphs. We also obtain two special classes of \( P_{\geq 3} \)-factor covered graphs. Furthermore, it is shown that these results are all sharp.

Keywords: path-factor, \( P_{\geq 2} \)-factor covered graph, \( P_{\geq 3} \)-factor covered graph, claw-free graph, isolated toughness.

2010 Mathematics Subject Classification: 05C70, 05C38.

1. Introduction

We consider only finite simple graph, unless explicitly stated. We refer to [6] for the notation and terminologies not defined here. Let \( G = (V(G), E(G)) \) be a simple graph, where \( V(G) \) and \( E(G) \) denote the vertex set and the edge set of \( G \), respectively. A subgraph \( H \) of \( G \) is called a spanning subgraph of \( G \) if \( V(H) = V(G) \) and \( E(H) \subseteq E(G) \). A subgraph \( H \) of \( G \) is called an induced subgraph of \( G \) if every pair of vertices in \( H \) which are adjacent in \( G \) are also adjacent in \( H \). For \( v \in V(G) \), the degree of \( v \) in \( G \) is denoted by \( d_G(v) \). A graph \( G \) is said to be claw-free if there is no induced subgraph of \( G \) isomorphic to \( K_{1,3} \).

For a family of connected graphs \( \mathcal{F} \), a spanning subgraph \( H \) of a graph \( G \) is called an \( \mathcal{F} \)-factor of \( G \) if each component of \( H \) is isomorphic to some graph in \( \mathcal{F} \). A spanning subgraph \( H \) of a graph \( G \) is called a \( P_{\geq k} \)-factor of \( G \) if every component of \( H \) is isomorphic to a path of order at least \( k \). For example, a \( P_{\geq 3} \)-factor means a graph factor in which every component is a path of order at least
A graph $G$ is called a $P_{\geq k}$-factor covered graph if there is a $P_{\geq k}$-factor of $G$ covering $e$ for any $e \in E(G)$.

Since Tutte proposed the well known Tutte 1-factor theorem [15], there are many results on graph factors [2, 3, 8, 9, 16] and $P_{\geq k}$-factors in claw-free graphs and cubic graphs [4, 12, 13]. More results on graph factors can be found in the survey papers and books in [2, 14, 18]. We use $\omega(G)$, $i(G)$ to denote the number of components and isolated vertices of a graph $G$, respectively. For a subset $X \subseteq V(G)$, $G - X$ denotes the graph obtained from $G$ by deleting all the vertices of $X$. Akiyama, Avis and Era [1] proved the following theorem, which is a criterion for a graph to have a $P_{\geq 2}$-factor.

**Theorem 1** (Akiyama et al. [2]). A graph $G$ has a $P_{\geq 2}$-factor if and only if $i(G - X) \leq 2|X|$ for all $X \subseteq V(G)$.

Kaneko [10] introduced the concept of a sun and gave a characterization for a graph with a $P_{\geq 3}$-factor. It is perhaps the first characterization of graphs which have a path factor not including $P_2$. Recently, Kano et al. [11] presented a simpler proof for Kaneko’s theorem [10].

A graph $H$ is called factor-critical if $H - \{v\}$ has a 1-factor for each $v \in V(H)$. Let $H$ be a factor-critical graph and $V(H) = \{v_1, v_2, \ldots, v_n\}$. By adding new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_i u_i : 1 \leq i \leq n\}$ to $H$, the resulting graph is called a sun. Note that, according to Kaneko [10], we regard $K_1$ and $K_2$ also as a sun, respectively. Usually, the suns other than $K_1$ are called big suns. It is called a sun component of $G - X$ if the component of $G - X$ is isomorphic to a sun. We denote by $\text{sun}(G - X)$ the number of sun components in $G - X$.

**Theorem 2** (Kaneko [10]). A graph $G$ has a $P_{\geq 3}$-factor if and only if $\text{sun}(G - X) \leq 2|X|$ for all $X \subseteq V(G)$.

Zhang and Zhou [19] proposed the concept of path-factor covered graph, which is a generalization of matching cover. They also obtained a characterization for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs, respectively.

**Theorem 3** (Zhang et al. [19]). Let $G$ be a connected graph. Then $G$ is a $P_{\geq 2}$-factor covered graph if and only if $i(G - S) \leq 2|S| - \varepsilon(S)$ for all $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by

$$
\varepsilon(S) = \begin{cases} 
2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\
1 & \text{if } S \neq \emptyset, S \text{ is an independent set and there exists a component of } G - S \text{ with at least two vertices,} \\
0 & \text{otherwise.}
\end{cases}
$$
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**Theorem 4** (Zhang et al. [19]). Let $G$ be a connected graph. Then $G$ is a $P_{\geq 3}$-factor covered graph if and only if $\text{sun}(G - S) \leq 2|S| - \varepsilon(S)$ for all $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by

$$\varepsilon(S) = \begin{cases} 
2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\
1 & \text{if } S \neq \emptyset, S \text{ is an independent set and there exists a} \\
& \text{non-sun component of } G - S, \\
0 & \text{otherwise.}
\end{cases}$$

For a connected graph $G$, its toughness, denoted by $t(G)$, was first introduced by Chvátal [7] as follows. If $G$ is complete, then $t(G) = +\infty$; otherwise,

$$t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subseteq V(G), \omega(G - S) \geq 2 \right\}.$$ 

Bazgan, Benhamdine, Li and Woźniak [5] showed a toughness condition for the existence of a $P_{\geq 3}$-factor in a graph.

**Theorem 5** (Bazgan, Benhamdine, Li and Woźniak [5]). Let $G$ be a graph with at least three vertices. If $t(G) \geq 1$, then $G$ includes a $P_{\geq 3}$-factor.

For a connected graph $G$, its isolated toughness, denoted by $I(G)$, was first introduced by Yang, Ma and Liu [17] as follows. If $G$ is complete, then $I(G) = +\infty$; otherwise,

$$I(G) = \min \left\{ \frac{|S|}{i(G - S)} : S \subseteq V(G), i(G - S) \geq 2 \right\}.$$ 


**Theorem 6** (Zhou and Wu [20]). A graph $G$ is a $P_{\geq 3}$-factor covered graph if one the following holds.

(i) $G$ is a connected graph with at least three vertices and $t(G) > 2/3$;
(ii) $G$ is a connected graph with at least three vertices and $I(G) > 5/3$;
(iii) $G$ is a $k$-regular graph with $k \geq 2$.

In this paper, we proceed to investigate $P_{\geq k}$-factor covered graphs. We respectively obtain two special classes of $P_{\geq 2}$-factor covered graphs and $P_{\geq 3}$-factor covered graphs. Our main results will be shown in Sections 2 and 3, respectively.
2. \( P_{\geq 2} \)-Factor Covered Graphs

In this section, we mainly obtain two special classes of \( P_{\geq 2} \)-factor covered graphs. First, we will give a sufficient condition for a connected claw-free graph to be a \( P_{\geq 2} \)-factor covered graph as following. Note that the result in Theorem 7 is sharp in the sense that there exists a connected claw-free graph of minimum degree 1, which is not a \( P_{\geq 2} \)-factor covered graph. An example can be seen in Figure 1.

![Figure 1. A connected claw-free graph of minimum degree 1 that does not contain any \( P_{\geq 2} \)-factor covering.](image)

**Theorem 7.** Let \( G \) be a connected claw-free graph of minimum degree at least 2. Then \( G \) is a \( P_{\geq 2} \)-factor covered graph.

**Proof.** Suppose \( G \) is not a \( P_{\geq 2} \)-factor covered graph. Then by Theorem 3, there exists a subset \( S \subseteq V(G) \) such that \( i(G - S) > 2|S| - \varepsilon(S) \). In terms of the integrality of \( i(G - S) \), we obtain that \( i(G - S) \geq 2|S| - \varepsilon(S) + 1 \). We will distinguish two cases below to show that \( G \) is a \( P_{\geq 2} \)-factor covered graph.

**Case 1.** \( |S| \leq 1 \). If \( S = \emptyset \), then \( \varepsilon(S) = |S| = 0 \) by the definition of \( \varepsilon(S) \). It follows easily that

\[
i(G) = i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.
\]

On the other hand, \( i(G) \leq \omega(G) = 1 \) since \( G \) is a connected graph. Combining the results above, we obtain \( i(G) = 1 \), which contradicts the connectivity of \( G \).

If \( |S| = 1 \), let \( S = \{s\} \). We obtain \( \varepsilon(S) \leq 1 \) by the definition of \( \varepsilon(S) \). If \( \varepsilon(S) = 0 \), then

\[
\omega(G - S) \geq i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 3.
\]

Therefore, if either \( \varepsilon(S) \) is 0 or 1, then there are at least three components of \( G - \{s\} \). It follows easily that there exists a claw with center vertex \( s \) in \( G \), a contradiction.
Case 2. $|S| \geq 2$. Let $|S| = k \geq 2$ and $S = \{s_1, s_2, \ldots, s_k\}$. By the definition of $\varepsilon(S)$, we have $\varepsilon(S) \leq 2$. It follows easily that

$$i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 2|S| - 1.$$ 

Let $i(G - S) = m \geq 2k - 1$ and $\{x_1, x_2, \ldots, x_m\}$ be the set of isolated vertices of $G - S$. Since the minimum degree of $G$ is at least two, we immediately obtain the number of edges incident with the vertices in $\{x_1, x_2, \ldots, x_m\}$ is at least $2m$.

Because $G$ does not have multiple edges and

$$\frac{2m}{|S|} = \frac{2m}{k} \geq \frac{2(2k - 1)}{k} = 4 - \frac{2}{k} \geq 3,$$

there must exist a vertex $s_i \in S$ adjacent to at least three vertices in $\{x_1, x_2, \ldots, x_m\}$ by pigeonhole principle. It follows easily that there exists a claw with center vertex $s_i$ in $G$, a contradiction.

Combining Case 1 and Case 2, Theorem 7 is proved.

Next, we study the relationship between isolated toughness and $P_{\geq 2}$-factor covered graphs, and obtain an isolated toughness condition for the existence of $P_{\geq 2}$-factor covered graphs. The example in Figure 2 shows the sharpness of the results in Theorem 8 in the sense that there exists a connected graph with $I(G) = 2/3$, which is not a $P_{\geq 2}$-factor covered graph.

![Figure 2](image_url)

Figure 2. A connected graph with $I(G) = 2/3$ that does not contain any $P_{\geq 2}$-factor covering $e = x_1x_5$.

**Theorem 8.** Let $G$ be a connected graph with at least two vertices. If $I(G) > 2/3$, then $G$ is a $P_{\geq 2}$-factor covered graph.

**Proof.** If $G$ is a complete graph with at least two vertices, obviously $G$ is a $P_{\geq 2}$-factor covered graph. Thus we may assume that $G$ is a connected graph with at least two vertices and not complete. Suppose $G$ is not a $P_{\geq 2}$-factor covered graph. Then by Theorem 3, there exists a subset $S \subseteq V(G)$ such that...
$i(G - S) > 2|S| - \varepsilon(S)$. Then, by the integrality of $i(G - S)$, we obtain that $i(G - S) \geq 2|S| - \varepsilon(S) + 1$.

Case 1. $|S| \leq 1$. If $|S| = 0$, by the definition of $\varepsilon(S)$, we have $S = \emptyset$ and $\varepsilon(S) = 0$. It follows immediately that

$$i(G) = i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1,$$

which contradicts the connectivity of $G$.

Thus we may assume $|S| = 1$, we have $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. It follows easily that

$$i(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S|.$$

By the definition of $I(G)$, we have that

$$I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{1}{2},$$

which contradicts $I(G) > 2/3$.

Case 2. $|S| \geq 2$. In this case, it follows from the definition of $\varepsilon(S)$ that $\varepsilon(S) \leq 2$, which implies that

$$i(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1 \geq 3.$$

Thus we immediately obtain

$$|S| \leq \frac{i(G - S) + 1}{2}.$$

By the definition of $I(G)$, we have

$$I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{i(G - S) + 1}{2} \leq \frac{1}{2} + \frac{1}{2i(G - S)} \leq \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

which contradicts $I(G) > 2/3$.

This completes the proof of Theorem 8.

3. $P_{\geq 3}$-Factor Covered Graphs

In this section, we mainly obtain two special classes of $P_{\geq 3}$-factor covered graphs. First, we give a minimum degree condition for a connected claw-free graph to be a $P_{\geq 3}$-factor covered graph as following. Note that the results in Theorem 9 is also sharp in the sense that there exists a connected claw-free graph of minimum degree 2, which is not a $P_{\geq 3}$-factor covered graph. It is shown by the example in Figure 3.
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Theorem 9. Let $G$ be a connected claw-free graph of minimum degree at least 3. Then $G$ is a $P_{\geq 3}$-factor covered graph.

Proof. Suppose $G$ is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 4, there exists a subset $S \subseteq V(G)$ such that $\text{sun}(G - S) > 2|S| - \varepsilon(S)$. In terms of the integrality of $\text{sun}(G - S)$, we obtain that $\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1$. We will distinguish two cases below to show that $G$ is a $P_{\geq 3}$-factor covered graph.

Case 1. $|S| \leq 1$. If $S = \emptyset$, then $\varepsilon(S) = |S| = 0$ by the definition of $\varepsilon(S)$. It follows easily that

$$\text{sun}(G) = \text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$  

On the other hand, $\text{sun}(G) \leq \omega(G) = 1$ since $G$ is a connected graph. Combining the results above, we obtain that $G$ is a big sun, which contradicts the minimum degree of $G$.

If $|S| = 1$, let $S = \{s\}$. We obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. If $\varepsilon(S) = 0$, then

$$\omega(G - S) \geq \text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 = 3.$$  

Otherwise $\varepsilon(S) = 1$, then there exists a non-sun component of $G - S$ and thus

$$\omega(G - S) \geq \text{sun}(G - S) + 1 \geq 2|S| - \varepsilon(S) + 1 + 1 = 3.$$  

Therefore, if either $\varepsilon(S)$ is 0 or 1, then there are at least three components of $G - \{s\}$. It follows easily that there exists a claw with center vertex $s$ in $G$, a contradiction.

This completes the proof of Case 1.

Case 2. $|S| \geq 2$. Let $|S| = k \geq 2$ and $S = \{s_1, s_2, \ldots, s_k\}$. By the definition of $\varepsilon(S)$, we have $\varepsilon(S) \leq 2$. It follows easily that

$$\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1.$$  

Figure 3. A connected claw-free graph of minimum degree 2 that does not contain any $P_{\geq 3}$-factor covering $e = x_2x_3$. 
Let \( \text{sun}(G - S) = m \geq 2k - 1 \) and \( \{C_1, C_2, \ldots, C_m\} \) be the set of sun components of \( G - S \). For any sun component \( C_i \) of \( G - S \), let \( L(C_i) \subseteq V(C_i) \) be the set of vertices with exactly one neighbour vertex in \( C_i \), and \( L(C_i) = V(C_i) \) if \( C_i \cong K_1 \), where \( 1 \leq i \leq m \). Let \( E(S, V(C_i) \setminus L(C_i)) \) be the set of edges in graph \( G \) between vertex \( a \) and \( b \) for any \( a \in S, b \in V(C_i) \setminus L(C_i) \) for \( 1 \leq i \leq m \). Then we construct a bipartite multigraph \( H \) from \( G \) by deleting all edges of \( E(G[S]) \cup \left( \bigcup_{i=1}^{m} E(S, V(C_i) \setminus L(C_i)) \right) \) and all vertices of \( V(G) \setminus S \cup \left( \bigcup_{i=1}^{m} V(C_i) \right) \) and contracting each \( C_i \) to a vertex \( c_i \) for \( 1 \leq i \leq m \).

**Claim 1.** For any vertex \( u, v \in V(H) \), there are at most two edges between \( u \) and \( v \) in \( H \).

**Proof.** Without loss of generality, we assume \( u = s_1 \) and \( v = c_1 \). Suppose there are three edges between \( u \) and \( v \) in \( H \). Then there are three vertices in \( L(C_1) \) corresponding to the vertex \( c_1 \), denoted by \( \{c_1^1, c_1^2, c_1^3\} \). By the definition of big sun, \( c_1^i c_1^j \notin E(G) \) for any \( 1 \leq i < j \leq 3 \), which implies a claw with center vertex \( u \) in \( G \). This is a contradiction. \( \Box \)

Since the minimum degree of \( G \) is at least three, it is clear that \( d_H(c_i) \geq 3 \) for \( 1 \leq i \leq m \). Trivially,

\[
|E(H)| \geq 3m \geq 3(2k - 1) = 6k - 3.
\]

By pigeonhole principle and

\[
\frac{|E(H)|}{|S|} \geq \frac{3m}{k} \geq \frac{6k - 3}{k} = 6 - \frac{3}{k} > 4,
\]

there must exist a vertex \( s_i \in S \) incident with at least five edges in \( E(H) \). According to Claim 1 and pigeonhole principle, there exists at least three vertices, denoted by \( \{c_1, c_2, c_3\} \), adjacent to \( s_i \). Since \( \{s_i, c_1, c_2, c_3\} \) induces a claw in \( H \), it follows easily that there exists a claw with center vertex \( s_i \) in \( G \), a contradiction. This completes the proof of Case 2.

Combining Case 1 and Case 2, Theorem 9 is proved. \( \blacksquare \)

Next, we investigate the relationship between planar graphs and \( P_{\geq 3} \)-factor covered graphs, and obtain a connectivity condition for a planar graph to be a \( P_{\geq 3} \)-factor covered graphs as following. The example in Figure 4 shows the sharpness of the results in Theorem 11 in the sense that there exists a 2-connected planar graph, which is not a \( P_{\geq 3} \)-factor covered graph.
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Figure 4. A 2-connected planar graph that does not contain any $P_{\geq 3}$-factor covering $e = x_3x_4$.

Lemma 10 [6]. Let $G$ be a connected planar graph with at least three vertices. If $G$ does not contain triangles, then $|E(G)| \leq 2|G| - 4$.

Theorem 11. Let $G$ be a 3-connected planar graph. Then $G$ is a $P_{\geq 3}$-factor covered graph.

Proof. Suppose $G$ is not a $P_{\geq 3}$-factor covered graph. By Theorem 4, there exists a subset $S \subseteq V(G)$ such that $\text{sun}(G - S) > 2|S| - \varepsilon(S)$. According to the integrality of $\text{sun}(G - S)$, we obtain that $\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1$. We distinguish three cases below to show that $G$ is a $P_{\geq 3}$-factor covered graph.

Case 1. $|S| = 0$. In this case, by the definition of $\varepsilon(S)$, we have $S = \emptyset$ and $\varepsilon(S) = 0$. Since $G$ is a connected graph, $\text{sun}(G) \leq \omega(G) = 1$. On the other hand, we obtain that

$$\text{sun}(G) = \text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$ 

It follows easily that $\text{sun}(G) = 1$, which is to say $G$ is a big sun. By the definition of sun, it contradicts the fact that $G$ is 3-connected. This completes the proof of Case 1.

Case 2. $|S| = 1$. In this case, we obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. It follows immediately that

$$\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2.$$ 

Let $S = \{x\} \subseteq V(G)$. Since $\omega(G - S) \geq \text{sun}(G - S) \geq 2$, $x$ is a cut-vertex of $G$, which contradicts the fact that $G$ is 3-connected. This completes the proof of Case 2.

Case 3. $|S| \geq 2$. In this case, we obtain $\varepsilon(S) \leq 2$ by the definition of $\varepsilon(S)$. It follows immediately that

$$\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1.$$
Set $|S| = s$. We denote by $\text{Sun}(G - S)$ the set of sun components in $G - S$. Since $\text{sun}(G - S) \geq 2|S| - 1$, let $C_1, C_2, \ldots, C_{2s-1}$ be $2s - 1$ distinct sun components where $C_i \in \text{Sun}(G - S)$ for any $1 \leq i \leq 2s - 1$. Then we construct a bipartite graph $H$ from $G$ by contracting each $C_i$ to a vertex $c_i$ for $1 \leq i \leq 2s - 1$ and deleting all edges of $E(G[S])$ and all vertices of
\[
V(G) \setminus (S \cup \bigcup_{i=1}^{2s-1} V(C_i)).
\]
Since $G$ is 3-connected, it is clear that $d_H(c_i) \geq 3$ for $1 \leq i \leq 2s - 1$. Trivially,
\[
|H| = s + (2s - 1) = 3s - 1 \geq 5,
\]
and
\[
|E(H)| \geq 3 (2s - 1) = 6s - 3.
\]
As $G$ is a 3-connected planar graph, it is easy to see that $H$ is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 10 implies that
\[
6s - 3 \leq |E(H)| \leq 2|H| - 4 = 2(3s - 1) - 4 = 6s - 6,
\]
which is a contradiction. This completes the proof of Case 3.

Combining Cases 1-3, Theorem 11 is proved.

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