ON CONDITIONAL CONNECTIVITY OF THE CARTESIAN PRODUCT OF CYCLES

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Abstract

The conditional $h$-vertex ($h$-edge) connectivity of a connected graph $H$ of minimum degree $k > h$ is the size of a smallest vertex (edge) set $F$ of $H$ such that $H - F$ is a disconnected graph of minimum degree at least $h$. Let $G$ be the Cartesian product of $r$ ≥ 1 cycles, each of length at least four and let $h$ be an integer such that $0 ≤ h ≤ 2r − 2$. In this paper, we determine the conditional $h$-vertex-connectivity and the conditional $h$-edge-connectivity of the graph $G$. We prove that both these connectivities are equal to $(2r - h)a_h^r$, where $a_h^r$ is the number of vertices of a smallest $h$-regular subgraph of $G$.

Keywords: fault tolerance, hypercube, conditional connectivity, cut, Cartesian product.

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1. Introduction

One of the features of a good interconnection network is its high fault tolerance capacity. Interconnection network can be modeled into a graph with the help of which we can study many properties of the network. Connectivity of a modeled graph measures the fault tolerance capacity of the interconnection network. High fault tolerance capacity of the network plays an important role in practice. Traditional connectivities have some limitations to measure the fault tolerance capacity of a network accurately. In order to compute traditional edge connectivity, one allows failure of all the links incident with the same processor, practically which is rare. One can overcome this limitation effectively by considering the conditional connectivity of graphs introduced by Harary [6].

Let $G$ be a connected graph with minimum degree at least $k \geq 1$ and let $h$ be an integer such that $0 \leq h < k$. A set $F$ of vertices (edges) of $G$ such that $G - F$ is disconnected and each component of it has minimum degree at least $h$ is an $h$-vertex (edge) cut of $G$. The conditional $h$-vertex (edge) connectivity of $G$, denoted by $\kappa^h(G)$ ($\lambda^h(G)$), is the minimum cardinality $|F|$ of an $h$-vertex (edge) cut $F$ of $G$. Clearly, $h = 0$ gives the traditional vertex (edge) connectivity.

Many researchers have worked on the problem of determining the conditional connectivities for various classes of graphs and determined these parameters for smaller values of $h$ [4, 5, 7, 9]. Exact values of one or both conditional connectivities are known for some classes of graphs. For the $n$-dimensional hypercube $Q_n$, the conditional connectivities $\lambda^h$ and $\kappa^h$ are same and their common value is $2^h(n - h)$; see [3, 7]. Li and Xu [10] proved that $\lambda^h$ of any $n$-dimensional hypercube-like network $G_n$ is also $2^h(n - h)$. Ye and Liang [16] established that $\kappa^h$ is also $2^h(n - h)$ for some members of hypercube-like networks such as Crossed cubes, Locally twisted cubes, Möbius cubes. Independently, Wei and Hsieh [14] determined $\kappa^h$ for the Locally twisted cubes. Ning [13] obtained $\kappa^h$ for the exchanged crossed cubes. Both $\lambda^h$ and $\kappa^h$ are determined for the class of $(n, k)$-star graphs by Li et al. [11].

An $r$-dimensional torus is the Cartesian product of $r$ cycles. The $k$-ary $r$-cube, denoted by $Q_k^r$, is the Cartesian product of $r$ cycles each of length $k$. In particular, the hypercube $Q_{2r}$ is $Q_2^4$. Hypercubes, $k$-ary $r$-cubes and multidimensional tori are widely used interconnection networks; see [2, 8, 12, 15].

It is easy to see that an $r$-dimensional torus is a $2r$-regular graph with traditional vertex connectivity and edge connectivity $2r$; see [15]. In this paper, we determine the conditional $h$-edge-connectivity as well as the conditional $h$-vertex-connectivity of the given multidimensional torus.

By $C_k$ we mean a cycle of length $k$. For integers $h, r, k_1, k_2, \ldots, k_r$ with $0 \leq h \leq 2r$ and $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$, we define a quantity $a^h_{k_1 \ldots k_r}$ as follows.
Definition 1.1.
\[
a^r_h = \begin{cases} 
2^h & \text{if } 0 \leq h \leq r, \\
2^{r-i} k_1 k_2 \cdots k_i & \text{if } h = r + i, \ 1 \leq i \leq r.
\end{cases}
\]

We prove that both the conditional connectivities \(\lambda^h\) and \(k^h\) are equal to \(a^r_h(2r - h)\) for the Cartesian product of cycles \(C_{k_1}, C_{k_2}, \ldots, C_{k_r}\).

The following is the main theorem of the paper.

Theorem 1.2. Let \(h, r, k_1, k_2, \ldots, k_r\) be integers such that \(0 \leq h \leq 2r - 2\) and \(4 \leq k_1 \leq k_2 \leq \cdots \leq k_r\) and let \(G\) be the Cartesian product of the cycles \(C_{k_1}, C_{k_2}, \ldots, C_{k_r}\). Then \(\lambda^h(G) = k^h(G) = a^r_h(2r - h)\).

Corollary 1.3. Let \(h, r, k\) be integers such that \(0 \leq h \leq 2r - 2\), \(4 \leq k\) and let \(Q^k_r\) be the k-ary r-cube. Then \(\lambda^h(Q^k_r) = k^h(Q^k_r) = a^r_h(2r - h)\), where \(a^r_h = 2^h\) if \(0 \leq h \leq r\) and \(a^r_h = 2^{r-i} k^i\) if \(h = r + i\) and \(1 \leq i \leq r\).

Corollary 1.4 [3, 7]. For integers \(h\) and \(r\) with \(0 \leq h \leq 2r - 2\), \(\lambda^h(Q_{2r}) = k^h(Q_{2r}) = 2^h(2r - h)\).

The proof of our main result, Theorem 1.2 is divided into three sections. In Section 2, we characterize the \(h\)-regular subgraph of the graph \(G\) with minimum number of vertices and explore some of its properties. Using these properties we determine the conditional \(h\)-vertex connectivity and the conditional \(h\)-edge connectivity of \(G\) in Sections 3 and 4, respectively.

2. Smallest \(h\)-Regular Subgraph

In this section, we define a smallest \(h\)-regular subgraph of the Cartesian product of \(r\) cycles and obtain some properties of it. We first introduce some notations.

For a graph \(K\), let \(V(K)\) denote the set of all vertices of \(K\). If \(H\) is a subgraph of \(K\), then \(\delta(K)\) is the minimum degree of \(K\) while \(\delta_K(H)\) is the minimum degree of \(H\) in \(K\). The Cartesian product of two graphs \(H\) and \(K\) is a graph \(H \square K\) with vertex set \(V(H) \times V(K)\). Two vertices \((x, y)\) and \((u, v)\) are adjacent in \(H \square K\) if and only if either \(x = u\) and \(y\) is adjacent to \(v\) in \(K\), or \(y = v\) and \(x\) is adjacent to \(u\) in \(H\). The hypercube \(Q_n\) is the Cartesian product of \(n\) copies of the complete graph \(K_2\).

We use the following notations about the structure of the multidimensional torus.

Notation.

Consider the graph \(G\) of Theorem 1.2. We have \(G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}\), where \(C_{k_i}\) is a cycle of length \(k_i\) for \(i = 1, 2, \ldots, r\) and \(4 \leq k_1 \leq k_2 \leq \cdots \leq k_r\). We can
write $G$ as $G = H \Box C_{k_r}$, where $H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_{r-1}}$. Label by $1, 2, \ldots, k_r$ the vertices of the cycle $C_{k_r}$ so that $i$ is adjacent to $(i + 1) \pmod{k_r}$. Hence $G$ can be obtained by replacing $i^{th}$ vertex of $C_{k_r}$ by a copy $H^i$ of $H$ and replacing edge joining $i$ and $i + 1$ of $C_{k_r}$ by the perfect matching $M_i$ between the corresponding vertices of $H^i$ and $H^{i+1}$. Thus $G = H^1 \cup H^2 \cup \cdots \cup H^{k_r} \cup (M_1 \cup M_2 \cup \cdots \cup M_{k_r})$; see Figure 1.

![Figure 1. $G = H \Box C_{k_r}$.](image)

Henceforth, by $G$ we mean the graph $C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$ with $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$, that is, the graph of Theorem 1.2.

From the following lemma, it is clear that $G$ is a $2r$-regular and $2r$-connected graph on $k_1 k_2 \cdots k_r$ vertices.

**Lemma 2.1.** If $G_i$ is an $m_i$-regular and $m_i$-connected graph on $n_i$ vertices for $i = 1, 2$, then $G_1 \Box G_2$ is an $(m_1 + m_2)$-regular and $(m_1 + m_2)$-connected graph on $n_1 n_2$ vertices.

We now define an $h$-regular subgraph, denoted by $W^r_h$, of the graph $G$.

**Definition 2.2.** For $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$ and $0 \leq h \leq 2r$, let

$$W^r_h = \begin{cases} Q_h & \text{if } 0 \leq h \leq r, \\ Q_{r-i} \Box C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_i} & \text{if } h = r + i \text{ and } 1 \leq i \leq r. \end{cases}$$

In the following figure, a 2-regular subgraph $W^2_2$ and a 3-regular subgraph $W^2_3$ of the graph $C_5 \Box C_5$ are shown by bold lines.

It is known that a smallest $h$-regular subgraph of the hypercube $Q_n$ is isomorphic to $Q_h$ (see [1]). We prove the analogous result for the Cartesian product of cycles. In fact, we establish that $W^r_h$ is a smallest $h$-regular subgraph of the above graph $G$. 
Lemma 2.5. Every subgraph of the graph $G$ of minimum degree at least $h$ has at least $a_h$ vertices.

Figure 2. The subgraph $W_2^2$ and $W_3^2$ of $C_5 \square C_5$.

The following lemma follows from Lemma 2.1, Definition 1.1 of the number $a_h^r$ and the fact that the hypercube $Q_h$ is an $n$-regular, $n$-connected graph on $2^n$ vertices for any integer $n \geq 0$.

Lemma 2.3. The graph $W_h^r$ is $h$-regular and $h$-connected with $a_h^r$ vertices.

We need the following lemma that gives relations between different values of $a_h^r$.

Lemma 2.4. Let $r \geq 2$ and let $a_h^r$ be the quantity given in Definition 1.1. Then the following statements hold.

1. $a_h^r = 2a_h^{r-1}$ if $1 \leq h \leq 2r - 1$;
2. $kr a_h^{r-1} \geq a_h^r$ if $2 \leq h \leq 2r$;
3. $a_h^{r-1} \geq a_h^r$ if $0 \leq h \leq 2r - 2$.

Proof. Recall that $a_h^r = 2^h$ if $0 \leq h \leq r$ and $a_h^r = 2^{r-i}k_1k_2\cdots k_i$ if $h = r + i$ with $1 \leq i \leq r$, where $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$.

(1) If $1 \leq h \leq r$, then $a_h^r = 2^h = 2(2^{h-1}) = 2a_h^{r-1}$. For $r + 1 \leq h \leq 2r - 1$, we have $h = r + i$ for some $1 \leq i \leq r - 1$. Hence $h - 1 = (r - 1) + i$ gives $a_h^{r-1} = 2(r-1-i)k_1k_2\cdots k_i$. Therefore $2a_h^{r-1} = a_h^r$.

(2) Suppose $2 \leq h \leq r + 1$. Then $a_h^{r-1} = 2^{h-2}$, and $a_h^r = 2^h$ if $h < r + 1$ and $a_h^r = 2^{r-1}k_1$ if $h = r + 1$. For $r + 2 \leq h \leq 2r$, we have $h - 2 = (r - 1) + (i - 1)$ for some $2 \leq i \leq r$ and so, $a_h^{r-1} = 2^{r-i}k_1k_2\cdots k_{i-1}$. Therefore, $kr a_h^{r-1} \geq a_h^r$ in each case as $k_r \geq k_i \geq k_1 \geq 4$.

(3) Note that $a_h^{r-1} = 2^h$ for $1 \leq h \leq r - 1$, and $a_h^{r-1} = 2^{(r-2)}k_1$ for $h = r = (r - 1) + 1$, and finally, $a_h^{r-1} = 2^{r-i-2}k_1k_2\cdots k_{i+1}$ for $h = (r - 1) + (i + 1)$ for $1 \leq i \leq r$. Since $k_{i+1} \geq k_i \geq 4$, we have $a_h^{r-1} \geq a_h^r$ in all the three cases. ■

Lemma 2.5. Every subgraph of the graph $G$ of minimum degree at least $h$ has at least $a_h^r$ vertices.
Proof. The graph $G$ is the product of $r$ cycles. We prove the result by induction on $r$. The result holds obviously for $h = 0$ and $h = 1$ and so it holds for $r = 1$. Suppose $r \geq 2$ and $h \geq 2$. Assume that the result holds for the product of $r - 1$ cycles. We have $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$, where $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$. Write $G$ as $G = H \Box C_{k_r}$, where $H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_{r-1}}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^{k_r} \cup (M_1 \cup M_2 \cup \cdots \cup M_{k_r})$, where $H^i$ is the copy of $H$ corresponding to vertex $i$ of $C_{k_r}$ and $M_i$ is the perfect matching between the corresponding vertices of $H^i$ and $H^{i+1}$.

Let $K$ be a subgraph of $G$ with $\delta(K) \geq h$. We prove that $|V(K)| \geq a_{h}^r$. Clearly, $K$ intersects at least one $H^i$. Let $K^i = K \cap H^i$ for $i = 1, 2, \ldots, k_r$. We have the following three cases.

1. Suppose only one $K^i$ is non-empty. Due to symmetry in $G$, we may assume $K^1$ is non-empty and $K^j$ is empty for every $j \neq 1$. Therefore $K$ is a subgraph of $H^1$ and it has minimum degree at least $h$ in $H^1$. Since $H^1$ is $(2(r - 1))$-regular, $h \leq 2r - 2$. Suppose $h = 2r - 2$. Then $K = H^1$ and so, $|V(K)| = k_1 k_2 \cdots k_{r-1}$. If $r = 2$, then $|V(K)| = k_1 \geq 4 = a_2^2 = a_h^r$. If $r \geq 3$, then $|V(K)| \geq 4 k_1 k_2 \cdots k_{r-2} = a_h^r$ as $k_{r-1} \geq 4$. If $h < 2r - 2$, then, by induction and Lemma 2.4(3), we have $|V(K)| \geq a_{h-1}^{r-1} \geq a_h^r$.

2. Suppose $K^i$ is non-empty for all $i$. Note that in the graph $G$, every vertex of $H^i$ has exactly one neighbour in $H^{i-1}$ and one in $H^{i+1}$. Hence the minimum degree of $K^i$ is at least $h - 2$. By induction, $|V(K^i)| \geq a_{h-2}^{r-1}$. Therefore, by Lemma 2.4(2),

$$|V(K)| = |V(K^1)| + |V(K^2)| + \cdots + |V(K^{k_r})| \geq k_r a_{h-2}^{r-1} \geq a_h^r.$$

3. Suppose at least two $K^i$ are non-empty and at least one $K^i$ is empty. Hence, we may assume that $K^1 \neq \emptyset$ but $K^{k_r} = \emptyset$. Further, we get an integer $1 < t < k_r$ such that $K^t \neq \emptyset$ but $K^{t+1} = \emptyset$. Then $\delta(K^j) \geq h - 1$ and so, by induction, $|V(K^j)| \geq a_{h-2}^{r-1}$ for $j = 1, t$. Now, by Lemma 2.4(1),

$$|V(K)| \geq |V(K^1)| + |V(K^t)| \geq 2a_{h-2}^{r-1} = a_h^r.$$

This completes the proof.

The following result is an immediate consequence of Lemmas 2.3 and 2.5.

Corollary 2.6. $W^r_h$ is a smallest subgraph of the graph $G$ of minimum degree at least $h$.

We obtain some more properties of the subgraph $W^r_h$ of $G$ to obtain an upper bound on the conditional connectivity of the graph $G$.

First, we introduce some notations. Let $K$ be a graph and let $Y$ be a subgraph of $K$. A neighbour of $Y$ in $K$ is a vertex in $V(K) \setminus V(Y)$ that is adjacent to
a vertex of $Y$. Let $N(Y)$ denote the set of all neighbours of $Y$ in $K$ and let $N[Y] = N(Y) \cup V(Y)$. Also, for a subgraph $H$ of $K$, let $N_H(Y)$ be the set of all neighbours of $Y$ that are present in $H$ and let $N_H[Y] = N_H(Y) \cup V(Y)$.

The following result is analogous to the result of hypercubes which states that if $K$ is a subgraph of the hypercube $Q_n$ isomorphic to $Q_h$, then every vertex of $Q_n$ which is not in $K$ has at most one neighbour in $K$; see [1].

**Lemma 2.7.** If $0 \leq h < 2r - 1$ and $K$ is a subgraph of $G$ isomorphic to the graph $W^r_h$, then every vertex of $G$ belonging to $V(G) \setminus V(K)$ has at most one neighbour in the subgraph $K$.

**Proof.** We argue by induction on $r$. If $r = 1$, then $G$ is just a cycle and so the result holds obviously. Suppose $r \geq 2$. Assume that the result holds for the product of any $r - 1$ cycles. We have $G = H \Box C_{k_r}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^{k_r} \cup (M_1 \cup M_2 \cup \cdots \cup M_{k_r})$, where $H^1$ is the copy of $H$ corresponding to vertex $i$ of $C_{k_r}$ and $M_i$ is the perfect matching between the corresponding vertices of $H^i$ and $H^{i+1}$. Since the graph $W^r_h$ is isomorphic to $W^r_{k_r-1}\Box K_2$, we may assume that $W^r_h$ is a subgraph of $H \Box K_2$ by considering $W^r_{h-1}$ as a subgraph of $H$. Hence, we may assume that $K$ is a subgraph of $H^2 \cup H^3 \cup M_2$, where $M_2$ is the perfect matching between $H^2$ and $H^3$.

Let $K^i = K \cap H^i$ for $i = 2, 3$. Then $K^i$ is isomorphic to $W^r_{k_r-1}$. Let $x$ be any vertex of $V(G) \setminus V(K)$. If $x$ is in $V(H^2)$, then, by induction, $x$ has at most one neighbour in $K^2$. Then $x$ has no neighbour in $K^3$ and so, it has at most one neighbour in $K$. Similarly, $x$ has at most one neighbour in $K$ if it belongs to $V(H^3)$. Suppose $x$ is in $H^j$ for some $j \notin \{2, 3\}$. Then $x$ has exactly one neighbour in $H^j+1$ and one in $H^{j-1}$ each and no neighbour in $H^i$ for any $i \notin \{j-1, j+1\}$. This shows that $x$ has at most one neighbour in $H^2 \cup H^3$ and hence in $K$ as $k_r \geq 4$. This completes the proof. 

**Lemma 2.8.** If $0 \leq h \leq 2r - 1$ and $Y = W^r_h$, then any vertex of $G$ which is not in $N[Y]$ has at most two neighbours in $N[Y]$.

**Proof.** We proceed by induction on $r$. The result holds trivially for $r = 1$ as $G$ is just a cycle in this case. Suppose $r \geq 2$. Assume that the result holds for the product of any $r - 1$ cycles. Write $G$ as $H \Box C_{k_r}$, where $H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_{r-1}}$. Since the graph $W^r_h$ is isomorphic to $W^r_{k_r-1}\Box K_2$, we may assume that $Y = W^r_h$ is a subgraph of $H^2 \cup H^3 \cup M_2$. Then $Y$ has neighbours in $H^1$ and $H^4$. Let $Y_i = W^r_h \cap H^i$ for $i = 2, 3$. Let $S_1 = N_{H^1}(Y_2)$, $S_2 = N_{H^2}(Y_2)$, $S_3 = N_{H^3}(Y_3)$ and $S_4 = N_{H^4}(Y_3)$. Then $N[Y] = S_1 \cup S_2 \cup S_3 \cup S_4$.

Let $x \in V(G) \setminus N[Y]$. Then $x$ is a vertex of $H^j$ for some $j$. If $j > 4$, then $x$ has at most two neighbours in the set $V(H^1) \cup V(H^2) \cup V(H^3) \cup V(H^4)$ and so in its subset $N[Y]$. Suppose $j \in \{1, 2, 3, 4\}$. Then $h \leq 2r - 2$ as for $h = 2r - 1$, we have $Y = H^2 \cup H^3 \cup M_2$ and so, $N[Y] = V(H^1) \cup V(H^2) \cup V(H^3) \cup V(H^4)$.
The subgraph of $G$ induced by the set $S_i$ is isomorphic to the graph $W_{r-1}^{r-1}$ for $i = 1, 4$. If $j \in \{1, 4\}$, then $x$ has at most one neighbour in $S_1 \cup S_4$ and at most one in $V(H^2) \cup V(H^3)$ by Lemma 2.7. If $j = 2$, then, by induction, $x$ has at most two neighbours in $S_2$ and no neighbour in $S_1 \cup S_3 \cup S_4$. Similarly, if $j = 3$, then $x$ has at most two neighbours in $S_2$ and no neighbour in $S_1 \cup S_2 \cup S_4$. Thus, in any case, $x$ has at most two neighbours in $N[Y]$.

**Lemma 2.9.** For $0 \leq h \leq 2r - 1$, the inequality $(2r - h + 1)a_h^r \leq k_1k_2 \cdots k_r$ holds. Moreover, the inequality is strict if $h < 2r - 1$.

**Proof.** Recall that $4$ holds. Moreover, the inequality is strict if $h < 2r - 1$.

Recall from Section 2 the graph $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}$ and its $r$-regular subgraph $W_h^r$ with $a_h^r$ vertices, where $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$. In this section, we prove that the conditional vertex connectivity $\kappa^h(G)$ the graph $G$ is $(2r - h)a_h^r$. Using Lemmas 2.7, 2.8 and 2.9, it easily follows that $\kappa^h(G) \leq (2r - h)a_h^r$.

**Lemma 3.1.** If $0 \leq h \leq 2r - 2$, then $\kappa^h(G) \leq (2r - h)a_h^r$.

**Proof.** We have $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}$. We simply denote the subgraph $W_h^r$ of $G$ by $Y$. Then $|V(Y)| = a_h^r$. Since $G$ is $2r$-regular and $Y$ is $h$-regular, every vertex of $Y$ has $2r - h$ neighbours in the $G - V(Y)$. By Lemma 2.7, $|N(Y)| = (2r - h)|V(Y)| = (2r - h)a_h^r$. This gives $|N[Y]| = |V(Y) \cup N(Y)| = |V(Y)| + |N(Y)| = (2r - h + 1)a_h^r$. Therefore, by Lemma 2.9, $|N[Y]| < k_1k_2 \cdots k_r = |V(G)|$. Hence $V(G) \setminus N[Y]$ is a non-empty set and by Lemma 2.8, every member of this set has at most two neighbours in $N[Y]$. Consequently, the minimum degree of the subgraph of $G$ induced by this set is at least $2r - 2 \geq h$. Already, the minimum degree of the graph $Y$ is $h$. Hence the graph $G - N(Y)$ is disconnected and every component of it has minimum degree at least $h$. Thus $N(Y)$ is an $h$-vertex cut of $G$. Therefore $\kappa^h(G) \leq |N(Y)| = (2r - h)a_h^r$.

To prove the reverse inequality for $\kappa^h(G)$, we obtain the following lemma.

**Lemma 3.2.** If $0 \leq h \leq 2r - 1$ and $Y$ is a subgraph of the graph $G$ with minimum degree at least $h$, then $|N[Y]| \geq a_h^r(2r - h + 1)$. 

3. Conditional Vertex Connectivity

Recall from Section 2 the graph $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}$ and its $r$-regular subgraph $W_h^r$ with $a_h^r$ vertices, where $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$. In this section, we prove that the conditional vertex connectivity $\kappa^h(G)$ the graph $G$ is $(2r - h)a_h^r$. Using Lemmas 2.7, 2.8 and 2.9, it easily follows that $\kappa^h(G) \leq (2r - h)a_h^r$.
Proof. If $N[Y] = V(G)$, then the result follows obviously from Lemma 2.9. Suppose $N[Y] \neq V(G)$. We prove the result by induction on $r$. Since $G$ is $2r$-regular, all $2r$ neighbours of any vertex of $Y$ belong to the set $N[Y]$. Hence $|N[Y]| \geq 2r + 1$. Therefore the result holds for $h = 0$. Also, the result trivially follows for $r = 1$ and $h = 1$ as in this case $G$ is a cycle of length $k_1 \geq 4$, $Y$ is a path on at least two vertices and $a_1^r = 2$.

Suppose $r \geq 2$ and $h \geq 1$. Assume that the result holds for a graph that is the product of $r - 1$ cycles. Let $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$. Then $G = H \Box H_{k_r}$, where $H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_{r - 1}}$. Then $G$ contains $k_r$ vertex-disjoint copies $H^1, H^2, \ldots, H^{k_r}$ of $H$. Then every vertex of $H^i$ has one neighbour in $H^{i - 1}$ and $H^{i + 1}$, where the addition and subtraction in the superscript is carried out modulo $k_r$. Let $Y$ be a subgraph of $G$ with $\delta(Y) \geq h$ and $N[Y] \neq V(G)$. Then $Y$ intersects at least one copy of $H^i$. Let $Y_i = Y \cap H^i$ for $i = 1, 2, \ldots, k_r$.

Case 1. $Y_i \neq \emptyset$ for only one value of $i$. Without loss of generality we may assume that only $Y_1$ is non-empty. Then $Y = Y_1$ is contained in the graph $H^1$. Since $H^1$ is $(2r - 2)$-regular, $h \leq 2r - 2$. Also, the minimum degree of $Y$ in $H^1$ is at least $h$. Hence, by Lemma 2.5, $Y$ has at least $a_{h - 1}^r$ vertices. We have $N[Y] = N_{H^1}[Y] \cup N_{H^2} \cup N_{H^2}(Y).$ If $h = 2r - 2$, then $H = H^1 + V(H^2) \cup V(H^{k_r}).$ Therefore

$$|N[Y]| \geq 3|V(H^1)| = 3k_1k_2 \cdots k_{r - 1} \geq 12k_1k_2 \cdots k_{r - 2} = (2r - h + 1)a_h^r.$$ 

Suppose $0 \leq h \leq 2r - 3 = (2r - 1) - 1$. Then, by induction, $|N_{H^1}[Y]| \geq a_{h - 1}^r(2r - h - 1)$. As $|N_{H^2}[Y]| = |N_{H^2}(Y)| = |V(Y)| \geq a_{h - 1}^r$, by Lemma 2.4(3) we have

$$|N[Y]| \geq a_{h - 1}^r(2r - h - 1) + 2a_{h - 1}^r = (2r - h + 1)a_{h - 1}^r \geq (2r - h + 1)a_h^r.$$ 

Case 2. $Y_i \neq \emptyset$ for all $i = 1, 2, \ldots, k_r$. In this case, $N[Y] \supseteq N_{H^1}[Y_1] \cup N_{H^2}[Y_2] \cup \cdots \cup N_{H^{k_r}}[Y_{k_r}].$ If $h = 1$, then $\delta_{H^1}(Y_1) \geq 0$ and so, by induction, $|N_{H^1}[Y_1]| \geq a_0^r(2r - 1) - 1 = 2r - 1$ implying

$$|N[Y]| \geq |N_{H^1}[Y_1]| + |N_{H^2}[Y_2]| + \cdots + |N_{H^{k_r}}[Y_{k_r}]| \geq k_r(2r - 1) \geq 8r - 4 \geq 4r \geq a_1^r(2r) = a_h^r(2r - h + 1).$$

Suppose $h \geq 2$. Then $\delta_{H^1}(Y_1) \geq 2 - 2 = 0$ and so, by induction, $|N_{H^1}[Y_1]| \geq a_{h - 1}^r(2r - h + 1)$ for all $i$. Therefore, by Lemma 2.4(2),

$$|N[Y]| \geq |N_{H^1}[Y_1]| + |N_{H^2}[Y_2]| + \cdots + |N_{H^{k_r}}[Y_{k_r}]| \geq k_r a_{h - 1}^r(2r - h + 1) \geq a_h^r(2r - h + 1).$$

Case 3. $Y_i \neq \emptyset$ for more than one but not all values of $i$. Without loss of generality, we may assume that $Y_1$ is non-empty but $Y_{k_r}$ is empty. Let $t$ be
the largest integer such that $Y_t$ is non-empty. Then $1 < t < k_r$; see Figure 3.
Suppose that $h = 2r - 1$. Then $Y_1 = H^1$ and $Y_t = H^t$. Hence $N[Y] \supseteq V(H^1) \cup V(H^2) \cup V(H^t) \cup V(H^{t+1}) \cup V(H^{k_r})$. Since $k_r \geq 4$, $t \neq 2$ or $t + 1 \neq k_r$ and $|V(H^1)| = |V(H^t)|$ for all $i > 1$. By Lemma 2.9,

$$|N[Y]| \geq 4|V(H^1)| = 4|V(H)| \geq 4k_1k_2 \cdots k_{r-1} = (2r - h + 1)a_h^r.$$ 

Figure 3. The graph $G$ with $Y_j = \emptyset$ for $t < j \leq k_r$.

Suppose that $0 \leq h \leq 2r - 2$. The graph $Y_t$ has $|V(Y_t)|$ neighbours in $H^{i-1}$ and $H^{i+1}$ for $i = 1, t$. Therefore $|N[Y]| \geq |N_{H^i}[Y_1]| + |N_{H^i}[Y_t]| + |V(Y_1)| + |V(Y_t)|$. If $i \in \{1, t\}$, then $\delta_{H^i}(Y_1) \geq h - 1$ and so, by induction, $|N_{H^i}[Y_1]| \geq a_h^{r-1}(2r - h)$. Also, by Lemma 2.5, $|V(Y_t)| \geq a_h^{r-1}$. Hence, by Lemma 2.4(1), we have

$$|N[Y]| \geq 2a_h^{r-1}(2r - h) + a_h^{r-1} = a_h^r(2r - h) + a_h = a_h^r(2r - h + 1).$$

Thus $|N[Y]| \geq a_h^r(2r - h + 1)$ in each case. This completes the proof. 

**Proposition 3.3.** If $0 \leq h \leq 2r - 2$ and $S$ is an $h$-vertex cut of the graph $G$, then $|S| \geq a_h^r(2r - h)$. 

**Proof.** We argue by induction on $r$. Suppose $h = 0$. Then $S$ is a traditional vertex cut of $G$. Therefore $|S| \geq 2r = a_0^r(2r - 0)$ as $G$ is $2r$-connected by Lemma 2.1. Hence the result holds for $h = 0$ and so for $r = 1$. Suppose $r \geq 2$ and $h \geq 1$. Assume that the result is true for the Cartesian product of $r - 1$ cycles, each of length at least 4. Let $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}$. Then $G = H \square C_{k_r}$, where $H = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_{r-1}}$. Then $G$ is obtained by replacing $i^{th}$ vertex of $C_{k_i}$ by the copy $H^i$ of $H$ and replacing each edge $C_{k_i}$ by the matching between the two copies of $H^i$ corresponding to the end vertices of that edge.
As $S$ is an $h$-vertex cut of $G$, the graph $G - S$ is disconnected and each component of it has minimum degree $h$. Let $Y$ be a subgraph of $G - S$ consisting of at least one but not all components of $G - S$ and let $Z$ be the subgraph consisting of the remaining components. Thus $G - S = Y \cup Z$ and further, $\delta(Y) \geq h$ and $\delta(Z) \geq h$. As $S$ is a cut, $N(Y) \subseteq S$ and $N(Z) \subseteq S$ and so, $|S| \geq |N(Y)|$ and $|S| \geq |N(Z)|$. Note that $Y$ and $Z$ each intersects $H^t$ for at least one $i$. Let $S_i = S \cap V(H^t)$, $Y_i = Y \cap V(H^t)$ and $Z_i = Z \cap V(H^t)$. Depending upon the nature of $Y$ and $Z$, the proof is divided into several cases.

**Case 1.** Suppose $Y_i \neq \emptyset$ for only one $i$. Without loss of generality, we may assume that only $Y_1$ is non-empty. Then $Y = Y_1$ is contained in $H^1$. Therefore $\delta_{H^1}(Y) \geq h - 1$. As $H^1$ is $(2r - 2)$-regular, $0 \leq h \leq 2r - 2$. If $h = 2r - 2$, then $Y = H^1$, $N(Y) = V(H_k^r) \cup V(H^2)$ and therefore,

$$|S| \geq |N(Y)| = |V(H_k^r)| + |V(H^2)| = 2k_1k_2 \cdots k_{r-2}k_{r-1}$$

$$\geq 8k_1k_2 \cdots k_{r-2} = a_h^r(2r - h).$$

Suppose $0 \leq h \leq 2r - 3 = 2(r - 1) - 1$. The graph $Y$ has $|V(Y)|$ neighbours in each of $H_k^r$ and $H^2$. Therefore $|N(Y)| = |N_{H^1}(Y)| + |V(Y)| + |V(Y)| = |N_{H^1}(Y)| + |V(Y)|$. By Lemmas 2.4(3), 2.5 and 3.2,

$$|S| \geq |N(Y)| \geq a_h^{r-1}(2r - h - 1) + a_h^{r-1} = a_h^{r-1}(2r - h) \geq a_h^r(2r - h).$$

**Case 2.** Suppose $Y_i \neq \emptyset$ for more than one but not all values of $i$. Without loss of generality, we may assume that $Y_1$ is non-empty but $Y_k$ is empty. Suppose there is an integer $t$ with $1 < t < k_r$ such that $Y_t$ is non-empty. Note that $\delta_{H^1}(Y_t) \geq h - 1$. Further, the set $S_{k_t}$ contains all $|V(Y_t)|$ neighbours of $Y_t$ present in $H_k^r$ and $S_1$ contains the set $N_{H^1}(Y_1)$ of neighbours of $Y_1$ in $H^1$. Therefore, by Lemma 3.2,

$$|S_1 \cup S_{k_t}| \geq |N_{H^1}(Y_1)| + |V(Y_1)| = |N_{H^1}(Y_1)| \geq (2r - h)a_h^{r-1}. $$

Suppose $Y_i$ is empty for more than one values of $i$. Suppose $Y_{k_t-1}$ is empty. Then we can choose $t$ so that $Y_{t+1}$ is empty. Then $\delta_{H^1}(Y_t) \geq h - 1$. The set $S$ contains $|N_{H^1}(Y_1)|$ neighbours of $Y_t$ present in $H^t$ and the $|V(Y_t)|$ neighbours of $Y_t$ that are present in $H^{t+1}$. Thus, by Lemmas 2.4(1) and 3.2,

$$|S| \geq |S_1 \cup S_{k_t}| + |N_{H^1}(Y_t)| + |V(Y_t)| \geq (2r - h)a_h^{r-1} + |N_{H^1}(Y_t)|$$

$$\geq 2(2r - h)a_h^{r-1} \geq a_h^r(2r - h).$$

Similarly, if $Y_{k_t-1}$ is non-empty, then we can choose $t$ so that $Y_{t-1}$ is empty and so, in this case $S$ contains $N_{H^1}(Y_t)$ and $N_{H^1-1}(Y_t)$ implying $|S| \geq a_h^r(2r - h)$.

Suppose $Y_i$ is non-empty for all $1 \leq i \leq k_r - 1$. Here we calculate $|S_i|$ by using Lemma 3.2 or induction. To use induction, we need to consider the nature
of the graph $Z$ also. If $Z_i \neq \emptyset$ for only one value of $i$, then result follows from
Case 1. Suppose $Z_i \neq \emptyset$ for more than one values of $i$. If $Z_i = \emptyset$ for at least two
values of $i$, then the result follows from the above paragraph by replacing $Y$ with
$Z$. It remains to consider the two subcases depending on whether $Z_i$ is empty for
exactly one value of $i$ or no value of $i$.

Subcase 1. $Z_i = \emptyset$ for exactly one value of $i$. We have two subcases depending
on $i = k_r$ or $i < k_r$.

(i) Suppose $Z_{k_r}$ is empty. Then $Z_j$ is non-empty like $Y_j$ for $1 \leq j < k_r$;
Figure 4(a). Suppose $h = 1$. Then $\delta_{H,1}(Y_i) \geq 0$ and $\delta_{H,1}(Z_i) \geq 0$ for all $i$. Hence
$|S_1 \cup S_{k_r}| \geq (2r-1)a_0^{r-1} = (2r-1)$ and by induction, $|S_i| \geq (2(r-1)-0)a_0^{r-1} = 2r-2$ for $i \in \{2,3,\ldots,k_r-1\}$. Therefore, as $S = S_1 \cup S_2 \cup \cdots \cup S_{k_r}$, we have

$$|S| = (|S_1 \cup S_{k_r}|) + \sum_{i=2}^{k_r-1} |S_i| \geq (2r-1) + \sum_{i=2}^{k_r-1} (2r-2)$$

$$= (2r-1) + (2r-2)(k_r-2) \geq 2(2r-1) = (2r-h)a_h^r.$$  

Suppose $h \geq 2$. Since $Y_{k_r}$ and $Z_{k_r}$ are empty, $\delta_{H^{k_r-1}}(Y_{k_r-1}) \geq h-1 > h-2$ and $\delta_{H^{k_r-1}}(Z_{k_r-1}) \geq h-1 > h-2$. Thus $S_{k_r-1}$ is an $(h-2)$-cut in $H^{k_r-1}$. For $i \in \{2,3,\ldots,k_r-2\}$, as both $Y_{i}$ and $Z_{i}$ are non-empty subgraphs of $H^{i}$ of
minimum degree at least $h-2$, $S_{i}$ is an $(h-2)$-cut in $H^{i}$. Hence, by induction,
$|S_i| \geq (2r-h)a_h^{r-1}$ for $i \in \{2,3,\ldots,k_r-1\}$. Therefore

$$|S| = (|S_1 \cup S_{k_r}|) + \sum_{i=2}^{k_r-1} |S_i| \geq (2r-h)a_h^{r-1} + \sum_{i=2}^{k_r-1} (2r-h)a_h^{r-1}$$

$$= (2r-h)a_h^{r-1} + (k_r-2)(2r-h)a_h^{r-1}$$

$\geq (2r-h)a_h^{r-1} + \frac{k_r}{2}a_h^{r-1}(2r-h) \geq \frac{1}{2}a_h^{r-1}(2r-h)$ 

$\geq (2r-h)a_h^{r-1} + a_h^{r-1}(2r-h) \geq 2a_h^{r-1}(2r-h)$

$= a_h^r(2r-h)$. (by Lemma 2.1)

(ii) Suppose $Z_{k_r}$ is non-empty. Then $Z_l$ is empty for some $l$ with $1 \leq l < k_r$
and $Z_l$ is non-empty for every $j \neq l$; see Figure 4(b). Then the minimum degree
of $Z_{l+1}$ is at least $h-1$ in $H^{l+1}$. Also, the neighbours of $Z_{l+1}$ present in $H^{l}$ are
contained in $S_l$ for $i = l, l+1$. Hence $|S_l \cup S_{l+1}| \geq |N_{H^{l+1}}(Z_{l+1})| \geq a_h^{r-1}(2r-h)$
by Lemma 3.2. Thus, if $l \notin \{1, k_r-1\}$, then

$$|S| \geq |S_l \cup S_{k_r}| + |S_l \cup S_{l+1}| \geq 2a_h^{r-1}(2r-h) = a_h^r(2r-h).$$
Figure 4. The graph $G$ with $Y_{k_r} = \emptyset$.

Suppose $l = 1$. Then by using similar arguments, we see that $S_1 \cup S_2 \supseteq N_{H^2}(Z_2) \cup N_{H^1}(Z_2)$ and $S_{k_r} \cup S_{k_r-1} \supseteq N_{H^{k_r-1}}(Y_{k_r-1}) \cup N_{H^{k_r}}(Y_{k_r-1})$. Hence

$$|S| \geq |S_1 \cup S_2| + |S_{k_r-1} \cup S_{k_r}| \geq 2a_{h-1}^{r-1}(2r-h) = a_h^r(2r-h).$$

Similarly, for $l = k_r - 1$,

$$|S| \geq |S_1 \cup S_{k_r}| + |S_{k_r-2} \cup S_{k_r-1}| \geq 2a_{h-1}^{r-1}(2r-h) = a_h^r(2r-h).$$

**Subcase 2.** Suppose that $Z_i \neq \emptyset$ for $i = 1, 2, \ldots, k_r$. Then $|S_1 \cup S_{k_r}| \geq a_{h-1}^{r-1}(2r-h)$ and $|S_i| \geq (2r-h)a_{h-2}^{r-1}$ for $i \in \{2, 3, \ldots, k_r-1\}$. As in Subcase 1(i), we have $|S| \geq a_h^r(2r-h)$.

**Case 3.** Suppose $Y_i \neq \emptyset$ for $i = 1, 2, \ldots, k_r$. If $Z$ does not intersect $H^i$ for some $i$, then the result follows by replacing $Y$ by $Z$ in Case 1 and Case 2. Suppose that $Z$ intersects $H^i$ for all $i = 1, 2, \ldots, k_r$. If $h = 1$, then the minimum degree of $Y_i$ and $Z_i$ is at least 0 and so, by induction, $|S_i| \geq a_0^{r-1}(2(r-1) - 0) = 2r-2$, also as $r \geq 2$ implies

$$|S| = \sum_{i=1}^{k_r} |S_i| \geq \sum_{i=1}^{k_r} (2r-2) = k_r(2r-2) \geq 4(2r-2)$$

$$= 8(r-1) > 2(2r-1) = a_1^r(2r-1).$$

Suppose $h \geq 2$. The minimum degree of $Y_i$ and $Z_i$ is at least $h-2 \geq 0$. This shows that $S_i$ is an $(h-2)$-vertex cut of the graph $H^i$ for $i = 1, 2, \ldots, k_r$. Therefore, by induction and by Lemma 2.4(2), we have

$$|S| = \sum_{i=1}^{k_r} |S_i| \geq \sum_{i=1}^{k_r} a_{h-2}^{r-1}(2r-h) = k_r a_{h-2}^{r-1}(2r-h) \geq a_h^r(2r-h).$$

Thus $|S| \geq a_h^r(2r-h)$ in all the above cases. This completes the proof.
Corollary 3.4. For the graph $G$ of Theorem 1.2, $\kappa^h(G) = a_h^r(2r - h)$.

**Proof.** By Lemma 3.1, $\kappa^h(G) \leq a_h^r(2r - h)$. Since $\kappa^h(G)$ is the cardinality of a smallest $h$-vertex cut of $G$, by Proposition 3.3, $\kappa^h(G) \geq a_h^r(2r - h)$. Hence $\kappa^h(G) = a_h^r(2r - h)$. ■

4. CONDITIONAL EDGE CONNECTIVITY

In this section, we prove that the conditional edge connectivity $\lambda^h(G)$ of the graph $G$ of Theorem 1.2 is same as its conditional vertex connectivity $\kappa^h(G)$.

Recall that $G = C_{k_1} \sqcup C_{k_2} \sqcup \cdots \sqcup C_{k_r}$ with $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$ and $W_h^r$ is an $h$-regular subgraph of $G$ with $a_h^r$ vertices. We get an upper bound for $\lambda^h(G)$ from the set of edges of $G$ each of which has exactly one end vertex in $W_h^r$. For such edge sets we introduce the following notation. For a subgraph $K$ of a graph $H$, let

$$E_H(K) = \{xy: x \in V(K) \text{ and } y \in V(H) \setminus V(K)\}.$$ 

**Lemma 4.1.** For $0 \leq h \leq 2r - 1$, $\lambda^h(G) \leq (2r - h)a_h^r$.

**Proof.** Let $K = W_h^r$. Then $K$ is $h$-regular and $G$ is $2r$-regular. Hence $|E_G(K)| = (2r - h)|V(K)|$ and $G - E_G(K)$ is disconnected with $K$ as one of its components. By Lemma 2.7, the minimum degree of every component of $G - E_G(K)$ other than $K$ is at least $2r - 1 \geq h$. Therefore $E_G(K)$ contains an $h$-edge cut of $G$. This shows that $\lambda^h(G) \leq |E_G(K)| = (2r - h)a_h^r$. ■

**Lemma 4.2.** For a subgraph $Y$ of $G$ of minimum degree at least $h$, $|V(Y)| + |E_G(Y)| \geq a_h^r(2r - h + 1)$.

**Proof.** If $Y$ spans $G$, then $|V(Y)| = k_1k_2 \cdots k_r \geq a_h^r(2r - h + 1)$ by Lemma 2.9. Suppose $Y$ is not a spanning subgraph of $G$. Since for every $x$ in $N(Y)$ there is a vertex $y$ of $Y$ adjacent to $x$ so that the edge $xy$ belongs to the edge set $E_G(Y)$. This implies that $|E_G(Y)| \geq |N(Y)|$. Hence, by Lemma 3.2, $|V(Y)| + |E_G(Y)| \geq |N(Y)| \geq a_h^r(2r - h + 1)$. ■

Using this lemma we now obtain the reverse inequality for $\lambda^h(G)$.

**Proposition 4.3.** Let $F$ be an $h$-edge cut of the graph $G$. Then $|F| \geq a_h^r(2r - h)$.

**Proof.** Since the graph $G$ is $2r$-regular, $0 \leq h \leq 2r$. The result holds obviously for $h = 2r$. Suppose $h = 0$. Then $F$ is a set of edges of $G$ such that $G - F$ is a disconnected graph. It follows from Lemma 2.1 that $G$ is $2r$-edge connected and so, $|F| \geq 2r = a_h^r(2r - 0)$. Thus the result holds for $h = 0$ also. Suppose $1 \leq h \leq 2r - 1$. We prove the result by induction on $r$. The result follows trivially
for \( r = 1 \). Suppose \( r \geq 2 \). Assume that the result holds for the product of \( r - 1 \) cycles. Let \( F \) be an \( h \)-edge cut of \( G \). Then \( G - F \) is disconnected and every component of it has minimum degree at least \( h \).

Let \( Y \) be a subgraph of \( G - F \) consisting of at least one but not all components of \( G - F \) and let \( Z \) be the subgraph consisting of the remaining components. Then \( Y \) and \( Z \) are vertex disjoint subgraphs of \( G - F \) of minimum degree at least \( h \) and their union is \( G - F \). Note that \( F \) contains both edge sets \( E_G(Y) \) and \( E_G(Z) \).

Hence \(|F| \geq |E_G(Y)|\) and \(|F| \geq |E_G(Z)|\).

Write \( G \) as \( H \Box C_k \), where \( H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r} \). Then \( G \) is obtained by replacing vertex \( i \) of the cycle \( C_k \) by a copy \( H^i \) of \( H \) and replacing the edge joining \( i \) and \( i + 1 \) (mod \( k_r \)) by the perfect matching \( M_i \) between the corresponding vertices of \( H^i \) and \( H^{i+1} \) (mod \( k_r \)). Then \( Y \) intersects at least one \( H^i \). Similarly, \( Z \) intersects at least one \( H^i \). Let \( Y_j = Y \cap H^i \) and \( Z_j = Z \cap H^i \) for \( i = 1, 2, \ldots, k_r \).

For a subgraph \( K \) of \( G \), let \( M_i(K) \) be the set of all edges in the matching \( M_i \) each having exactly one end vertex in \( K \).

**Case 1.** Suppose \( Y_i \neq \emptyset \) for only one value of \( i \). Without loss of generality, we may assume that \( Y_i \) is non-empty for only \( i = 1 \). Then \( Y \) is contained in the graph \( H^1 \) and \( \delta_{H^1}(Y) \geq h \). Since \( H^1 \) is \((2r - 2)\)-regular, \( h \leq 2r - 2 \). If \( h = 2r - 2 \), then \( Y = H^1 \) and so, \(|E_G(Y)| = |M_1| + |M_k_r| = 2|V(H^1)| = 2k_1k_2 \cdots k_{r-1} \). As \( 4 \leq k_{r-1} \), we have

\[
a^r_h(2r - h) = 2a^r_h2(2r-(r-2)k_1k_2 \cdots k_{r-2}) = 2(4k_1k_2 \cdots k_{r-2})
\]

\[
\leq 2k_1k_2 \cdots k_{r-2}k_{r-1} = |E_G(Y)| \leq |F|.
\]

Suppose \( h < 2r - 2 \). Then \( E_G(Y) \supseteq E_{H^1}(Y) \cup M_1(Y) \cup M_k_r(Y) \). As \(|M_1(Y)| = |M_k_r(Y)| = |V(Y)|\), by Lemmas 2.4(3), 2.5 and 4.2, we have

\[
|E_G(Y)| \geq \left( |E_{H^1}(Y)| + |V(Y)| \right) + |V(Y)| \geq a^{r-1}_h(2r - h - 1) + a^{r-1}_h
\]

\[
= a^{r-1}_h(2r - h) \geq a^r_h(2r - h).
\]

**Case 2.** Suppose \( Y_i \neq \emptyset \) for more than one but not all values of \( i \). Without loss of generality, we may assume that \( Y_i \) is non-empty but \( Y_{kr} \) is empty. Let \( t \) be the largest integer such that \( Y_i \) is non-empty. Then \( 1 < t < k_r \). The minimum degree of \( Y_i \) in \( H^t \) is at least \( h - 1 \) for \( i = 1, t \). The graph \( Y_1 \) has \(|V(Y_1)| \) neighbours in \( H^{k_r-t} \) and \( Y_t \) has \(|V(Y_t)| \) neighbours in \( H^{t+1} \). Hence \( E_G(Y) \supseteq E_{H^1}(Y_1) \cup E_{H^t}(Y_t) \cup M_{k_r}(Y_1) \cup M_t(Y_t) \).

Suppose \( h = 2r - 1 \). Then \( Y_j = H^j \) for \( j = 1, t \) giving \( M_{k_r}(Y_1) = M_{k_r}(H^t) = M_{k_r} \) and \( M_t(Y_t) = M_t(H^t) = M_t \). Hence

\[
a^r_h(2r - h) = a^r_h = 2k_1k_2 \cdots k_{r-1} = |V(H^t)| + |V(H^1)|
\]

\[
= |M_{k_r}| + |M_t| \leq |E_G(Y)| \leq |F|.
\]
Suppose $h \leq 2r - 2$. Then $h - 1 \leq 2r - 3$ and so, by Lemmas 4.2 and 2.4(1),

$$|F| \geq |E_G(Y)| \geq (|E_{H^i}(Y_i)| + |V(Y_i)|) + (|E_{H^i}(Y_i)| + |V(Y_i)|)$$

$$\geq 2a_{h-1}^{r-1}(2r - h) = (2r - h)a_h^r.$$

**Case 3.** Suppose $Y_i \neq \emptyset$ for all $i = 1, 2, \ldots, k_r$. If the graph $Z$ does not intersect $H^i$ for some $i$, then the result follows easily by replacing $Y$ by $Z$ in Case 1 and Case 2. Suppose $Z$ intersects $H^i$ for all $i = 1, 2, \ldots, k_r$. Suppose $h = 1$. As $r \geq 2$, $\delta(Y_i) \geq 0$ and $\delta(Z_i) \geq 0$, by induction, we have

$$|E_G(Y)| = \sum_{i=1}^{k_r} |E_{H^i}(Y_i)| \geq \sum_{i=1}^{k_r} (2r - 2) \geq k_r(2r - 2)$$

$$\geq 4(2r - 2) = 8(r - 1) > 2(2r - 1) = a_1^r(2r - 1).$$

Suppose $h \geq 2$. The minimum degree of $Y_i$ and $Z_i$ is at least $h - 2 \geq 0$. Therefore the edge set $E_{H^i}(Y_i)$ is an $(h - 2)$-edge cut of $H^i$. By induction, $|E_{H^i}(Y_i)| \geq a_{h-2}^{r-1}(2r - h)$ for $i = 1, 2, \ldots, k_r$. By Lemma 2.4(2),

$$|F| \geq |E_G(Y)| = \sum_{i=1}^{k_r} |E_{H^i}(Y_i)| \geq \sum_{i=1}^{k_r} a_{h-2}^{r-1}(2r - h)$$

$$\geq k_r a_{h-2}^{r-1}(2r - h) \geq a_h^r(2r - h).$$

This completes the proof. ■

**Corollary 4.4.** For the graph $G$ of Theorem 1.2, $\lambda^h(G) = a_h^r(2r - h) = \kappa^h(G)$.

**Proof.** By Proposition 4.3, $\lambda^h(G) \geq a_h^r(2r - h)$ and by Lemma 4.1, $\lambda^h(G) \leq a_h^r(2r - h)$. Hence $\lambda^h(G) = a_h^r(2r - h) = \kappa^h(G)$ by Corollary 3.4. ■

This completes the proof of Theorem 1.2.

It is worth mentioning that the edge connectivity part of Theorem 1.2 proves that the following conjecture of Xu [7] holds for the classes of multidimensional tori and $k$-ary $r$-cubes.

**Conjecture 4.5.** Let $k, h$ be two non-negative integers and $G$ be a connected graph with minimum degree at least $k$ and $a_h(G)$ be the minimum cardinality of a vertex set of an $h$-regular subgraph of $G$. If $\lambda^h(G)$ exists, then $\lambda^h(G) \leq a_h(G)(k - h)$.

**Concluding Remarks.**

We determine the conditional $h$-vertex connectivity and the conditional $h$-edge connectivity of a multidimensional torus $G$ which is the Cartesian product of $r$
cycles each of length at least four, for all possible values of $h$. We first characterize the $h$-regular subgraph of $G$ with minimum number of vertices and then establish that both these conditional connectivities of $G$ are equal to $(2r - h)$ times the number of vertices of this subgraph.

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