ANTIMAGIC LABELING OF SOME BIREGULAR
BIPARTITE GRAPHS

KECAI DENG
School of Mathematical Sciences
Huqiao University
Quanzhou 362000, Fujian, P.R. China
e-mail: kecaideng@126.com

AND

YUNFEI LI
School of Accounting and Finance
Xiamen University Tan Kah Kee College
Zhangzhou 363000, Fujian, P.R. China
e-mail: lyfdkc@xujc.com

Abstract

An antimagic labeling of a graph $G = (V,E)$ is a one-to-one mapping from $E$ to $\{1, 2, \ldots, |E|\}$ such that distinct vertices receive different label sums from the edges incident to them. $G$ is called antimagic if it admits an antimagic labeling. It was conjectured that every connected graph other than $K_2$ is antimagic. The conjecture remains open though it was verified for several classes of graphs such as regular graphs. A bipartite graph is called $(k,k')$-biregular, if each vertex of one of its parts has the degree $k$, while each vertex of the other parts has the degree $k'$. This paper shows the following results. (1) Each connected $(2,k)$-biregular $(k \geq 3)$ bipartite graph is antimagic; (2) Each $(k,pk)$-biregular $(k \geq 3, p \geq 2)$ bipartite graph is antimagic; (3) Each $(k,k^2+y)$-biregular $(k \geq 3, y \geq 1)$ bipartite graph is antimagic.

Keywords: antimagic labeling, bipartite, biregular.

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1Corresponding author.
1. Introduction

Let $G = (V,E)$ be a graph. Suppose $f$ is a one-to-one mapping from $E$ to \{1,2,...,|E|\}. For each vertex $v$ in $V$, the vertex sum $\varphi_f(v)$ at $v$ under $f$ is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of edges incident to $v$. If $\varphi_f(u) \neq \varphi_f(v)$ for any vertex pair $u,v \in V$, then $f$ is called an antimagic labeling of $G$. A graph $G$ is called antimagic if $G$ admits an antimagic labeling. The antimagic labeling of graphs was introduced by Hartsfield and Ringel [8] in 1989 (also in [9]), who verified the antimagicness of paths, 2-regular graphs and complete graphs. Moreover, they put forth the following conjecture.

Conjecture 1 [9]. Every connected graph other than $K_2$ is antimagic.

The conjecture has received much attention, but remains open. It was proved by Alon et al. [1] that there is an absolute constant $c$ such that graphs with minimum degree $\delta(G) \geq c \log |V|$ are antimagic, and graphs with maximum degree at least $|V| - 2$ and complete bipartite graphs except $K_2$ are antimagic. And then graphs of large linear size were shown to be antimagic [6]. For regular graphs, the antimagicnesses of $k$-regular ($k \geq 2$) bipartite graphs [3], cubic graphs [12], odd degree regular graphs [4], and finally even regular graphs [2] were verified, respectively. For more results on antimagic labeling such as those about trees, one can refer to [5, 10, 11, 13, 14, 17] and the survey of Gallian [7].

A bipartite graph is called $(k,k')$-biregular, if each vertex in one of its two parts has the degree $k$, while each vertex in the other part has the degree $k'$. This paper shows the following results. (1) Each connected $(2,k)$-biregular $(k \geq 3)$ bipartite graph is antimagic; (2) Each $(k,pk)$-biregular $(k \geq 3, p \geq 2)$ bipartite graph is antimagic; (3) Each $(k,k^2 + y)$-biregular $(k \geq 3, y \geq 1)$ bipartite graph is antimagic. The first result is shown in Section 2, where we treat each connected $(2,k)$-biregular $(k \geq 3)$ bipartite graph as the subdivision graph of a connected $k$-regular graph. A subdivision graph $G_s$ of a graph $G$, is obtained from $G$ by replacing each edge with a path of length two. The second and the third results are shown in Section 3, based on an extended version of Hall’s matching theorem [15, 16].

2. Connected $(2,k)$-Biregular $(k \geq 3)$ Bipartite Graph

With respect to a given labeling, two vertices are in conflict if they have a common vertex sum. When we have labeled a subset of the edges, we call the resulting sum at each vertex a partial vertex-sum. For short, we denote by $[i,j]$ the integer set \{i,i+1,...,j\} for integers $i$ and $j$ (where $i < j$).

Theorem 2. The subdivision graph $G_s$ of every connected $k$-regular $(k \geq 3)$ graph $G$ is antimagic.
Proof. Choose an arbitrary vertex $v^*$ in $G$ as a root. Let $\alpha$ be the longest distance of a vertex from $v^*$ in $G$. Suppose $i \in [1, \alpha]$. Denote by $V_i$ the sets of vertices at distance exactly $i$ from $v^*$, by $G[V_i]$ the subgraph induced by $V_i$, and by $G[V_{i-1}; V_i]$ (here we suppose $V_0 = \{v^*\}$) the induced bipartite subgraph with parts $V_{i-1}$ and $V_i$, respectively. For $v \in V_i$, let $\sigma(v)$ be an arbitrary edge in $G[V_{i-1}; V_i]$ which is incident to $v$. Let $\sigma(V_i) = \{\sigma(v) \mid v \in V_i\}$ and $G[\sigma[V_{i-1}; V_i]] = G[V_{i-1}; V_i] \setminus \sigma(V_i)$.

Now subdivide $G$ into $G_s$. Then every vertex in $V_i$ is at distance exactly $2i$ from $v^*$ in $G_s$. Denote by $S_i$, $U_i$ and $W_i$ the newly added vertex sets on the edges of $G[V_i]$, $G[\sigma[V_{i-1}; V_i]]$ and $\sigma(V_i)$, respectively, when subdividing $G$ into $G_s$. Let $X = \bigcup_{i=1}^\alpha X_i$ for $X = V, S, U, W$. For a vertex $v \in V_i$, let $\nu(v)$ be the vertex in $W_i$ which is adjacent to $v$. For every vertex $x \in (S_i \cup U_i \cup W_i)$, let $e^x$ and $\nu_e$ be the two edges incident to $x$. If $x \in (U_i \cup W_i)$, we suppose $e^x$ is incident to some vertex in $V_i$, while $\nu_e$ is incident to some vertex in $V_{i-1}$. For $X = S, U, W$, let $E^X_1 = \{e^x \mid x \in X_1\}$, $E^X_2 = \{\nu_e \mid x \in X_1\}$ and $E^X_i = E^X_{i-1} \cup E^X_i$.

Respect to a labeling $f$ on $E(G_s)$, if $v \in V_i$, we denote the partial sum at $v$ (omitting the label on $\nu_e$) by $p(v) = \sum_{e \in E(v) \setminus \{\nu_e\}} f(e) = \nu_f(v) - f(\nu_e)$. Let $p(v^*) = \nu_f(v^*) - f(e^*)$ where $e^*$ is the edge in $E(v^*)$ which receives the greatest label among $E(v^*)$.

Note that $V(G_s) = V \cup S \cup U \cup W \cup \{v^*\}$. To show $G_s$ is antimagic, we will construct a labeling $f$ which satisfies the following conditions.

1. The vertex sums in $X_i$ are all odd and pairwise different, for $X \in \{S, U, W\}$ and $i \in [1, \alpha]$.
2. The vertex sums in $V_i$ are all even and pairwise different for $i \in [1, \alpha]$.
3. The vertex sums in $(S_i \cup U_i \cup W_i)$ are smaller than those in $(S_{i-1} \cup U_{i-1} \cup W_{i-1})$ for $i \in [2, \alpha]$.
4. The vertex sums in $S_i$ are smaller than those in $U_i$, while the later ones are smaller than those in $W_i$ for $i \in [1, \alpha]$.
5. The vertex sums in $V_i$ are smaller than those in $V_{i-1}$ for $i \in [2, \alpha]$.
6. The vertex sum at $v^*$ is greater than those in $V_i$ and those in $W_i$.

Conditions (1) and (2) make sure there is no conflict between $V$ and $(S \cup U \cup W)$. Conditions (1), (3), (4) make sure there is no conflict inside $(S \cup U \cup W)$. Conditions (2) and (5) make sure there is no conflict inside $V$. Conditions (3), (4), (5) and (6) make sure there is no conflict between $v^*$ and any other vertex in $G_s$. So these conditions imply that $f$ is antimagic.

Note that $E(G_s) = \bigcup_{i=1}^\alpha (E^S_i \cup E^U_i \cup E^W_i)$. We will label $E(G_s)$ in the order $E^S_\alpha$, $E^U_\alpha$, $E^W_\alpha$, $E^S_{\alpha-1}$, $(E^U_{\alpha-1} \cup E^W_{\alpha-1})$, ..., $E^S_1$, $(E^U_1 \cup E^W_1)$, using the smallest unused labels on each edge set when we come to it. This labeling assignment immediately implies that (3) holds, and that the vertex sums in $S_i$ are smaller than those in $(U_i \cup W_i)$ for $i \in [1, \alpha]$.

Suppose $i \in [1, \alpha]$ in the following. Note that $|E^X_i| = 2|X_i|$, for $X = S, U, W$. Antimagic Labeling of Some Biregular Bipartite Graphs 3
(I) The labeling of $E^S$. We first label $E^S_i$ arbitrarily using the $|S_i|$ odd labels from the $2|S_i|$ assigned labels for $E^S_i$. Secondly let $f(\overline{v}) = f(\overline{e}) + 1$ for each $s \in S$. Then the vertex sums in $S_i$ are odd and pairwise different.

(II) The labeling of $(E^U \cup E^W)$. If $|U_i|$ is odd, then $i \in [2, \alpha]$, since $U_1$ is an empty set. We will label $(E^U_i \cup E^W_i)$ in the order $E^U_i, E^U_i, E^W_i$ using the smallest unused assigned labels on each edge subset when we come to it. This sub-assignment (based on our global assignment), gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, which implies $\varphi_f(v) = p(v) + f(\overline{v}w(v)) < p(v') + f(\overline{v}'w(v')) = \varphi_f(v')$, since $f(\overline{v}w(v)) < f(\overline{v}'w(v'))$ by our global assignment. So (5) holds for those $i$ with $|U_i|$ being odd. It gives that the vertex sums in $U_i$ are smaller than those in $W_i$. So (4) holds for those $i$ with $|U_i|$ being odd. We first label $E^U_i$ arbitrarily using its assigned labels. Secondly let $f(\overline{v}) = f(\overline{e}) + |U_i|$ for each $u \in U_i$. This gives that the vertex sums in $U_i$ are odd and pairwise different. Third, suppose $V_i = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\overline{e}_r(v_r)$ with the $r$-th smallest label among the odd (even) assigned labels for $E^W_i$, when $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) holds for those $i$ with $|U_i|$ being odd. Fourth, let $f(\overline{v}) = f(\overline{e}) + 1$ when $f(\overline{e})$ is odd, while $f(\overline{v}) = f(\overline{e}) - 1$ when $f(\overline{e})$ is even. This implies that vertex sums in $W_i$ are odd and pairwise different. So (1) holds for those $i$ with $|U_i|$ being odd.

If $|U_i|$ is even ($|U_i|$ may equal to 0), then $i \in [1, \alpha]$. We will label edges in $E^U_i$ using the smallest $(2|U_i| + 1)$ assigned labels for $E^U_i \cup E^W_i$ except the $(|U_i| + 1)$-th smallest one (denoted by $\xi_{|U_i|+1}$). We first label the edges of $E^U_i$ arbitrarily using the $|U_i|$ smallest assigned labels. This gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$. And then, if $i \neq 1$, one has $\varphi_f(v) = p(v) + f(\overline{v}w(v)) < p(v') + f(\overline{v}'w(v')) = \varphi_f(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, since $f(\overline{v}w(v)) < f(\overline{v}'w(v'))$ by our global assignment. So (5) also holds for those $i (i \neq 1)$ with $|U_i|$ being odd. Secondly let $f(\overline{v}) = f(\overline{e}) + |U_i| + 1$ for each $u \in U_i$. This implies that the vertex sums in $U_i$ are odd and pairwise different. It also implies that the vertex sums in $U_i$ are smaller than those in $W_i$, since any pair of the rest assigned labels left for $W_i$ has a sum greater than any vertex sum in $U_i$. So (4) also holds for those $i$ with $|U_i|$ being even. Note that, $\xi_{|U_i|+1}$ and $\xi_{|U_i|+1} + |U_i| + 1$ have distinct parity, and so far, they are the smallest two unused assigned labels for $W_i$. Third, suppose $V_i = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\overline{e}_r(v_r)$ with the $r$-th smallest label among the rest odd (even) assigned labels, if $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) also holds for those $i$ with $|U_i|$ being even. And note that either $\xi_{|U_i|+1}$ or $\xi_{|U_i|+1} + |U_i| + 1$ is assigned to $w(v_1)$ by our labeling way. Fourth, let $f(\overline{v}) = f(\overline{e}) + 1$ if $f(\overline{v}) = f(\overline{e}) + 1, 1, 2, \ldots, |U_i| + 1$ if
\( f(\xi^w(v)) = \xi_{|U_i|+1}, \) so that \( \{ f(\xi^w(v)), f(\tau^w(v)) \} = \{ \xi_{|U_i|+1}, \xi_{|U_i|+1} + |U_i| + 1 \}. \)

And for \( r \in [2,|V_i|], \) let \( f(\tau^w(v)) = f(\xi^w(v)) + 1 \) if \( f(\xi^w(v)) \) is odd, while \( f(\tau^w(v)) = f(\xi^w(v)) - 1 \) if \( f(\xi^w) \) is even. This implies that vertex sums in \( W_i \) are odd and pairwise different. So (1) also holds for those \( i \) with \( |U_i| \) being even.

For (6), note that the process of the labeling of \( E(v^*) = \overline{E}_1^W \) is discussed in the case when \( |U_i| \) is even (since \( U_1 = \emptyset \) and \( |U_1| = 0 \)). Recall that, \( |E_1^W| = 2k \) and \( E_1^W \) are assigned with the greatest 2\( k \) labels, i.e., those labels in \( L_{2k} = \{|E(G_s)|, |E(G_s)| - 1, \ldots, |E(G_s)| - 2k + 1 \}. \) More precisely, \( \overline{E}_1^W = E(v^*) \) are assigned with the labels in \( \{i_1, i_2, \ldots, i_k\} \subseteq L_{2k} \) where either \( i_j = |E(G_s)| - 2j + 1 \) or \( i_j = |E(G_s)| - 2j + 2 \) for \( j = 1, 2, \ldots, k \). So \( p(v^*) \geq p(v_1) + 1 + 3 + \cdots + (2k - 3) > p(v_1) + 3 \) for arbitrary \( v_1 \in V_1 \) (recall that \( k \geq 3 \)). Then \( \varphi_f(v^*) = p(v^*) + f(e^*) \geq p(v^*) + |E(G_s)| - 1 > p(v_1) + |E(G_s)| + 2 > p(v_1) + |E(G_s)| \geq p(v_1) + f(v_1 w(v)) = \varphi_f(v_1) \) for each \( v_1 \in V_1 \). On the other hand, \( \varphi_f(v^*) \geq (|E(G_s)| - 1) + (|E(G_s)| - 3) + (|E(G_s)| - 5) = 3|E(G_s)| - 9, \) since \( k \geq 3 \). Thus, each vertex in \( W_i \) receives a sum at most \( (2|E(G_s)| - 1) \). So \( \varphi_f(v^*) \geq 3|E(G_s)| - 9 > 2|E(G_s)| - 1 \geq \varphi_f(w_1) \) for each \( w_1 \in W_1 \) (one has \( |E(G_s)| \geq 12 \), because \( k \geq 3 \)). So (6) holds.

Thus, \( G_s \) is antimagic. This completes our proof. \( \blacksquare \)

It is interesting to consider the case when \( G \) is \( k \)-regular \( (k \geq 3) \) but disconnected. In the proof of Theorem 2, suppose \( G \) has \( m \) edges. Then \( G_s \) has \( m \) 2-vertices. Note that the total sum of all the vertex sums is even, since each label contributes to the total sum twice. Thus, each 2-vertex contributes an odd value to the total sum, while each \( k \)-vertex other than \( v^* \) contributes an even value, under our labeling way in the proof of Theorem 2. Thus, \( \varphi_f(v^*) \) is odd if and only if \( m \) is odd.

**Theorem 3.** Let \( G \) be an disconnected \( k \)-regular \((k \geq 3)\) graph, which has at most one connected component with an odd number of edges. Then \( G_s \) is antimagic.

**Proof.** Suppose \( G \) consists of the connected components \( H_1, H_2, \ldots, H_\beta \) \((\beta \geq 2)\), where \( H_i \) has an even number of edges for each \( i \in [1,\beta - 1] \). We can label \( E(G_s) \) in the order \( E((H_1)_s), E((H_2)_s), \ldots, E((H_\beta)_s) \) using the smallest unused labels on each edge set when we come to it. Next, we label each connected component of \( G_s \) in the same way to that in Theorem 2, choosing a root for each component of \( G \). Then there is no conflict among each \( (H_i)_s \) for \( i \in [1,\beta] \). Each 2-vertex receives an odd sum, while each \( k \)-vertex other than the root of \( (H_\beta)_s \) receives an even sum. Each 2-vertex in \( (H_1)_s \) receives a smaller sum than each 2-vertex in \( (H_j)_s \), while each \( k \)-vertex in \( (H_1)_s \) receives a smaller sum than each \( k \)-vertex in \( (H_j)_s \), whenever \( i < j \leq \beta \) holds. And the root vertex in \( (H_\beta)_s \) receives a greater sum than those of any other vertex in \( G_s \). So we obtain an antimagic labeling. \( \blacksquare \)

Since \( m = \frac{n^2}{2} \), for each \( k \)-regular graph with \( n \) vertices and \( m \) edges, we have the following corollary.
Corollary 4. Let $G$ be an disconnected $k$-regular ($k \geq 3$) graph. Then $G_s$ is antimagic if one of the following holds.

1. $k = 4t$ ($t \geq 1$);
2. $k$ is even and at most one of the connected components of $G$ has an odd number of vertices;
3. At most one of the connected components of $G$ has a number of vertices which is not a multiple of 4.

3. $(k, pk)$-Biregular ($k \geq 3, p \geq 2$) Bipartite Graph

For a bipartite graph $G(A, B)$, a complete $p$-claw matching $CM$ from $A$ to $B$ is a set of edges of $G$ that induce a subgraph $G[CM]$ such that each vertex of $A$ in $G$ is also a vertex in $G[CM]$ and each component of $G[CM]$ is a copy of $K_{1,p}$ where the vertex of degree $p$ is in $A$, while the vertices of degree 1 are in $B$. For $A_0 \subseteq A$, denote by $N(A_0)$ the set of vertices in $B$ each of which has a neighbor in $A_0$. Let $E_1, E_2, \ldots, E_k \subseteq E(G)$ be disjoint edge sets. If $E_1 \cup E_2 \cup \cdots \cup E_k = E(G)$, then we say $G$ decomposes into $E_1, E_2, \ldots, E_k$.

Lemma 5 (An extended version of Hall’s theorem, [15, 16]). A bipartite graph $G[A, B]$ admits a complete $p$-claw matching from $A$ to $B$, if and only if $p|A_0| \leq |N(A_0)|$ for every subset $A_0$ of $A$.

Lemma 6. Let $G[A, B]$ be a $(k, pk)$-biregular $(k \geq 3, p \geq 2)$ bipartite graph where the degree of each vertex in $A$ is $kp$, while each vertex in $B$ has degree $k$. Then $G$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

**Proof.** Let $A_0 \subseteq A$. Let $G[A_0, N(A_0)]$ be the graph induced by $A_0 \cup N(A_0)$. Then each vertex of $A_0$ in $G[A_0, N(A_0)]$ has the degree $kp$ while each vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most $k$. So there are exactly $kp|A_0|$ edges in $G[A_0, N(A_0)]$. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number of edges in $G[A_0, N(A_0)]$ is less than $k \cdot p|A_0|$, a contradiction. So $|N(A_0)| \geq p|A_0|$. By Lemma 5, there exists a complete $p$-claw matching $CM_1$ from $A$ to $B$ in $G[A, B]$. Then $G_1 = G[A, B] - CM_1$ is a $(k-1, p(k-1))$-biregular bipartite graph. So we can use Lemma 5 repeatedly until we obtain a $(1, p)$-biregular bipartite graph $G_{k-1}$ which is also a complete $p$-claw matching from $A$ to $B$. Thus, $G[A, B]$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Lemma 7. Let $I = [i + 1, i + 2q]$. Then, there exist partitions $P_1$ (when $q$ is odd) and $\{P_2, P_3, P_4\}$ (when $q$ is even) of $I$, such that under $P_j$, $j \in [1, 4]$, $I$ is departed into $q$ parts where each part has 2 integers, integers in $[i + (x - 1)q + 1, i + xq]$ ($x \in [1, 2]$) are in distinct parts and the following conditions are satisfied.
(1) Under $P_1$, the $q$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q - 1)/2, (2i + 2q + 1) + (q - 1)/2]$;

(2) Under $P_2$, $q/2$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q/2 - 1), 2i + 2q + 1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2i + 2q + 1, (2i + 2q + 1) + (q/2 - 1)]$;

(3) Under $P_3$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q + 2) - (q/2 - 1), (2i + 2q + 2) + (q/2 - 1)]$ and the other part has the sum $2i + q + 2$;

(4) Under $P_4$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q) - (q/2 - 1), (2i + 2q) + (q/2 - 1)]$ and the other part has the sum $2i + 3q$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $q$ is odd, let $\{2j - 1, -j + (3q + 1)/2 + 1\}$ be in the same partition for $j \in [1, (q + 1)/2]$, and let $\{2j, -j + 2q + 1\}$ be in the same partition for $j \in [1, (q - 1)/2]$, which is the desired partition $P_1$.

(2) If $q$ is even, let $\{2j, -j + 3q/2 + 1\}$ be in the same partition and let $\{2j - 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2]$, which is the desired partition $P_2$.

(3) If $q$ is even, let $\{2j, -j + 3q/2 + 2\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j + 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{1, q + 1\}$ be in the same partition, which is the desired partition $P_3$.

(4) If $q$ is even, let $\{2j - 1, -j + 3q/2 + 1\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j, -j + 2q\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{q, 2q\}$ be in the same partition, which is the desired partition $P_4$. ■

**Lemma 8.** Let $I = [i + 1, i + zq]$ $(z \geq 3)$. Then, there exist partitions $P_1$ (when $z$ is even or $q$ is odd) and $P_2, P_3$ (when $z$ is odd and $q$ is even) of $I$, such that under $P_j$, $j \in [1, 3]$, $I$ is departed into $q$ parts where each part has $z$ integers, integers in $[i + (x - 1)q + 1, i + xq]$ $(x \in [1, z])$ are in distinct parts and the following conditions are satisfied.

(1) Under $P_1$, the $q$ parts have the same sum $(2i + zq + 1)z/2$;

(2) Under $P_2$, $q/2$ parts have the same sum $(2i + zq + 1)z/2 + 1/2$ and the other $q/2$ parts have the same sum $(2i + zq + 1)z/2 - 1/2$;

(3) Under $P_3$, $(q - 1)$ parts have the same sum $(2i + zq + 1)z/2 + 3/2$ and the other part has the sum $(2i + zq + 1)z/2 - 3q/2 + 3/2$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $z$ is even, let $\{(j - 1)q + \ell \mid j \in [1, z/2]\} \cup \{jq - l + 1 \mid j \in [z/2 + 1, z]\}$ be in the partition for $\ell \in [1, q]$, which is the desired partition $P_1$ and (1) holds in this case.
If $z$ is odd (then $z-3$ is even) and $q$ is odd, we first assign the $(z-3)q$ integers in $[2q+1,(z-1)q]$ to the $q$ parts (suppose $I_1, I_2, \ldots, I_q$ are the $q$ parts) such that these $q$ parts receive the same partial sum $(qz + q + 1)(z - 3)/2$. We can do this since $(z - 3)$ is even. Second, assign $[(z-1)q+l]$ to $I_l$ for $l \in [1,q]$ such that the $q$ parts have distinct partial sums and attain all values in $[(qz + q + 1)(z-3)/2 + (z-1)q + 1, (qz + q + 1)(z-3)/2 + qz]$. Third, partition $[1,2q]$ into $q$ parts (denoted by $I_{1}', I_{2}', \ldots, I_{q}'$) which have distinct sums which attain all the values in $[(2q + 1) - (q-1)/2, (2q + 1) + (q-1)/2]$. We can do this owing to the partition in Lemma 7(1). Then assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $[(2q + 1) + (q-1)/2 - l + 1)$ for $l \in [1,q]$. Then the final sum of $I_l$ equals to $[(qz + 1)(z-3)/2 + (z-1)q + l + [(2q + 1) + (q-1)/2 - l + 1) = (qz + 1)z/2$ for each $l \in [1,q]$. So (1) also holds in this case.

(2) If $z$ is odd and $q$ is even, we first partition $[2q+1, zq]$ into $q$ parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in $[(qz + q + 1)(z-3)/2 + (z-1)q + 1, (qz + q + 1)(z-3)/2 + qz]$. We can do this owing to the discussion in (1). Then partition $[1,2q]$ into $q$ parts (denoted by $I_{1}', I_{2}', \ldots, I_{q}'$) such that $q/2$ parts have distinct sums which attain all the values in $[(2q + 1) - (q-1)/2, 2q + 1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2q+1, (2q+1) + (q-1)/2]$. We can do this owing to the partition in Lemma 7(2). Denote by $I_{i}'_{q/2,1}$ and $I_{i}'_{q/2,2}$ the two parts each of which admits the sum $(2q + 1)$. Then assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $[(2q + 1) + (q-1)/2 - l + 1] for $l \in [1,q/2-1]$. Assign $I_{i}'_{q/2,1}$ to $I_{q/2}$, while assign $I_{i}'_{q/2,2} to $I_{q/2+1}$. And assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $(2q + 1) + (q/2 - 1) - l + 2$ for $l \in [q/2 + 2,q]$. Then for $l \in [1,q/2-1]$ the final sum of $I_l$ equals to $[(qz + q + 1)(z-3)/2 + (z-1)q + l + [(2q + 1) + (q-1)/2 - l + 1] = (qz + 1)z/2 - 1/2$. The final sum of $I_{q/2}$ equals to $[(qz + q + 1)(z-3)/2 + (z-1)q + q/2] + [(2q + 1)] = (qz+1)z/2 - 1/2$, while the final sum of $I_{q/2+1}$ equals to $[(qz + q + 1)(z-3)/2 + (z-1)q + q/2 + 1] + [(2q + 1)] = (qz+1)z/2 + 1/2$. Thus, for $l \in [q/2 + 2,q]$ the final sum of $I_l$ equals to $[(qz+q+1)(z-3)/2 + (z-1)q + l + [(2q + 1) + (q/2 - 1) - l + 2] = (qz + 1)z/2 + 1/2$. So (2) holds.

(3) If $z$ is odd and $q$ is even, we first partition $[2q+1, zq]$ into $q$ parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in $[(qz + q + 1)(z-3)/2 + (z-1)q + 1, (qz + q + 1)(z-3)/2 + qz]$. We can do this owing to the discussion in (1). Then partition $[1,2q]$ into $q$ parts (denoted by $I_{1}', I_{2}', \ldots, I_{q}'$) such that the $(q-1)$ parts have distinct sums which attain all the values in $[(2q+2) - (q-1)/2, (2q+2) + (q-1)/2]$ and the other part has the sum $(q+2)$. We can do this owing to the partition in Lemma 7(3). Denote by $I_{i}'$ the part with the sum $(q+2)$. Then assign $I_{i}'$ to $I_1$, and assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $[(2q+2) + (q/2 - 1) - l + 2] for $l \in [2,q]$. Then the final sum of $I_l$ equals to $[(qz + q + 1)(z-3)/2 + (z-1)q + l + [q + 2] = (qz + 1)z/2 - 3q/2 + 3/2$, and
for \( l \in [2, q] \), the final sum of \( I_l \) equals to \([(qz + q + 1)(z - 3)/2 + [(z - 1)q + l] + [(2q + 2) + (q/2 - 1) - l + 2] = (qz + 1)z/2 + 3/2 \). So (3) holds.

\[ \text{(3)} \]

**Theorem 9.** Every \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph is antimagic.

**Proof.** Let \( G[A, B] \) be a \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph, where each vertex in \( A \) has the degree \( pk \), while each vertex in \( B \) has the degree \( k \). Suppose \(|A| = n \) \((n \geq k)\) and \(|B| = pn\). Let \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_{pn}\} \). By Lemma 6, \( G \) decomposes into \( k \) complete \( p \)-claw matchings \( CM_1, CM_2, \ldots, CM_k \) from \( A \) to \( B \). Denote by \( CM_i(V_0) \) \((i \in [1, k])\) the edges in \( CM_i \), which are incident to some vertex in \( V_0 \) for \( V_0 \subseteq V(G) \).

**Step 1.** Label \( \bigcup_{i=1}^{k-1} CM_i \) with \([1, (k - 1)pn]\).

First, label \( CM_{k-1} \) with \([(k - 2)pn + 1, (k - 1)pn]\), i.e., \([(k - 2)pn + 1, (k - 2)pn + pn] \) such that the following conditions are satisfied.

\[ (1.1) \text{Within } CM_{k-1}, \text{ vertices in } A \text{ have the same partial sum } [(2k - 3)pn + 1]p/2 \text{ if } p \text{ is even or } n \text{ is odd. We can do this owing to the partition in Lemma 8(1).} \]

\[ (1.2) \text{Within } CM_{k-1}, n/2 \text{ vertices in } A \text{ have the same partial sum } [(2k - 3)pn + 1]p/2 + 1/2 \text{ and the other } n/2 \text{ vertices in } A \text{ have the same partial sum } [(2k - 3)pn + 1]p/2 - 1/2 \text{ if } p \text{ is odd and } n \text{ is even. We can do this owing to the partition in Lemma 8(2).} \]

Second, based on the labeling to \( CM_{k-1} \), for each \( i \in [1, k - 2] \), label \( CM_i \) with \([(i - 1)pn + 1, ipn]\), such that the following conditions are satisfied.

\[ (1.3) \text{Within } \bigcup_{i=1}^{k-1} CM_i \text{, the vertices in } B \text{ have the same partial sum } [(k - 1)pn + 1)(k - 1)/2 \text{ if } (k - 1) \text{ even or } pn \text{ is odd. We can do this owing to the partition in Lemma 8(1).} \]

\[ (1.4) \text{Within } \bigcup_{i=1}^{k-1} CM_i \text{, } (pn - 1) \text{ vertices in } B \text{ have the same partial sum } [(k - 1)pn + 1)(k - 1)/2 + 3/2 \text{ while the other vertex (denoted by } b_0) \text{ has the partial sum } [(k - 1)np + 1)(k - 1)/2 + 3/2 - 3pn/2 \text{ if } (k - 1) \text{ is odd and } pn \text{ is even. We can do this owing to the partition in Lemma 8(3).} \]

Note that, (1.3) implies the vertices in \( B \) will receive distinct final vertex sums, when \((k - 1) \text{ is even or } pn \text{ is odd, if we label the rest edges } CM_k \text{ using the rest labels } [(k - 1)pn + 1, kpn]. \text{ Thus in (1.4), the partial sum of } b_0 \text{ is at least } 3pn/2 \text{ smaller than those of the vertices in } (B \setminus \{b_0\}). \text{ So the final vertex sum of } b_0 \text{ will still be smaller than those of the vertices in } (B \setminus \{b_0\}), \text{ if we label } CM_k \text{ with } [(k - 1)pn + 1, kpn]. \text{ Hence, the final vertex sums of in } (B \setminus \{b_0\}) \text{ will be pairwise different. That is, all vertices in } B \text{ will also receive distinct final vertex sums when } (k - 1) \text{ is odd and } pn \text{ is even.} \]

**Step 2.** Label \( CM_k \) with \([(k - 1)pn + 1, kpn]\), i.e., \([(k - 1)pn + 1, (k - 1)pn + pn] \).

Suppose \( f_1(a_{i1}) \leq f_1(a_{i2}) \leq \cdots \leq f_1(a_{in}) \) where \( f_i(a_{ij}) \) is the partial vertex sum of \( a_{ij} \) within \( \bigcup_{i=1}^{k-1} CM_i \) for \( j \in [1, n]. \)
(2.1) If $p$ is odd (then $(p-1)$ is even) or $n$ is odd, let $\sigma(a)$ be an edge in $CM_k(a)$ for each $a \in A$. Label $\left[CM_k \setminus \left( \bigcup_{a \in A} \{\sigma(a)\} \right) \right]$ with $[(k-1)pn+1, (k-1)pn+(p-1)n]$ such that, within $\left[CM_k \setminus \left( \bigcup_{a \in A} \{\sigma(a)\} \right) \right]$, the vertices in $A$ have the same partial sum $[(2k-1)pn - n + 1](p-1)/2$. We can do this owing to the partition in Lemma 8(1). Next label $\sigma(a_t)$ with $(kpn - n + j)$ for $j \in [1, n]$. Then the vertex sums in $A$ are pairwise different.

(2.2) If $p$ is even (then $(p-2)$ is also even) and $n$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM_k(a)$ for each $a \in A$. Label $\left[CM_k \setminus \left( \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\} \right) \right]$ using the labels in $[(k-1)pn + 1, (k-1)pn + (p-2)n]$ such that, within $\left[CM_k \setminus \left( \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\} \right) \right]$, the vertices in $A$ have the same partial sum $[(2k-1)pn - 2n + 1](p-2)/2$. We can do this owing to the partition in Lemma 8(1). Then label $\left( \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\} \right)$ with $[(kpn - 2n + 1, (kpn - 2n + 2n)] such that $f(\sigma_1(a_t)) + f(\sigma_2(a_t)) = 2kpn - 5n/2 + j$ for $j \in [1, n - 1]$ while $f(\sigma_1(a_{t_n})) + f(\sigma_2(a_{t_n})) = 2kpn - n$. We can do this owing to the partition in Lemma 7(4).

Then the vertex sums in $A$ are also pairwise different.

Recall that, owing to Step 1 (1.3) and (1.4), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{[(k-1)pn + 1](k-1)}{2} + \frac{3}{2} + kpn.$$

On the other hand, owing to the labeling way in Step 1 (1.3) and (1.4), the labels assigned to $CM_i$ are those in $[(i-1)pn+1, ipn]$ for $i \in [1, k-2]$. Let $a \in A$. Then the sum of the labels in $CM_i(a)$ is at least $\sum_{j=1}^{p} [(i-1)pn + j]$ for $i \in [1, k-2]$. So the sum of the labels in $\left( \bigcup_{i=1}^{k-2} CM_i(a) \right)$ is at least $\sum_{i=1}^{k-2} \sum_{j=1}^{p} [(i-1)pn + j]$. Recall that, owing to Step 1 (1.1) and (1.2), the sum of labels in $CM_{k-1}(a)$ is at least $[(2k-3)pn + 1]/2 - 1/2$. Next recall that, owing to Step 2 (2.1), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn - n + 1](p-1)/2 + (kpn - n + 1)$ if $p$ is odd (then $(p-1)$ is even) or $n$ is odd, while owing to Step 2 (2.2), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn - 2n + 1](p-2)/2 + (2kpn - 5n/2 + 1)$ if $p$ is even (then $(p-2)$ is also even) and $n$ is even. Thus, the later lower bound is $1/2$ smaller than the first lower bound. So

$$\varphi_f(a) \geq \sum_{i=1}^{k-2} \sum_{j=1}^{p} [(i-1)pn + j] + \left\{ \frac{[(2k-3)pn + 1]p}{2} - \frac{1}{2} \right\} + \left\{ \frac{[(2k-1)pn - 2n + 1](p-2)}{2} + \left( 2kpn - \frac{5n}{2} + 1 \right) \right\}.$$

Then for each $a \in A$ and $b \in B$, one has

$$\varphi_f(a) - \varphi_f(b) \geq \frac{1}{2}\left[ \left( \frac{1}{2}k - 1 \right)p^2kn + (k-3)p^2 + k^2 \left( \frac{1}{2}p - 1 \right)pn \right. \\
+ \left. (p-1)(np + k) + (p^2 - 1)n + (p^2 - 3) \right] > 0,$$
since \( k \geq 3 \) and \( p \geq 2 \).
Thus, we obtain an antimagic labeling. This completes our proof. \( \square \)

**Theorem 10.** Every \((k, k^2 + y)\)-biregular \((k \geq 3, y \geq 1)\) bipartite graph is antimagic.

**Proof.** Let \( G[A,B] \) be a \((k, k')\)-biregular \((k' = k^2 + y)\) bipartite graph, where each vertex in \( A \) has the degree \( k' \), while each vertex in \( B \) has the degree \( k \).

Suppose \(|A| = k\eta\) and \(|B| = k'\eta\) where \( \eta \) may be not an integer. It is sufficient to consider the case when \( k' = kp + r \) for some integers \( p \) and \( r \) satisfying \( p \geq k \) and \( 1 \leq r \leq k - 1 \) (note that \( r\eta \) is an integer since \( k\eta \) and \( k'\eta \) are integers).

Let \( A = \{a_1, a_2, \ldots, a_k\eta\} \) and \( B = \{b_1, b_2, \ldots, b_{k'}\eta\} \). For \( A_0 \subseteq A \), the graph \( G[A_0, N(A_0)] \) has \( k'|A_0| \) edges, since each vertex of \( A_0 \) in \( G[A_0, N(A_0)] \) has the degree \( k' \). On the other hand, suppose \(|N(A_0)| < p|A_0|\). Then the number of edges in \( G[A_0, N(A_0)] \) is at most \( k|N(A_0)| < pk|A_0| < k'|A_0| \), since each vertex of \( N(A_0) \) in \( G[A_0, N(A_0)] \) has the degree at most \( k \), a contradiction. So \(|N(A_0)| \geq p|A_0|\). So, by Lemma 5, \( G \) admits a complete \( p \)-claw matching \( CM \) from \( A \) to \( B \). Suppose \( B = B_1 \cup B_2 \) where \( B_1 = V(CM) \cap B \) and \( B_2 = B \setminus B_1 \). Then \(|B_1| = kp\eta \) and \(|B_2| = r\eta \).

**Step 1.** Label \((E(G) - CM - \sigma(B_2))\) with \([1, (k-1)k'\eta]\).

\((1.1)\) If \((k-1)\) is even or \( k'\eta \) is odd, label \((E(G) - CM - \sigma(B_2))\) with \([1, (k-1)k'\eta]\) such that, within \((E(G) - CM - \sigma(B_2))\), the vertices in \( B \) have the same partial sum \(((k-1)k'\eta + 1)(k-1)/2\). We can do this owing to the partition in Lemma 8(1).

\((1.2)\) If \((k-1)\) is odd and \( k'\eta \) is even, label \((E(G) - CM - \sigma(B_2))\) with \([1, (k-1)k'\eta]\) such that, within \((E(G) - CM - \sigma(B_2))\), the vertices in \( B \) have the same partial sum \(((k-1)k'\eta + 1)(k-1)/2 + 3/2\) except one (denoted by \( b_0 \)) which equals to \(((k-1)k'\eta + 1)(k-1)/2 - 3k'\eta/2 + 3/2\). We can do this owing to the partition in Lemma 8(3).

Note that \((1.1)\) implies the final vertex sums in \( B \) will be pairwise different when \((k-1)\) is even or \( k'\eta \) is odd, if we label the rest edges \((CM \cup \sigma(B_2))\) with the rest labels \([(k-1)k'\eta + 1, kk'\eta]\). Then in \((1.2)\), the partial sum of \( b_0 \) is at least \( 3k'\eta/2 \) smaller than those of the vertices in \((B \setminus \{b_0\})\). So the final vertex sum of \( b_0 \) will be smaller than those of the vertices in \((B \setminus \{b_0\})\), if we label \((CM \cup \sigma(B_2))\) with \([(k-1)k'\eta + 1, kk'\eta]\). Next, the final vertex sums of in \((B \setminus \{b_0\})\) will be pairwise different. That is, vertices in \( B \) will also receive distinct final vertex sums, when \((k-1)\) is odd and \( k'\eta \) is even.

**Step 2.** Label \( \sigma(B_2) \) with \([(k-1)k'\eta + 1, (k-1)k'\eta + r\eta]\) arbitrarily.
**Step 3.** Label CM with \([(k-1)k\eta+r\eta+1, kk\eta]\), i.e., \([(kk\eta-pk\eta)+1, (kk\eta-pk\eta)+(p-1)k\eta]\). Suppose \(f_1(a_{t_1}) \leq f_1(a_{t_2}) \leq \cdots \leq f_1(a_{t_n})\), where \(f_1(a_{t_j})\) is the partial vertex sum of \(a_{t_j}\) after Steps 1 and 2, for \(j \in [1, k\eta]\).

(3.1) If \(p\) is odd (then \((p-1)\) is even) or \(k\eta\) is odd, let \(\sigma(a)\) be an edge in \(CM(a)\) for each \(a \in A\). Label \([CM \setminus (\bigcup_{a \in A} \{\sigma(a)\})]\) with \([(kk\eta-pk\eta)+1, (kk\eta-pk\eta)+(p-2)k\eta]\) such that, within \([CM \setminus (\bigcup_{a \in A} \{\sigma(a)\})]\), vertices in \(A\) have the same vertex sum \([2kk\eta-pk\eta]+(p-1)k\eta+1(p-2)/2\). We can do this owing to the partition in Lemma 8(1). And label \(\sigma(a_{t_j})\) with \((kk\eta-k\eta+j)\) for \(j \in [1, k\eta]\). Then the vertex sums in \(A\) are pairwise different.

(3.2) If \(p\) is even (then \((p-2)\) is also even) and \(k\eta\) is even, let \(\sigma_1(a)\) and \(\sigma_2(a)\) be two distinct edges in \(CM(a)\) for each \(a \in A\). Label \([CM \setminus (\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\})]\) with \([(kk\eta-pk\eta)+1, (kk\eta-pk\eta)+(p-2)k\eta]\) such that, within \([CM \setminus (\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\})]\), vertices in \(A\) have the same vertex sum \([2kk\eta-pk\eta]+(p-2)k\eta+1(p-2)/2\). We can do this owing to the partition in Lemma 8(1). Next, label \((\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\})\) with \([(kk\eta-2k\eta)+1, (kk\eta-2k\eta)+2k\eta]\) such that \(f(\sigma_1(a_{t_1})) + f(\sigma_2(a_{t_1})) = 2kk\eta-5k\eta/2+j\) for \(j \in [1, k\eta-1]\), while \(f(\sigma_1(a_{t_{k\eta}})) + f(\sigma_2(a_{t_{k\eta}})) = 2kk\eta-k\eta\). We can do this owing to the partition in Lemma 7(4). Then the vertex sums in \(A\) are also pairwise different.

Recall that, owing to Step 1 (1.1) and (1.2), for each \(b \in B\), one has

\[
\phi_f(b) \leq \frac{((k-1)k\eta+1)(k-1)}{2} + \frac{3}{2} + kk'p.
\]

On the other hand, let \(a \in A\). Recall that, owing to Step 3 (3.1) the sum of the labels in \(CM(a)\) is at least \([2kk\eta-pk\eta]+(p-1)k\eta+1(p-1)/2+(kk\eta-k\eta+1)\) if \(p\) is odd (then \((p-1)\) is even) or \(k\eta\) is odd. Then owing to Step 3 (3.2), the sum of the labels in \(CM(a)\) is at least \([2kk\eta-pk\eta]+(p-2)k\eta+1(p-2)/2+(2kk\eta-5k\eta/2+1)\) if \(p\) is even (then \((p-2)\) is also even) and \(k\eta\) is even. And the lower bound is 1/2 smaller than the first lower bound. So

\[
\phi_f(a) > \frac{2kk\eta-pk\eta+(p-2)k\eta+1(p-2)}{2} + \left(2kk\eta-\frac{5k\eta}{2}+1\right).
\]

Then for each \(a \in A\) and \(b \in B\) one has

\[
\phi_f(a) - \phi_f(b) > \frac{1}{2}[(p-k)kk\eta+(k'-2p-1)pk\eta+(p^2-3)k\eta
\]
\[
+ (pk+k-k')\eta+(\eta-1)k+(p-2)] > 0,
\]

since \(p \geq k \geq 3\) and \(2p+1 < k' = pk + r \leq pk + k\).

Thus, we obtain an antimagic labeling. This completes our proof. 

\[\blacksquare\]
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