ANTIMAGIC LABELING OF SOME BIREGULAR BIPARTITE GRAPHS

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Abstract

An antimagic labeling of a graph $G = (V,E)$ is a one-to-one mapping from $E$ to $\{1, 2, \ldots, |E|\}$ such that distinct vertices receive different label sums from the edges incident to them. $G$ is called antimagic if it admits an antimagic labeling. It was conjectured that every connected graph other than $K_2$ is antimagic. The conjecture remains open though it was verified for several classes of graphs such as regular graphs. A bipartite graph is called $(k,k')$-biregular, if each vertex of one of its parts has the degree $k$, while each vertex of the other parts has the degree $k'$. This paper shows the following results. (1) Each connected $(2,k)$-biregular ($k \geq 3$) bipartite graph is antimagic; (2) Each $(k,pk)$-biregular ($k \geq 3, p \geq 2$) bipartite graph is antimagic; (3) Each $(k,k^2 + y)$-biregular ($k \geq 3, y \geq 1$) bipartite graph is antimagic.

Keywords: antimagic labeling, bipartite, biregular.

2010 Mathematics Subject Classification: 05C69.
1. Introduction

Let $G = (V,E)$ be a graph. Suppose $f$ is a one-to-one mapping from $E$ to $\{1,2,\ldots,|E|\}$. For each vertex $v$ in $V$, the vertex sum $\varphi_f(v)$ at $v$ under $f$ is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of edges incident to $v$. If $\varphi_f(u) \neq \varphi_f(v)$ for any vertex pair $u,v \in V$, then $f$ is called an antimagic labeling of $G$. A graph $G$ is called antimagic if $G$ admits an antimagic labeling. The antimagic labeling of graphs was introduced by Hartsfield and Ringel [8] in 1989 (also in [9]), who verified the antimagicnesses of paths, 2-regular graphs and complete graphs. Moreover, they put forth the following conjecture.

**Conjecture 1** [9]. Every connected graph other than $K_2$ is antimagic.

The conjecture has received much attention, but remains open. It was proved by Alon et al. [1] that there is an absolute constant $c$ such that graphs with minimum degree $\delta(G) \geq c \log |V|$ are antimagic, and graphs with maximum degree at least $|V| - 2$ and complete bipartite graphs except $K_2$ are antimagic. And then graphs of large linear size were shown to be antimagic [6]. For regular graphs, the antimagicnesses of $k$-regular ($k \geq 3$) bipartite graphs [3], cubic graphs [12], odd degree regular graphs [4], and finally even regular graphs [2] were verified, respectively. For more results on antimagic labeling such as those about trees, one can refer to [5, 10, 11, 13, 14, 17] and the survey of Gallian [7].

A bipartite graph is called $(k,k')$-biregular, if each vertex in one of its two parts has the degree $k$, while each vertex in the other part has the degree $k'$. This paper shows the following results. (1) Each connected $(2,k)$-biregular ($k \geq 3$) bipartite graph is antimagic; (2) Each $(k, pk)$-biregular ($k \geq 3$, $p \geq 2$) bipartite graph is antimagic; (3) Each $(k, k^2 + y)$-biregular ($k \geq 3$, $y \geq 1$) bipartite graph is antimagic. The first result is shown in Section 2, where we treat each connected $(2,k)$-biregular ($k \geq 3$) bipartite graph as the subdivision graph of a connected $k$-regular graph. A subdivision graph $G_s$ of a graph $G$, is obtained from $G$ by replacing each edge with a path of length two. The second and the third results are shown in Section 3, based on an extended version of Hall’s matching theorem [15, 16].

2. Connected $(2,k)$-Biregular ($k \geq 3$) Bipartite Graph

With respect to a given labeling, two vertices are in conflict if they have a common vertex sum. When we have labeled a subset of the edges, we call the resulting sum at each vertex a partial vertex-sum. For short, we denote by $[i,j]$ the integer set $\{i,i + 1,\ldots,j\}$ for integers $i$ and $j$ (where $i < j$).

**Theorem 2.** The subdivision graph $G_s$ of every connected $k$-regular ($k \geq 3$) graph $G$ is antimagic.
Proof. Choose an arbitrary vertex \( v^* \) in \( G \) as a root. Let \( \alpha \) be the longest distance of a vertex from \( v^* \) in \( G \). Suppose \( i \in [1, \alpha] \). Denote by \( V_i \) the sets of vertices at distance exactly \( i \) from \( v^* \), by \( G[V_i] \) the subgraph induced by \( V_i \), and by \( G[V_{i-1}; V_i] \) (here we suppose \( V_0 = \{v^*\} \)) the induced bipartite subgraph with parts \( V_{i-1} \) and \( V_i \), respectively. For \( v \in V_i \), let \( \sigma(v) \) be an arbitrary edge in \( G[V_{i-1}; V_i] \) which is incident to \( v \). Let \( \sigma(V_i) = \{ \sigma(v) \mid v \in V_i \} \) and \( G_\alpha[V_{i-1}; V_i] = G[V_{i-1}; V_i] \setminus \sigma(V_i) \).

Now subdivide \( G \) into \( G_s \). Then every vertex in \( V_i \) is at distance exactly \( 2i \) from \( v^* \) in \( G_s \). Denote by \( S_i, U_i \) and \( W_i \) the newly added vertex sets on the edges of \( G[V_i], G_\alpha[V_{i-1}; V_i] \) and \( \sigma(V_i) \), respectively, when subdividing \( G \) into \( G_s \). Let \( X = \bigcup_{i=1}^\alpha X_i \) for \( X = V, S, U, W \). For a vertex \( v \in V_i \), let \( w(v) \) be the vertex in \( W_i \) which is adjacent to \( v \). For every vertex \( x \in (S_i \cup U_i \cup W_i) \), let \( e^x \) and \( \overline{e}^x \) be the two edges incident to \( x \). If \( x \in (U_i \cup W_i) \), we suppose \( e^x \) is incident to some vertex in \( V_i \), while \( \overline{e}^x \) is incident to some vertex in \( V_{i-1} \). For \( X = S, U, W \), let \( E^X_i = \{ e^x \mid x \in X_i \} \), \( \overline{E}^X_i = \{ \overline{e}^x \mid x \in X_i \} \) and \( E^X_i = E^X_i \cup \overline{E}^X_i \).

Respect to a labeling \( f \) on \( E(G_s) \), if \( v \in V_i \), we denote the partial sum at \( v \) (omitting the label on \( e^{w(v)} \)) by \( p(v) = \sum_{e \in E(v)} \{ e^{w(v)} \} f(e) = \varphi_f(v) - f(e^{w(v)}) \). Let \( p(v^*) = \varphi_f(v^*) - f(e^*) \) where \( e^* \) is the edge in \( E(v^*) \) which receives the greatest label among \( E(v^*) \).

Note that \( V(G_s) = V \cup S \cup U \cup W \cup \{v^*\} \). To show \( G_s \) is antimagic, we will construct a labeling \( f \) which satisfies the following conditions.

1. The vertex sums in \( X_i \) are all odd and pairwise different, for \( X \in \{ S, U, W \} \) and \( i \in [1, \alpha] \).
2. The vertex sums in \( U_i \) are all even and pairwise different for \( i \in [1, \alpha] \).
3. The vertex sums in \( S_i \cup U_i \cup W_i \) are smaller than those in \( S_{i-1} \cup U_{i-1} \cup W_{i-1} \) for \( i \in [2, \alpha] \).
4. The vertex sums in \( S_i \) are smaller than those in \( U_i \), while the later ones are smaller than those in \( W_i \) for \( i \in [1, \alpha] \).
5. The vertex sums in \( V_i \) are smaller than those in \( V_{i-1} \) for \( i \in [2, \alpha] \).
6. The vertex sum at \( v^* \) is greater than those in \( V_i \) and those in \( W_i \).

Conditions (1) and (2) make sure there is no conflict between \( V \) and \( S \cup U \cup W \). Conditions (1), (3), (4) make sure there is no conflict inside \( S \cup U \cup W \). Conditions (2) and (5) make sure there is no conflict inside \( V \). Conditions (3), (4), (5) and (6) make sure there is no conflict between \( v^* \) and any other vertex in \( G_s \). So these conditions imply that \( f \) is antimagic.

Note that \( E(G_s) = \bigcup_{i=1}^\alpha (E^S_i \cup E^U_i \cup E^W_i) \). We will label \( E(G_s) \) in the order \( E^S_\alpha \), \( (E^U_\alpha \cup E^W_\alpha) \), \( E^S_{\alpha-1} \), \( (E^U_{\alpha-1} \cup E^W_{\alpha-1}) \), ..., \( E^S_1 \), \( (E^U_1 \cup E^W_1) \), using the smallest unused labels on each edge set when we come to it. This label assignment immediately implies that (3) holds, and that the vertex sums in \( S_i \) are smaller than those in \( (U_i \cup W_i) \) for \( i \in [1, \alpha] \).

Suppose \( i \in [1, \alpha] \) in the following. Note that \( |E^X_i| = 2|X_i| \), for \( X = S, U, W \).
(I) The labeling of $E^S$. We first label $E^S_i$ arbitrarily using the $|S_i|$ odd labels from the $2|S_i|$ assigned labels for $E^S_i$. Secondly let $f(v^e) = f(e^s) + 1$ for each $s \in S_i$. Then the vertex sums in $S_i$ are odd and pairwise different.

(II) The labeling of $(E^U \cup E^W)$. If $|U_i|$ is odd, then $i \in [2, \alpha]$, since $U_1$ is an empty set. We will label $(E^U_i \cup E^W_i)$ in the order $E^U_i$, $E^U_i$, $E^W_i$ using the smallest unused assigned labels on each edge subset when we come to it. This sub-assignment (based on our global assignment), gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, which implies $\varphi_f(v) = p(v) + f(vw(v)) < p(v') + f(v'w(v')) = \varphi_f(v')$, since $f(vw(v)) < f(v'w(v'))$ by our global assignment. So (5) holds for those $i$ with $|U_i|$ being odd. It gives that the vertex sums in $U_i$ are smaller than those in $W_i$. So (4) holds for those $i$ with $|U_i|$ being odd. We first label $E^U_i$ arbitrarily using its assigned labels. Secondly let $f(v^w) = f(e^w) + |U_i|$ for each $u \in U_i$. This gives that the vertex sums in $U_i$ are odd and pairwise different. Third, suppose $V_i = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $e^{w(v_r)}$ with the $r$-th smallest label among the odd (even) assigned labels for $E^W_i$, when $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) holds for those $i$ with $|U_i|$ being odd. Fourth, let $f(v^e) = f(e^w) + 1$ when $f(e^w)$ is odd, while $f(v^e) = f(e^w) - 1$ when $f(e^w)$ is even. This implies that vertex sums in $W_i$ are odd and pairwise different. So (1) holds for those $i$ with $|U_i|$ being odd.

If $|U_i|$ is even ($|U_i|$ may equal to 0), then $i \in [1, \alpha]$. We will label edges in $E^U_i$ using the smallest $(2|U_i| + 1)$ assigned labels for $E^U_i \cup E^W_i$ except the $|U_i| + 1$-th smallest one (denoted by $\xi_{|U_i|+1}$). We first label the edges of $E^U_i$ arbitrarily using the $|U_i|$ smallest assigned labels. This gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$. And then, if $i \neq 1$, one has $\varphi_f(v) = p(v) + f(vw(v)) < p(v') + f(v'w(v')) = \varphi_f(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, since $f(vw(v)) < f(v'w(v'))$ by our global assignment. So (5) also holds for those $i$ ($i \neq 1$) with $|U_i|$ being even. Secondly let $f(v^e) = f(e^w) + |U_i| + 1$ for each $u \in U_i$. This implies that the vertex sums in $U_i$ are odd and pairwise different. It also implies that the vertex sums in $U_i$ are smaller than those in $W_i$, since any pair of the rest assigned labels left for $W_i$ has a sum greater than any vertex sum in $U_i$. So (4) also holds for those $i$ with $|U_i|$ being even. Note that, $\xi_{|U_i|+1}$ and $(\xi_{|U_i|+1} + |U_i| + 1)$ have distinct parity, and so far, they are the smallest two unused assigned labels for $W_i$. Third, suppose $V_i = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $e^{w(v_r)}$ with the $r$-th smallest label among the rest odd (even) assigned labels, if $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) also holds for those $i$ with $|U_i|$ being even. And note that either $\xi_{|U_i|+1}$ or $(\xi_{|U_i|+1} + |U_i| + 1)$ is assigned to $w(v_1)$ by our labeling way. Fourth, let $f(\pi^{w(v_1)}) = \xi_{|U_i|+1}$ if $f(e^{w(v_1)}) = \xi_{|U_i|+1} + |U_i| + 1$, while $f(\pi^{w(v_1)}) = \xi_{|U_i|+1} + |U_i| + 1$ if
And for $r \in [2, |V_i|]$, let $f(\tau^{w(v_i)}) = f(\xi^{w(v_i)}) + 1$ if $f(\xi^{w(v_i)})$ is odd, while $f(\tau^{w(v_i)}) = f(\xi^{w(v_i)}) - 1$ if $f(\xi^{w})$ is even. This implies that vertex sums in $W_i$ are odd and pairwise different. So (1) also holds for those $i$ with $|U_i|$ being even.

For (6), note that the process of the labeling of $E(v^*) = E^W_1$ is discussed in the case when $|U_1|$ is even (since $U_1 = \emptyset$ and $|U_1| = 0$). Recall that, $|E^W_1| = 2k$ and $E^W_i$ are assigned with the greatest $2k$ labels, i.e., those labels in $L_{2k} = \{|E(G_s)|, |E(G_s)| - 1, \ldots, |E(G_s)| - 2k + 1\}$. More precisely, $E^W_1 = E(v^*)$ are assigned with the labels in $\{i_1, i_2, \ldots, i_k\} \subseteq L_{2k}$ where either $i_j = |E(G_s)| - 2j + 1$ or $i_j = |E(G_s)| - 2j + 2$ for $j = 1, 2, \ldots, k$. So $p(v^*) \geq p(v_1) + 1 + 3 + \ldots + (2k - 3) > p(v_1) + 3$ for arbitrary $v_1 \in V_1$ (recall that $k \geq 3$). Then $\varphi_f(v^*) = p(v^*) + f(e^*) \geq p(v^*) + |E(G_s)| - 1 > p(v_1) + |E(G_s)| - 2 > p(v_1) + |E(G_s)| \geq p(v_1) + f(v_1 w(v_1)) = \varphi_f(v_1)$ for each $v_1 \in V_1$. On the other hand, $\varphi_f(v^*) \geq |E(G_s)| - 1 + |E(G_s)| - 3 + |E(G_s)| - 5 = 3|E(G_s)| - 9$, since $k \geq 3$. Thus, each vertex in $W_1$ receives a sum at most $(2|E(G_s)| - 1)$. So $\varphi_f(v^*) \geq 3|E(G_s)| - 9 > 2|E(G_s)| - 1 \geq \varphi_f(w_1)$ for each $w_1 \in W_1$ (one has $|E(G_s)| \geq 12$, because $k \geq 3$). So (6) holds.

Thus, $G_s$ is antimagic. This completes our proof. ■

It is interesting to consider the case when $G$ is $k$-regular ($k \geq 3$) but disconnected. In the proof of Theorem 2, suppose $G$ has $m$ edges. Then $G_s$ has $m$ 2-vertices. Note that the total sum of all the vertex sums is even, since each label contributes to the total sum twice. Thus, each 2-vertex contributes an odd value to the total sum, while each $k$-vertex other than $v^*$ contributes an even value, under our labeling way in the proof of Theorem 2. Thus, $\varphi_f(v^*)$ is odd if and only if $m$ is odd.

**Theorem 3.** Let $G$ be an disconnected $k$-regular ($k \geq 3$) graph, which has at most one connected component with an odd number of edges. Then $G_s$ is antimagic.

**Proof.** Suppose $G$ consists of the connected components $H_1, H_2, \ldots, H_\beta$ ($\beta \geq 2$), where $H_i$ has an even number of edges for each $i \in [1, \beta - 1]$. We can label $E(G_s)$ in the order $E((H_1)_s), E((H_2)_s), \ldots, E((H_\beta)_s)$ using the smallest unused labels on each edge set when we come to it. Next, we label each connected component of $G_s$ in the same way to that in Theorem 2, choosing a root for each component of $G$. Then there is no conflict among each $(H_i)_s$ for $i \in [1, \beta]$. Each 2-vertex receives an odd sum, while each $k$-vertex other than the root of $(H_\beta)_s$ receives an even sum. Each 2-vertex in $(H_i)_s$ receives a smaller sum than each 2-vertex in $(H_j)_s$, while each $k$-vertex in $(H_i)_s$ receives a smaller sum than each $k$-vertex in $(H_j)_s$, whenever $i < j \leq \beta$ holds. And the root vertex in $(H_\beta)_s$ receives a greater sum than those of any other vertex in $G_s$. So we obtain an antimagic labeling. ■

Since $m = \frac{nk}{2}$, for each $k$-regular graph with $n$ vertices and $m$ edges, we have the following corollary.
Corollary 4. Let $G$ be an disconnected $k$-regular ($k \geq 3$) graph. Then $G_s$ is antimagic if one of the following holds.

(1) $k = 4t$ ($t \geq 1$);

(2) $k$ is even and at most one of the connected components of $G$ has an odd number of vertices;

(3) At most one of the connected components of $G$ has a number of vertices which is not a multiple of 4.

3. $(k,pk)$-Biregular ($k \geq 3, p \geq 2$) Bipartite Graph

For a bipartite graph $G(A, B)$, a complete $p$-claw matching $CM$ from $A$ to $B$ is a set of edges of $G$ that induce a subgraph $G[CM]$ such that each vertex of $A$ in $G$ is also a vertex in $G[CM]$ and each component of $G[CM]$ is a copy of $K_{1,p}$ where the vertex of degree $p$ is in $A$, while the vertices of degree 1 are in $B$. For $A_0 \subseteq A$, denote by $N(A_0)$ the set of vertices in $B$ each of which has a neighbor in $A_0$. Let $E_1, E_2, \ldots, E_k \subseteq E(G)$ be disjoint edge sets. If $E_1 \cup E_2 \cup \cdots \cup E_k = E(G)$, then we say $G$ decomposes into $E_1, E_2, \ldots, E_k$.

Lemma 5 (An extended version of Hall’s theorem, [15, 16]). A bipartite graph $G[A, B]$ admits a complete $p$-claw matching from $A$ to $B$, if and only if $p|A_0| \leq |N(A_0)|$ for every subset $A_0$ of $A$.

Lemma 6. Let $G[A, B]$ be a $(k,pk)$-biregular ($k \geq 3, p \geq 2$) bipartite graph where the degree of each vertex in $A$ is $kp$, while each vertex in $B$ has degree $k$. Then $G$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Proof. Let $A_0 \subseteq A$. Let $G[A_0, N(A_0)]$ be the graph induced by $A_0 \cup N(A_0)$. Then each vertex of $A_0$ in $G[A_0, N(A_0)]$ has the degree $kp$, while each vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most $k$. So there are exactly $kp|A_0|$ edges in $G[A_0, N(A_0)]$. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number of edges in $G[A_0, N(A_0)]$ is less than $k \cdot p|A_0|$, a contradiction. So $|N(A_0)| \geq p|A_0|$. By Lemma 5, there exists a complete $p$-claw matching $CM_1$ from $A$ to $B$ in $G[A, B]$. Then $G_1 = G[A, B] - CM_1$ is a $(k-1, p(k-1))$-biregular bipartite graph. So we can use Lemma 5 repeatedly until we obtain a $(1,p)$-biregular bipartite graph $G_{k-1}$ which is also a complete $p$-claw matching from $A$ to $B$. Thus, $G[A, B]$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Lemma 7. Let $I = [i + 1, i + 2q]$. Then, there exist partitions $P_1$ (when $q$ is odd) and $\{P_2, P_3, P_4\}$ (when $q$ is even) of $I$, such that under $P_j$, $j \in [1, 4]$, $I$ is divided into $q$ parts where each part has 2 integers, integers in $[i + (x - 1)q + 1, i + xq]$ ($x \in [1, 2]$) are in distinct parts and the following conditions are satisfied.
Under $P_1$, the $q$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q - 1)/2, (2i + 2q + 1) + (q - 1)/2]$;

(2) Under $P_2$, $q/2$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q/2 - 1), 2i + 2q + 1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2i + 2q + 1, (2i + 2q + 1) + (q/2 - 1)]$;

(3) Under $P_3$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q + 2) - (q/2 - 1), (2i + 2q + 2) + (q/2 - 1)]$ and the other part has the sum $2i + q + 2$;

(4) Under $P_4$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q) - (q/2 - 1), (2i + 2q) + (q/2 - 1)]$ and the other part has the sum $2i + 3q$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $q$ is odd, let $\{2j - 1, -j + (3q + 1)/2 + 1\}$ be in the same partition for $j \in [1, (q + 1)/2]$, and let $\{2j, -j + 2q + 1\}$ be in the same partition for $j \in [1, (q - 1)/2]$, which is the desired partition $P_1$.

(2) If $q$ is even, let $\{2j, -j + 3q/2 + 1\}$ be in the same partition and let $\{2j - 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2]$, which is the desired partition $P_2$.

(3) If $q$ is even, let $\{2j, -j + 3q/2 + 2\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j + 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{1, q + 1\}$ be in the same partition, which is the desired partition $P_3$.

(4) If $q$ is even, let $\{2j - 1, -j + 3q/2 + 1\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j, -j + 2q\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{q, 2q\}$ be in the same partition, which is the desired partition $P_4$.

**Lemma 8.** Let $I = [i + 1, i + zq]$ ($z \geq 3$). Then, there exist partitions $P_1$ (when $z$ is even or $q$ is odd) and $P_2, P_3$ (when $z$ is odd and $q$ is even) of $I$, such that under $P_j$, $j \in [1, 3]$, $I$ is departed into $q$ parts where each part has $z$ integers, integers in $[i + (x - 1)q + 1, i + xq]$ ($x \in [1, z]$) are in distinct parts and the following conditions are satisfied.

(1) Under $P_1$, the $q$ parts have the same sum $(2i + zq + 1)z/2$;

(2) Under $P_2$, $q/2$ parts have the same sum $(2i + zq + 1)z/2 + 1/2$ and the other $q/2$ parts have the same sum $(2i + zq + 1)z/2 - 1/2$;

(3) Under $P_3$, $(q - 1)$ parts have the same sum $(2i + zq + 1)z/2 + 3/2$ and the other part has the sum $(2i + zq + 1)z/2 - 3q/2 + 3/2$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $z$ is even, let $\{(j - 1)q + l|j \in [1, z/2]\} \cup \{jq - l + 1|j \in [z/2 + 1, z]\}$ be in the partition for $l \in [1, q]$, which is the desired partition $P_1$ and (1) holds in this case.
If \( z \) is odd (then \((z - 3)\) is even) and \( q \) is odd, we first assign the \((z - 3)q\) integers in \([2q + 1, (z - 1)q]\) to the \( q \) parts (suppose \( I_1, I_2, \ldots, I_q \) are the \( q \) parts) such that these \( q \) parts receive the same partial sum \((qz + q + 1)(z - 3)/2\). We can do this since \((z - 3)\) is even. Second, assign \([(z - 1)q + l]\) to \( I_l \) for \( l \in [1, q]\) such that the \( q \) parts have distinct partial sums and attain all values in \([(qz + q + 1)(z - 3)/2 + (z - 1)q + 1, (qz + q + 1)(z - 3)/2 + qz]\). Third, partition \([1, 2q]\) into \( q \) parts (denoted by \( I_{1}', I_{2}', \ldots, I_q'\)) which have distinct sums which attain all the values in \([(2q + 1) - (q - 1)/2, (2q + 1) + (q - 1)/2\]. We can do this owing to the partition in Lemma 7(1). Then assign \( I_{q+1}' \) to \( I_l \) if the sum of \( I_{q+1}' \) equals to \([(qz + q + 1)(z - 3)/2 + (z - 1)q + l + [(2q + 1) + (q - 1)/2 - l + 1] = (qz + 1)z/2\) for each \( l \in [1, q]\). So (1) also holds in this case.

(2) If \( z \) is odd and \( q \) is even, we first partition \([2q + 1, zq]\) into \( q \) parts \( I_1, I_2, \ldots, I_q \) which have distinct partial sums and attain all values in \([(qz + q + 1)(z - 3)/2 + (z - 1)q + 1, (qz + q + 1)(z - 3)/2 + qz]\). We can do this owing to the discussion in (1). Then partition \([1, 2q]\) into \( q \) parts (denoted by \( I_{1}', I_{2}', \ldots, I_q'\)) such that \( q/2 \) parts have distinct sums which attain all the values in \([(2q + 1) - (q/2 - 1), 2q + 1]\), while the other \( q/2 \) parts have distinct sums which attain all the values in \([2q + 1, (2q + 1) + (q/2 - 1)]\). We can do this owing to the partition in Lemma 7(2). Denote by \( I_{q/2+1}' \) and \( I_{q/2+2}' \) the two parts each of which admits the sum \((2q + 1)\). Then assign \( I_{q/2+1}' \) to \( I_l \) if the sum of \( I_{q/2+1}' \) equals to \([(2q + 1) + (q/2 - 1) - l + 1] \) for \( l \in [1, q/2 - 1]\). Assign \( I_{q/2+1}' \) to \( I_{q/2} \), while assign \( I_{q/2+2}' \) to \( I_{q/2+1}' \). And assign \( I_{q/2+1}' \) to \( I_l \) if the sum of \( I_{q/2+1}' \) equals to \((2q + 1) + (q/2 - 1) - l + 2\) for \( l \in [q/2 + 2, q]\). Then for \( l \in [1, q/2 - 1]\) the final sum of \( I_l \) equals to \([(qz + q + 1)(z - 3)/2 + (z - 1)q + l + [(2q + 1) + (q/2 - 1) - l + 1] = (qz + 1)z/2 - 1/2\). The final sum of \( I_{q/2} \) equals to \([(qz + q + 1)(z - 3)/2 + (z - 1)q + q/2 + [(2q + 1)] = (qz + 1)z/2 - 1/2\), while the final sum of \( I_{q/2+1} \) equals to \([(qz + q + 1)(z - 3)/2 + (z - 1)q + q/2 + 1 + [(2q + 1)] = (qz + 1)z/2 + 1/2\). Thus, for \( l \in [q/2 + 2, q]\) the final sum of \( I_l \) equals to \([(qz + q + 1)(z - 3)/2 + (z - 1)q + l + [(2q + 1) + (q/2 - 1) - l + 2] = (qz + 1)z/2 + 1/2\). So (2) holds.

(3) If \( z \) is odd and \( q \) is even, we first partition \([2q + 1, zq]\) into \( q \) parts \( I_1, I_2, \ldots, I_q \) which have distinct partial sums and attain all values in \([(qz + q + 1)(z - 3)/2 + (z - 1)q + 1, (qz + q + 1)(z - 3)/2 + qz]\). We can do this owing to the discussion in (1). Then partition \([1, 2q]\) into \( q \) parts (denoted by \( I_{1}', I_{2}', \ldots, I_q'\)) such that \( (q - 1) \) parts have distinct sums which attain all the values in \([(2q + 2) - (q/2 - 1), 2q + 2 + (q/2 - 1)]\) and the other part has the sum \((q + 2)\). We can do this owing to the partition in Lemma 7(3). Denote by \( I_{q+1}' \) the part with the sum \((q + 2)\). Then assign \( I_{q+1}' \) to \( I_1 \), and assign \( I_{q+2}' \) to \( I_l \) if the sum of \( I_{q+2}' \) equals to \([(2q + 2) + (q/2 - 1) - l + 2] \) for \( l \in [2, q]\). Then the final sum of \( I_l \) equals to \([(qz + q + 1)(z - 3)/2 + (z - 1)q + l + [q + 2] = (qz + 1)z/2 - 3q/2 + 3/2\, and
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for \( l \in [2, q] \), the final sum of \( I_l \) equals to \( [(qz + q + 1)(z - 3)/2] + [(z - 1)q + l] + [(2q + 2) + (q/2 - 1) - l + 2] = (qz + 1)z/2 + 3/2 \). So (3) holds.

\[ \square \]

**Theorem 9.** Every \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph is antimagic.

**Proof.** Let \( G[A, B] \) be a \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph, where each vertex in \( A \) has the degree \( pk \), while each vertex in \( B \) has the degree \( k \).

Suppose \(|A| = n (n \geq k)\) and \(|B| = pn\). Let \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_{pn}\} \). By Lemma 6, \( G \) decomposes into \( k \) complete \( p \)-claw matchings \( CM_1, CM_2, \ldots, CM_k \) from \( A \) to \( B \). Denote by \( CM_i(V_0) \) \((i \in [1, k])\) the edges in \( CM_i \) which are incident to some vertex in \( V_0 \) for \( V_0 \subseteq V(G) \).

**Step 1.** Label \( \bigcup_{i=1}^{k-1} CM_i \) with \([1, (k - 1)pn]\).

First, label \( CM_{k-1} \) with \([(k - 2)pn + 1, (k - 1)pn] \), i.e., \([(k - 2)pn + 1, (k - 2)pn + pn] \) such that the following conditions are satisfied.

1. Within \( CM_{k-1} \), vertices in \( A \) have the same partial sum \([(2k - 3)pn + 1]p/2 \) if \( p \) is even or \( n \) is odd. We can do this owing to the partition in Lemma 8(1).
2. Within \( CM_{k-1} \), \( n/2 \) vertices in \( A \) have the same partial sum \([(2k - 3)pn + 1]p/2 + 1/2 \) and the other \( n/2 \) vertices in \( A \) have the same partial sum \([(2k - 3)pn + 1]p/2 - 1/2 \) if \( p \) is odd and \( n \) is even. We can do this owing to the partition in Lemma 8(2).

Second, based on the labeling to \( CM_{k-1} \), for each \( i \in [1, k - 2] \), label \( CM_i \) with \([(i - 1)pn + 1, ipn] \), such that the following conditions are satisfied.

1. Within \( \bigcup_{i=1}^{k-1} CM_i \), the vertices in \( B \) have the same partial sum \([(k - 1)pn + 1]p/2 \) if \( k - 1 \) even or \( pn \) is odd. We can do this owing to the partition in Lemma 8(1).
2. Within \( \bigcup_{i=1}^{k-1} CM_i \), \( (pn - 1) \) vertices in \( B \) have the same partial sum \([(k - 1)pn + 1](k - 1)/2 + 3/2 \) while the other vertex (denoted by \( b_0 \)) has the partial sum \([(k - 1)pn + 1](k - 1)/2 - 3pn/2 \) if \( (k - 1) \) is odd and \( pn \) is even. We can do this owing to the partition in Lemma 8(3).

Note that, (1.3) implies the vertices in \( B \) will receive distinct final vertex sums, when \( (k - 1) \) is even or \( pn \) is odd, if we label the rest edges \( CM_k \) using the rest labels \([(k - 1)pn + 1, kpn] \). Thus in (1.4), the partial sum of \( b_0 \) is at least \( 3pn/2 \) smaller than those of the vertices in \( (B \setminus \{b_0\}) \). So the final vertex sum of \( b_0 \) will still be smaller than those of the vertices in \( (B \setminus \{b_0\}) \), if we label \( CM_k \) with \([(k - 1)pn + 1, kpn] \). Hence, the final vertex sums of in \( (B \setminus \{b_0\}) \) will be pairwisely different. That is, all vertices in \( B \) will also receive distinct final vertex sums when \( (k - 1) \) is odd and \( pn \) is even.

**Step 2.** Label \( CM_k \) with \([(k - 1)pn + 1, kpn] \), i.e., \([(k - 1)pn + 1, (k - 1)pn + pn] \).

Suppose \( f_1(a_{i_1}) \leq f_1(a_{i_2}) \leq \cdots \leq f_1(a_{i_n}) \) where \( f_1(a_{i_j}) \) is the partial vertex sum of \( a_{i_j} \) within \( \bigcup_{i=1}^{k-1} CM_i \) for \( j \in [1, n] \).
(2.1) If \( p \) is odd (then \((p-1)\) is even) or \( n \) is odd, let \( \sigma(a) \) be an edge in \( CM_k(a) \) for each \( a \in A \). Label \([CM_k \setminus \left( \bigcup_{a \in A} \{ \sigma(a) \} \right)]\) with \([(k-1)pn+1, (k-1)pn+(p-1)n]\) such that, within \([CM_k \setminus \left( \bigcup_{a \in A} \{ \sigma(a) \} \right)]\), the vertices in \( A \) have the same partial sum \([(2k-1)pn-n+1](p-1)/2\). We can do this owing to the partition in Lemma 8(1). Next label \( \sigma(a_t) \) with \((kpm-n+j)\) for \( j \in [1,n] \). Then the vertex sums in \( A \) are pairwise different.

(2.2) If \( p \) is even (then \((p-2)\) is also even) and \( n \) is even, let \( \sigma_1(a) \) and \( \sigma_2(a) \) be two distinct edges in \( CM_k(a) \) for each \( a \in A \). Label \([CM_k \setminus \left( \bigcup_{a \in A} \{ \sigma_1(a), \sigma_2(a) \} \right)]\) using the labels in \([(k-1)pn+1, (k-1)pn+(p-2)n]\) such that, within \([CM_k \setminus \left( \bigcup_{a \in A} \{ \sigma_1(a), \sigma_2(a) \} \right)]\), the vertices in \( A \) have the same partial sum \([(2k-1)pn-2n+1](p-2)/2\). We can do this owing to the partition in Lemma 8(1). Then label \([\bigcup_{a \in A} \{ \sigma_1(a), \sigma_2(a) \}]\) with \([(kpm-2n)+1, (kpm-2n)+2n]\) such that \( f(\sigma_1(a_t))+f(\sigma_2(a_t))) = 2kpm-5n/2+j \) for \( j \in [1,n-1] \) while \( f(\sigma_1(a_{t_n}))+f(\sigma_2(a_{t_n})) = 2kpm-n \). We can do this owing to the partition in Lemma 7(4). Then the vertex sums in \( A \) are also pairwise different.

Recall that, owing to Step 1 (1.3) and (1.4), for each \( b \in B \), one has

\[
\varphi_f(b) \leq \frac{[(k-1)pn+1](k-1)}{2} + \frac{3}{2} + kpm.
\]

On the other hand, owing to the labeling way in Step 1 (1.3) and (1.4), the labels assigned to \( CM_i \) are those in \([(i-1)pn+1, ipn]\) for \( i \in [1,k-2] \). Let \( a \in A \). Then the sum of the labels in \( CM_i(a) \) is at least \( \sum_{j=1}^{p}[((i-1)pn+j] \) for \( i \in [1,k-2] \). So the sum of the labels in \( \left( \bigcup_{i=1}^{k-2} CM_i(a) \right) \) is at least \( \frac{1}{2}\sum_{i=1}^{k-2} \sum_{j=1}^{p}[(i-1)pn+j] \) Recall that, owing to Step 1 (1.1) and (1.2), the sum of labels in \( CM_k(a) \) is at least \( [(2k-3)pn+1]/2+(p-1)/2 \). Next recall that, owing to Step 2 (2.1), the sum of labels in \( CM_k(a) \) is at least \( [(2k-1)pn-n+1](p-1)/2+(kpm-n+1) \) if \( p \) is odd (then \((p-1)\) is even) or \( n \) is odd, while owing to Step 2 (2.2), the sum of labels in \( CM_k(a) \) is at least \( [(2k-1)pn-2n+1](p-2)/2+(2kpm-5n/2+1) \) if \( p \) is even (then \((p-2)\) is also even) and \( n \) is even. Thus, the later lower bound is \( 1/2 \) smaller than the first lower bound. So

\[
\varphi_f(a) \geq \sum_{i=1}^{k-2} \sum_{j=1}^{p} [(i-1)pn+j] + \left\{ \frac{[(2k-3)pn+1]}{2} - \frac{1}{2} \right\}
+ \left\{ \frac{[(2k-1)pn-2n+1](p-2)}{2} + \left( \frac{2kpm-5n}{2} + 1 \right) \right\}.
\]

Then for each \( a \in A \) and \( b \in B \), one has

\[
\varphi_f(a) - \varphi_f(b) \geq \frac{1}{2} \left[ \left( \frac{1}{2} k - 1 \right) p^2 kn + (k-3)p^2 + k^2 \left( \frac{1}{2} p - 1 \right) pn 
+ (p-1)(np+k) + (p^2 - 1) n + (p^2 - 3) \right] > 0,
\]
since \( k \geq 3 \) and \( p \geq 2 \).

Thus, we obtain an antimagic labeling. This completes our proof. \( \blacksquare \)

**Theorem 10.** Every \((k, k^2 + y)\)-biregular \((k \geq 3, y \geq 1)\) bipartite graph is antimagic.

**Proof.** Let \( G[A, B] \) be a \((k, k')\)-biregular \((k' = k^2 + y)\) bipartite graph, where each vertex in \( A \) has the degree \( k' \), while each vertex in \( B \) has the degree \( k \).

Suppose \(|A| = k\eta\) and \(|B| = k'\eta\) where \( \eta \) may be not an integer. It is sufficient to consider the case when \( k' = kp + r \) for some integers \( p \) and \( r \) satisfying \( p \geq k \) and \( 1 \leq r \leq k - 1 \) (note that \( r\eta \) is an integer since \( k\eta \) and \( k'\eta \) are integers).

Let \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_k\} \). For \( A_0 \subseteq A \), the graph \( G[A_0, N(A_0)] \) has \( k'\eta \) edges, since each vertex of \( A_0 \) in \( G[A_0, N(A_0)] \) has the degree \( k' \).

On the other hand, suppose \(|N(A_0)| < p|A_0|\). Then the number of edges in \( G[A_0, N(A_0)] \) is at most \( k|N(A_0)| < pk|A_0| < k'\eta \) since each vertex of \( N(A_0) \) in \( G[A_0, N(A_0)] \) has the degree at most \( k \), a contradiction.

So, by Lemma 5, \( G \) admits a complete \( p \)-claw matching \( CM \) from \( A \) to \( B \).

Suppose \( B = B_1 \cup B_2 \) where \( B_1 = V(CM) \cap B \) and \( B_2 = B \setminus B_1 \).

Then \(|B_1| = k\eta p\) and \(|B_2| = r\eta\). Let \( \sigma(b) \) be an edge incident to \( b \) for each \( b \in B_2 \), and let \( \sigma(B_2) = \{\sigma(b) | b \in B_2\} \).

**Step 1.** Label \((E(G) - CM - \sigma(B_2))\) with \( [1, (k - 1)k'\eta]\).

(1.1) If \((k - 1)\) is even or \(k'\eta\) is odd, label \((E(G) - CM - \sigma(B_2))\) with \([1, (k - 1)k'\eta]\) such that, within \((E(G) - CM - \sigma(B_2))\), the vertices in \( B \) have the same partial sum \(((k - 1)k'\eta + 1)(k - 1)/2\). We can do this owing to the partition in Lemma 8(1).

(1.2) If \((k - 1)\) is odd and \(k'\eta\) is even, label \((E(G) - CM - \sigma(B_2))\) with \([1, (k - 1)k'\eta]\) such that, within \((E(G) - CM - \sigma(B_2))\), the vertices in \( B \) have the same partial sum \[((k - 1)k'\eta + 1)(k - 1)/2 + 3/2\] except one (denoted by \(b_0\)) which equals to \[((k - 1)k'\eta + 1)(k - 1)/2 - 3k'\eta/2 + 3/2\]. We can do this owing to the partition in Lemma 8(3).

Note that (1.1) implies the final vertex sums in \( B \) will be pairwise different when \((k - 1)\) is even or \(k'\eta\) is odd, if we label the rest edges \((CM \cup \sigma(B_2))\) with the rest labels \([(k - 1)k'\eta + 1, kk'\eta]\). Then in (1.2), the partial sum of \(b_0\) is at least \(3k'\eta/2\) smaller than those of the vertices in \((B \setminus \{b_0\})\). So the final vertex sum of \(b_0\) will be smaller than those of the vertices in \((B \setminus \{b_0\})\), if we label \((CM \cup \sigma(B_2))\) with \([(k - 1)k'\eta + 1, kk'\eta]\). Next, the final vertex sums of in \((B \setminus \{b_0\})\) will be pairwise different. That is, vertices in \( B \) will also receive distinct final vertex sums, when \((k - 1)\) is odd and \(k'\eta\) is even.

**Step 2.** Label \( \sigma(B_2) \) with \([(k - 1)k'\eta + 1, (k - 1)k'\eta + r\eta]\) arbitrarily.
Step 3. Label $CM$ with $[(k - 1)k'\eta + r\eta + 1, kk'\eta]$, i.e., $[(kk'\eta - pk\eta) + 1, (kk'\eta - pk\eta) + r\eta + 1, kk'\eta]$. Suppose $f_1(a_{t_1}) \leq f_1(a_{t_2}) \leq \cdots \leq f_1(a_{t_m})$, where $f_1(a_{t_j})$ is the partial vertex sum of $a_{t_j}$ after Steps 1 and 2, for $j \in [1, kn]$.

(3.1) If $p$ is odd (then $(p - 1)$ is even) or $k\eta$ is odd, let $\sigma(a)$ be an edge in $CM(a)$ for each $a \in A$. Label $[CM \setminus \{\bigcup_{a \in A} \{\sigma(a)\}\}]$ with $[(kk'\eta - pk\eta) + 1, (kk'\eta - pk\eta) + (p - 1)kn]$ such that, within $[CM \setminus \{\bigcup_{a \in A} \{\sigma(a)\}\}]$, vertices in $A$ have the same vertex sum $2(kk'\eta - pk\eta) + (p - 1)kn + 1(p - 2)/2$. We can do this owing to the partition in Lemma 7(4). And label $\sigma(a_{t_j})$ with $(kk'\eta - k\eta + j)$ for $j \in [1, kn]$. Then vertex sums in $A$ are pairwise different.

(3.2) If $p$ is even (then $(p - 2)$ is also even) and $k\eta$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM(a)$ for each $a \in A$. Label $[CM \setminus \{\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}\}]$ with $[(kk'\eta - pk\eta) + 1, (kk'\eta - pk\eta) + (p - 2)\eta]$ such that, within $[CM \setminus \{\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}\}]$, vertices in $A$ have the same vertex sum $2(kk'\eta - pk\eta) + (p - 2)k\eta + 1(p - 2)/2$. We can do this owing to the partition in Lemma 7(1). Next, label $\{\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}\} with $[(kk'\eta - 2k\eta) + 1, (kk'\eta - 2k\eta) + 2k\eta]$ such that $f(\sigma_1(a_{t_j})) + f(\sigma_2(a_{t_j})) = 2kk'\eta - 5k\eta/2 + j$ for $j \in [1, kn - 1]$, while $f(\sigma_1(a_{t_{kn}})) + f(\sigma_2(a_{t_{kn}})) = 2kk'\eta - kn$. We can do this owing to the partition in Lemma 7(4). Then vertex sums in $A$ are also pairwise different.

Recall that, owing to Step 1 (1.1) and (1.2), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{((k - 1)k'\eta + 1)(k - 1)}{2} + \frac{3}{2} + kk'p.$$ 

On the other hand, let $a \in A$. Recall that, owing to Step 3 (3.1) the sum of the labels in $CM(a)$ is at least $\{(2kk'\eta - pk\eta) + (p - 1)k\eta + 1(p - 1)/2 + (kk'\eta - k\eta + 1)$ if $p$ is odd (then $(p - 1)$ is even) or $k\eta$ is odd. Then owing to Step 3 (3.2), the sum of the labels in $CM(a)$ is at least $\{2kk'\eta - pk\eta + (p - 2)k\eta + 1(p - 2)/2\} + (2kk'\eta - 5k\eta/2 + 1)$ if $p$ is even (then $(p - 2)$ is also even) and $k\eta$ is even. And the lower bound is $1/2$ smaller than the first lower bound. So

$$\varphi_f(a) > \frac{2kk'\eta - pk\eta + (p - 2)k\eta + 1(p - 2)}{2} + \left(\frac{2kk'\eta - 5k\eta/2 + 1}{2}\right).$$

Then for each $a \in A$ and $b \in B$ one has

$$\varphi_f(a) - \varphi_f(b) > \frac{1}{2}((p - k)kk'\eta + (k' - 2p - 1)pk\eta + (p^2 - 3)k\eta + (pk + k - k')\eta + (\eta - 1)k + (p - 2)] > 0,$$

since $p \geq k \geq 3$ and $2p + 1 < k' = pk + r \leq pk + k$.

Thus, we obtain an antimagic labeling. This completes our proof.
Acknowledgments

The authors would like to thank very much the anonymous referees for valuable suggestions, corrections and comments which results in a great improvement of the original manuscript. The first author is supported by NSFC (No. 11701195) and by the Scientific Research Funds of Huaqiao University (No. 16BS808).

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Received 12 August 2019
Revised 1 June 2020
Accepted 1 June 2020