THE TURÁN NUMBER FOR 4 · Sℓ

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Abstract

The Turán number of a graph $H$, denoted by $ex(n, H)$, is the maximum number of edges of an $n$-vertex simple graph having no $H$ as a subgraph. Let $S_ℓ$ denote the star on $ℓ + 1$ vertices, and let $k \cdot S_ℓ$ denote $k$ disjoint copies of $S_ℓ$. Erdős and Gallai determined the value $ex(n, k \cdot S_1)$ for all positive integers $k$ and $n$. Yuan and Zhang determined the value $ex(n, k \cdot S_2)$ and characterized all extremal graphs for all positive integers $k$ and $n$. Recently, Lan et al. determined the value $ex(n, 2 \cdot S_3)$ for all positive integers $n$, and Li and Yin determined the values $ex(n, k \cdot S_ℓ)$ for $k = 2, 3$ and all positive integers $ℓ$ and $n$. In this paper, we further determine the value $ex(n, 4 \cdot S_ℓ)$ for all positive integers $ℓ$ and almost all $n$, improving one of the results of Lidický et al.

Keywords: Turán number, disjoint copies, $k \cdot S_ℓ$.

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1. Introduction

Graphs in this paper are finite and simple. Terms and notation not defined here are from [1]. Let $S_ℓ$ denote the star on $ℓ + 1$ vertices and let $P_ℓ$ denote the path on $ℓ$ vertices. For a graph $G$ and a vertex $v \in V(G)$, the degree of $v$ in $G$ is the number of edges incident to $v$, is denoted by $d_G(v)$, and the set of neighbors of $v$
in $G$ is denoted by $N_G(v)$. Moreover, we define $N_G[v] = N_G(v) \cup \{v\}$. The vertex with degree $\ell$ in $S_\ell$ is called the center of $S_\ell$. For a set $S$ by $|S|$ we denote the cardinality of $S$. Clearly, $d_G(v) = |N_G(v)|$. For graphs $G$ and $H$, $G \cup H$ denotes the disjoint union of $G$ and $H$, $p \cdot G$ denotes the disjoint union of $p$ copies of $G$, and $G \vee H$ denotes the join of $G$ and $H$, that is, the graph obtained from $G \cup H$ by joining each vertex of $G$ to each vertex of $H$. For $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$.

The Turán number $ex(n, H)$ of the graph $H$ is the maximum number of edges of an $n$-vertex simple graph having no $H$ as a subgraph. Let $H_{ex}(n, H)$ denote a graph on $n$ vertices with $ex(n, H)$ edges not containing $H$. We call this graph an extremal graph for $H$. Let $T_r(n)$ denote the complete $r$-partite graph on $n$ vertices in which all parts are as equal in size as possible. Turán [9] determined the value $ex(n, K_{r+1})$ and showed that $T_r(n)$ is the unique extremal graph for $K_{r+1}$, where $K_{r+1}$ is the complete graph on $r + 1$ vertices. Turán’s theorem is regarded as the basis of a significant branch of graph theory known as extremal graph theory. It was shown by Simonovits [8] that if $n$ is sufficiently large, then $K_{p-1} \cup T_r(n-p+1)$ is the unique extremal graph for $p \cdot K_{r+1}$. Gorgol [3] further considered the Turán number for $p$ disjoint copies of any connected graph $T$ on $t$ vertices and gave a lower bound for $ex(n, p \cdot T)$ by simply counting the number of edges of the graphs $H_{ex}(n-2t+1, T) \cup K_{pt-1}$ and $H_{ex}(n-p+1, T) \cup K_{p-1}$ which do not contain $p \cdot T$.

**Theorem 1** [3]. Let $T$ be an arbitrary connected graph on $t$ vertices, $p$ be an arbitrary positive integer and $n$ be an integer such that $n \geq pt$. Then $ex(n, p \cdot T) \geq \max \{ex(n-2t+1, T) + \binom{p-1}{2}, ex(n-p+1, T) + (p-1)n - \binom{p}{2}\}$.

Lidický et al. [7] investigated the Turán number of a star forest (a forest whose connected components are stars), and determined the value $ex(n, F)$ for sufficiently large $n$, where $F = S_{d_1} \cup S_{d_2} \cup \cdots \cup S_{d_k}$ and $d_1 \geq d_2 \geq \cdots \geq d_k$. Lidický et al. [7] also pointed out that they make no attempt to minimize the bound on $n$ in their proof. Yin and Rao [10] improved the result of Lidický et al. by determining the value $ex(n, k \cdot S_\ell)$ for $n \geq \frac{1}{2} \ell^2 k(k-1)+k-2+\max\{\ell k, \ell^2+2\ell\}$. Lan et al. [4] further improved these results by determining the value $ex(n, k \cdot S_\ell)$ for $n \geq k(\ell^2 + \ell + 1) - \frac{\ell}{2}(\ell - 3)$. However, there are very few cases when the Turán number $ex(n, k \cdot S_\ell)$ is known exactly for all positive integers $k$, $\ell$ and $n$. Erdős and Gallai [2] determined $ex(n, k \cdot S_1)$ for all positive integers $k$ and $n$. Yuan and Zhang [11] determined $ex(n, k \cdot S_2)$ (i.e., $ex(n, k \cdot P_3)$) and characterized all extremal graphs for all positive integers $k$ and $n$. Lan et al. [4] determined $ex(n, 2 \cdot S_3)$ for all positive integers $n$. Li and Yin [6] determined $ex(n, k \cdot S_\ell)$ for $k = 2, 3$ and all positive integers $\ell$ and $n$. Recently, Lan et al. [5] studied the degree powers for forbidding star forests, which is a classical generalization of the Turán number for star forests.
The Turán Number for $4 \cdot S_\ell$

**Theorem 2** [2].

$$ ex(n, k \cdot S_1) = \begin{cases} \binom{n}{2}, & \text{if } n < 2k, \\ \binom{2k-1}{2}, & \text{if } 2k \leq n < \frac{5k}{2} - 1, \\ \binom{k-1}{2} + (n - k + 1)(k - 1), & \text{if } n \geq \frac{5k}{2} - 1. \end{cases} $$

**Theorem 3** [11].

$$ ex(n, k \cdot S_2) = \begin{cases} \binom{n}{2}, & \text{if } n < 3k; \\ \binom{3k-1}{2} + \left\lfloor \frac{n-3k+1}{2} \right\rfloor, & \text{if } 3k \leq n < 5k - 1, \\ \binom{k-1}{2} + (n - k + 1)(k - 1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor, & \text{if } n \geq 5k - 1. \end{cases} $$

Furthermore, all extremal graphs for $k \cdot S_2$ are characterized.

**Theorem 4** [4].

$$ ex(n, 2 \cdot S_3) = \begin{cases} \binom{n}{2}, & \text{if } n < 8, \\ n + 14, & \text{if } 8 \leq n < 16, \\ 2(n - 1), & \text{if } n \geq 16. \end{cases} $$

**Theorem 5** [6].

$$ ex(n, 2 \cdot S_\ell) = \begin{cases} \binom{n}{2}, & \text{if } n < 2(\ell + 1), \\ \left\lfloor \frac{(\ell-1)n+(\ell+1)(\ell+1)}{2} \right\rfloor, & \text{if } 2(\ell + 1) \leq n < (\ell + 1)^2, \\ \left\lfloor \frac{(\ell+1)n-(\ell+1)}{2} \right\rfloor, & \text{if } n \geq (\ell + 1)^2. \end{cases} $$

**Theorem 6** [6].

$$ ex(n, 3 \cdot S_\ell) = \begin{cases} \binom{n}{2}, & \text{if } n < 3(\ell + 1), \\ \left\lfloor \frac{(\ell-1)n+(3\ell+2)(2\ell+2)}{2} \right\rfloor, & \text{if } 3(\ell + 1) \leq n < \frac{3\ell^2+6\ell+4}{2}, \\ \left\lfloor \frac{(\ell+3)n-2(\ell+2)}{2} \right\rfloor, & \text{if } n \geq \frac{3\ell^2+6\ell+4}{2}. \end{cases} $$

In this paper, we further determine the Turán number $ex(n, 4 \cdot S_\ell)$ for all positive integers $\ell$ and almost all $n$.

**Theorem 7.**

$$ ex(n, 4 \cdot S_\ell) = \begin{cases} \binom{n}{2}, & \text{if } n < 4(\ell + 1), \\ \left\lfloor \frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rfloor, & \text{if } 5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3, \\ \left\lfloor \frac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor, & \text{if } n \geq 2\ell^2 + 4\ell + 3. \end{cases} $$
2. Proof of Theorem 7

For \( \ell = 1 \) and 2, Theorem 7 follows from Theorems 2–3 (the case \( k = 4 \)). Assume \( \ell \geq 3 \). Note that the extremal graph \( K_n \) gives the lower and upper bounds for \( ex(n, 4 \cdot S_\ell) \) in the case \( n \leq 4\ell + 3 \). Thus, we consider only the case \( n \geq 5(\ell + 1) \).

Denote \( f(\ell, n) = \max \left\{ \left\lceil \frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rceil, \left\lfloor \frac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor \right\} \). Clearly,

\[
f(\ell, n) = \begin{cases} \left\lceil \frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rceil, & \text{if } 5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3, \\ \left\lfloor \frac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor, & \text{if } n \geq 2\ell^2 + 4\ell + 3. \\
\end{cases}
\]

The lower bound \( ex(n, 4 \cdot S_\ell) \geq f(\ell, n) \) follows from \( ex(n, S_\ell) = \left\lfloor n\frac{\ell-1}{2} \right\rfloor \) and Theorem 1. To show the upper bound, we assume that \( G \) is a graph on \( n \geq 5(\ell + 1) \) vertices with \( e(G) \geq f(\ell, n) + 1 \) and \( G \) contains no \( 4 \cdot S_\ell \) as a subgraph. The degree sequence of \( G \) is denoted by \( (d_1, d_2, \ldots, d_n) \), where \( d_1 \geq d_2 \geq \cdots \geq d_n \). By \( \frac{(\ell+5)n-3(\ell+3)}{2} \geq \frac{(\ell+3)n+2(\ell+1)-(3\ell+3)}{2} \), we can see that

\[
e(G) \geq \max \left\{ \left\lceil \frac{(\ell-1)n+(3\ell+2)(2\ell+2)}{2} \right\rceil, \left\lfloor \frac{(\ell+3)n-2(\ell+2)}{2} \right\rfloor \right\}.
\]

It follows from Theorem 6 that \( G \) contains three disjoint copies of \( S_\ell \), denoted \( F_1, F_2 \) and \( F_3 \). For convenience, we let \( V(F_i) = \{v_{i0}, v_{i1}, \ldots, v_{i\ell}\} \) and \( E(F_i) = \{v_{i0}v_{i1}, v_{i0}v_{i2}, \ldots, v_{i0}v_{i\ell}\} \), for \( i = 1, 2, 3 \). Denote \( H = G \setminus (V(F_1) \cup V(F_2) \cup V(F_3)) \), and \( H' = G[V(F_1) \cup V(F_2) \cup V(F_3)] \). We first have the following Claims 1–4.

**Claim 1.** \( d_3 \geq 2\ell + 3 \).

**Proof.** Note that \( G - S_\ell \) contains no \( 3 \cdot S_\ell \). Let \( m_0 \) be the number of edges incident to \( S_\ell \) in \( G \). Thus, we have

\[
m_0 = e(G) - e(G - S_\ell) \geq e(G) - ex(n - \ell - 1, 3 \cdot S_\ell).
\]

If \( n \geq 2\ell^2 + 4\ell + 3 \), then \( n - \ell - 1 \geq 2\ell^2 + 3\ell + 2 \geq \frac{3\ell^2 + 6\ell + 4}{2} \). By Theorem 6, we have

\[
m_0 \geq \frac{(\ell+5)n-3(\ell+3)}{2} + 1 - \frac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \\
\geq \frac{(\ell+5)n-3(\ell+3)-1}{2} + 1 - \frac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \\
= \frac{2n+\ell^2+3\ell-1}{2} \geq \frac{2(2\ell^2+4\ell+3)+\ell^2+3\ell-1}{2} = \frac{5(\ell+1)^2+\ell}{2}.
\]

Assume \( 5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3 \), that is, \( 4(\ell + 1) \leq n - \ell - 1 < 2\ell^2 + 3\ell + 2 \).
If $4(\ell + 1) \leq n - \ell - 1 < \frac{3\ell^2 + 6\ell + 4}{2}$, by Theorem 6, then we have

$$m_0 \geq \frac{(\ell - 1)n + (4\ell + 3)(3\ell + 3)}{2} + 1 - \frac{(\ell - 1)(n - \ell - 1) + (3\ell + 2)(2\ell + 2)}{2}$$

$$\geq \frac{(\ell - 1)n + (4\ell + 3)(3\ell + 3) - 1}{2} + 1 - \frac{(\ell - 1)(n - \ell - 1) + (3\ell + 2)(2\ell + 2)}{2}$$

$$= \frac{7\ell^2 + 11\ell + 5}{2} \geq \frac{5(\ell + 1)^2}{2} + \ell.$$

If $\frac{3\ell^2 + 6\ell + 4}{2} \leq n - \ell - 1 < 2\ell^2 + 3\ell + 2$, by Theorem 6, then we have

$$m_0 \geq \frac{(\ell - 1)n + (4\ell + 3)(3\ell + 3)}{2} + 1 - \frac{(\ell + 3)(n - \ell - 1) - 2(\ell + 2)}{2}$$

$$\geq \frac{(\ell - 1)n + (4\ell + 3)(3\ell + 3) - 1}{2} + 1 - \frac{(\ell + 3)(n - \ell - 1) - 2(\ell + 2)}{2}$$

$$= \frac{13\ell^2 + 27\ell + 17 - 4n}{2} \geq \frac{13\ell^2 + 27\ell + 17 - 4(2\ell^2 + 4\ell + 3)}{2} = \frac{5(\ell + 1)^2 + \ell}{2}.$$ 

Hence each $S_\ell$ must contain a vertex of degree at least

$$\frac{m_0}{\ell + 1} \geq \frac{5(\ell + 1)^2 + \ell}{2(\ell + 1)} \geq 2\ell + 3.$$ 

This implies that $G$ contains three vertices of degree at least $2\ell + 3$, which proves Claim 1. □

**Claim 2.** $d_4 \geq \ell + 3$.

**Proof.** If $d_4 \leq \ell + 2$, then $e(G) \leq \frac{3(n-1)+(\ell+2)(n-3)}{2} = \frac{(\ell+5)n-3(\ell+3)}{2} < f(\ell, n) + 1$, a contradiction, which proves Claim 2. □

**Claim 3.** If $1 \leq |N_H(v_{i_0})| \leq \ell$ for some $i \in \{1, 2, 3\}$, then $|N_H(v_{i_j})| \leq \ell$ for all $j \in \{1, \ldots, \ell\}$.

**Proof.** Assume $|N_H(v_{i_j})| \geq \ell + 1$ for some $j \in \{1, \ldots, \ell\}$. Let $v \in N_H(v_{i_0})$; we can find an $S_\ell$ in $G[(V(F_i) \setminus \{v_{i_j}\}) \cup \{v\}]$ whose center is $v_{i_0}$. By $|N_H(v_{i_j}) \setminus \{v\}| \geq \ell + 1 - 1 = \ell$, we can find another $S_\ell$ in $G[N_H(v_{i_j}) \setminus \{v\}]$ whose center is $v_{i_j}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 3. □

**Claim 4.** If $|N_H(v_{i_0})| \geq \ell + 1$ for some $i \in \{1, 2, 3\}$, then $|N_H(v_{i_j})| \leq \ell - 1$ for all $j \in \{1, \ldots, \ell\}$.

**Proof.** If $|N_H(v_{i_j})| \geq \ell$ for some $j \in \{1, \ldots, \ell\}$, then we can find an $S_\ell$ in $G[N_H[v_{i_j}]]$ whose center is $v_{i_j}$. This $S_\ell$ is denoted by $F$. Let $v \in N_H(v_{i_0}) \setminus V(F)$; we can find another $S_\ell$ in $G[(V(F_i) \setminus \{v_{i_j}\}) \cup \{v\}]$ whose center is $v_{i_0}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 4. □

We consider the following two cases in terms of the value of $d_1$. 

Case 1. $d_1 \geq 4\ell + 3$. If $d_2 \geq 3\ell + 3$, by Claims 1–2, then $G$ contains $4 \cdot S_\ell$. Hence $d_2 \leq 3\ell + 2$. By Claim 1, we may take $v_{i0}$ to be the vertex with degree $d_i$, for $i = 1, 2, 3$. Denote $H_1 = G \setminus V(F_3)$.

**Claim 5.** $|N_{H_1}(v_{j3})| \leq 2\ell + 2$ for all $j \in \{1, \ldots, \ell\}$.

**Proof.** Assume $N_{H_1}(v_{j3}) \geq 2\ell + 3$ for some $j \in \{1, \ldots, \ell\}$. By Claim 1, $|N_G(v_{j0}) \setminus (\{v_{j1}, \ldots, v_{j3}\} \cup V(F_2) \cup \{v_{10}\})| \geq d_3 - \ell - (\ell + 1) - 1 \geq 2\ell + 3 - (2\ell + 2) = 1$. Let $v \in N_G(v_{j0}) \setminus (\{v_{j1}, \ldots, v_{j3}\} \cup V(F_2) \cup \{v_{10}\})$, we can find the first $S_\ell$ (denoted $F$) in $G(V(F_3) \setminus \{v_{j3}\} \cup \{v\})$ whose center is $v_{j0}$. By $|N_G(v_{j3}) \setminus (V(F_2) \cup \{v_{10}, v\})| \geq 2\ell + 3 - (\ell + 1 + 1 + 1) = \ell$; we can find the second $S_\ell$ (denoted $F'$) in $G[N_H[v_{j3}] \setminus (V(F_2) \cup \{v_{10}, v\})]$ whose center is $v_{j3}$. By $|N_G(v_{j0}) \setminus (V(F_2) \cup V(F) \cup V(F'))| \geq d_1 - 3(\ell + 1) \geq 4\ell + 3 - 3\ell - 3 = \ell$, we can find the third $S_\ell$ in $G[N_G[v_{10}] \setminus (V(F_2) \cup V(F) \cup V(F'))]$ whose center is $v_{10}$. Thus $G$ contains $4 \cdot S_\ell$ if we view $F_2$ as the fourth $S_\ell$, a contradiction which proves Claim 5. □

Now by $|N_{H_1}(v_{j0})| = |N_G(v_{j0}) \setminus \{v_{j1}, \ldots, v_{j3}\}| = d_3 - \ell \leq 3\ell + 2 - \ell = 2\ell + 2$ and Claim 5, we have

$$e(H_1) = e(G) - e(G[V(F_3)]) - |N_{H_1}(v_{j0})| - \sum_{j=1}^\ell |N_{H_1}(v_{j3})|$$

$$\geq e(G) - \frac{(\ell+1)\ell}{2} - (2\ell + 2) - (2\ell + 2)\ell = e(G) - \frac{5\ell^2 + 9\ell + 4}{2}.
$$

If $5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3$, i.e., $4(\ell + 1) \leq n - \ell - 1 < 2\ell^2 + 3\ell + 2$, then

$$e(H_1) \geq \frac{(\ell-1)n + (4\ell + 3)(3\ell + 1)}{2} + 1 - \frac{5\ell^2 + 9\ell + 4}{2} \geq \frac{(\ell-1)n + 5\ell^2 + 10\ell + 5}{2},$$

However, since $H_1$ contains no 3-$S_\ell$, we have that if $4(\ell + 1) \leq n - \ell - 1 < \frac{3\ell^2 + 6\ell + 4}{2}$, by Theorem 6, then $e(H_1) \leq ex(n - \ell - 1, 3 \cdot S_\ell) = \left[\frac{(\ell-1)(n-\ell-1) + (3\ell + 2)(2\ell + 2)}{2}\right] = \left[\frac{(\ell-1)n + 5\ell^2 + 10\ell + 5}{2}\right]$, a contradiction; and if $\frac{3\ell^2 + 6\ell + 4}{2} \leq n - \ell - 1 < 2\ell^2 + 3\ell + 2$, by Theorem 6, then

$$e(H_1) \leq ex(n - \ell - 1, 3 \cdot S_\ell) = \left[\frac{(\ell+3)(n-\ell-1-2\ell+2)}{2}\right] = \left[\frac{(\ell-1)n + 4n - \ell^2 - 6\ell - 7}{2}\right],$$

a contradiction.

If $n \geq 2\ell^2 + 4\ell + 3$, i.e., $n - \ell - 1 \geq 2\ell^2 + 3\ell + 2 \geq \frac{3\ell^2 + 6\ell + 4}{2}$, then

$$e(H_1) \geq \frac{(\ell+5)n - 3(\ell+3)}{2} + 1 - \frac{5\ell^2 + 9\ell + 4}{2} \geq \frac{(\ell+5)n - 3(\ell+3) - 1}{2} + 1 - \frac{5\ell^2 + 9\ell + 4}{2}$$

$$= \frac{(\ell+3)n + 2\ell^2 - 12\ell - 12}{2} \geq \frac{(\ell+3)n + 2(2\ell^2 + 4\ell + 3) - 5\ell^2 - 12\ell - 12}{2} = \frac{(\ell+3)n - \ell^2 - 4\ell - 6}{2}.$$
However, \( e(H_1) \leq ex(n-\ell-1, 3\cdot S_\ell) = \left[ \frac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \right] = \left[ \frac{(\ell+3)n^2-6\ell-7}{2} \right] \), a contradiction.

**Case 2.** \( d_1 \leq 4\ell + 2 \).

**Case 2.1.** \( d_3 \geq 3\ell + 3 \). Let \( v_{i0} \) be the vertex with degree \( d_i \) for \( i = 1, 2, 3 \), and let \( \{v_{31}, \ldots, v_{3\ell}\} \subseteq N_G(v_{30}), \{v_{21}, \ldots, v_{2\ell}\} \subseteq N_G(v_{20}) \setminus \{v_{30}, v_{31}, \ldots, v_{3\ell}\} \) and \( \{v_{11}, \ldots, v_{1\ell}\} \subseteq N_G(v_{10}) \setminus \{v_{20}, v_{21}, \ldots, v_{2\ell}, v_{30}, v_{31}, \ldots, v_{3\ell}\} \).

We take \( F_i \) to be the graph with \( V(F_i) = \{v_{i0}, v_{i1}, \ldots, v_{i\ell}\} \) and \( E(F_i) = \{v_{i0}v_{i1}, v_{i0}v_{i2}, \ldots, v_{i0}v_{i\ell}\} \) for \( i = 1, 2, 3 \). Then \( F_i \) is the \( S_\ell \) whose center is \( v_{i0} \) for \( i = 1, 2, 3 \). Moreover, \( |N_H(v_{i0})| \geq d_i - (3\ell + 2) \geq 1 \) for all \( i \in \{1, 2, 3\} \). Let \( I = \{i \mid i \in \{1, 2, 3\} \text{ and } 1 \leq |N_H(v_{i0})| \leq \ell\}; \ J = \{1, 2, 3\} \setminus I; \ A = \bigcup_{i \in I} V(F_i) \), \( B = \bigcup_{i \in J} V(F_i) \), \( B_1 = \{v \in B \setminus \{v_{10}, v_{20}, v_{30}\} \text{ and } 1 \leq |N_H(v)| \leq \ell - 1\} \) and \( B_2 = B \setminus (B_1 \cup \{v_{10}, v_{20}, v_{30}\}) \). Clearly, \( |A| = (\ell + 1)|I|, |I| + |J| = 3 \) and \( |B_1| + |B_2| = \ell|J| \). By Claim 4, \( |N_H(v)| = 0 \) for \( v \in B_2 \).

**Claim 6.** If \( v \in B_1 \), then \( d_{H'}(v) \leq 3\ell + 1 \), where \( H' = G[V(F_1) \cup V(F_2) \cup V(F_3)] \).

**Proof.** We may assume \( v = v_{ij} \) for some \( i \in J \) and some \( j \in \{1, \ldots, \ell\} \). If \( d_{H'}(v_{ij}) = 3\ell + 2 \), let \( u \in N_H(v_{ij}) \), then we can find an \( S_\ell \) in \( G[\{u\} \cup (V(F_i) \setminus \{v_{i0}\})] \) whose center is \( v_{ij} \). By \( |N_H(v_{i0})| \setminus \{u\} |\geq \ell + 1 - 1 = \ell \), we can find another \( S_\ell \) in \( G[N_H[v_{i0}] \setminus \{u\}] \) whose center is \( v_{i0} \). Therefore, \( G \) contains \( 4 \cdot S_\ell \), a contradiction. This proves Claim 6.

Now by \( |N_H(v_{i0})| \leq |N_G(v_{i0})| \setminus \{v_{i1}, \ldots, v_{i\ell}\} \) \( d_1 - \ell \leq 4\ell + 2 - \ell = 3\ell + 2 \) for \( i \in J, \ell \geq 3 \) and Claims 3, 4 and 6, we have

\[
e(H) = e(G) - e(H') - \sum_{i=1}^{3} \sum_{j=0}^{\ell} |N_H(v_{ij})|
\]

\[
= e(G) - \sum_{v \in A} d_{H'}(v) + \sum_{v \in B_1} d_{H'}(v_0) + \sum_{v \in B_2} d_{H'}(v) - \sum_{v \in B_1} |N_H(v_0)| - \sum_{v \in B_2} |N_H(v)| - \sum_{v \in B_1} |N_H(v_0)| - \sum_{v \in B_2} |N_H(v)|
\]

\[
ge \geq e(G) - \frac{(3\ell+2)|A|+(\sum_{i \in J}(d_i-|N_H(v_{i0})|)+(3\ell+1)|B_1|+(3\ell+2)|B_2|}{2}
\]

\[
ge \geq e(G) - \frac{(3\ell+2)|A|+(\sum_{i \in J}(d_i+|N_H(v_{i0})|)+(5\ell-1)|B_1|+(3\ell+2)|B_2|}{2}
\]

\[
ge \geq e(G) - \frac{(5\ell+2)(\ell+1)|I|+(4\ell+2+3\ell+2)|J|+(5\ell-1)(|B_1|+|B_2|)}{2}
\]

\[
ge \geq e(G) - \frac{(5\ell^2+7\ell+2)|I|+(5\ell^2+7\ell+2)|J|}{2} = e(G) - \frac{15\ell^2+21\ell+6}{2}.
\]
If $5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3$, then
\[
e(H) \geq \left(\frac{\ell-1)n+(4\ell+3)(3\ell+3)}{2}\right) + 1 - \frac{1}{2}(15\ell^2 + 21\ell + 6) \\
\geq \frac{(\ell-1)n+(4\ell+3)(3\ell+3)-1}{2} + 1 - \frac{1}{2}(15\ell^2 + 21\ell + 6) = \frac{(\ell-1)n-3\ell^2+4}{2}.
\]

However, since $H$ contains no $S_\ell$, by $ex(n, S_\ell) = \left\lceil \frac{n(\ell-1)}{2} \right\rceil$, then $e(H) \leq ex(n - 3\ell - 3, S_\ell) = \left\lceil \frac{(\ell-1)n-3\ell^2+3}{2} \right\rceil$, a contradiction.

If $n \geq 2\ell^2 + 4\ell + 3$, then
\[
e(H) \geq \left(\frac{(\ell+5)n-3(\ell+3)}{2}\right) + 1 - \frac{1}{2}(15\ell^2 + 21\ell + 6) \\
\geq \frac{(\ell+5)n-3(\ell+3)-1}{2} + 1 - \frac{1}{2}(15\ell^2 + 21\ell + 6) \\
= \frac{(\ell-1)n+6\ell-15\ell^2-24\ell-14}{2} \\
\geq \frac{(\ell-1)n+6(2\ell^2+4\ell+3)-15\ell^2-24\ell-14}{2} = \frac{(\ell-1)n-3\ell^2+4}{2}.
\]

However, $e(H) \leq ex(n - 3\ell - 3, S_\ell) = \left\lceil \frac{(\ell-1)n-3\ell^2+3}{2} \right\rceil$, a contradiction.

**Case 2.2.** $d_3 \leq 3\ell + 2$. If $d_1 \geq 3\ell + 3$, by Claim 1, we take $F_1, F_2$ and $F_3$ to be the same as Case 2.1. Clearly, $d_G(v) \leq d_3 \leq 3\ell + 2$ for all $v \in V(H') \setminus \{v_{10}, v_{20}, v_{30}\}$. This implies that $d_H(v) \leq 3\ell + 1$ for all $v \in V(H') \setminus \{v_{10}, v_{20}, v_{30}\}$.

Let $I = \{i \mid i \in \{1, 2, 3\}$ and $|N_H(v_0)| \geq \ell + 1\}$, $J = \{1, 2, 3\} \setminus I$, $A = \bigcup_{i \in I} V(F_i)$, $A_1 = A \setminus \{v_{10}, v_{20}, v_{30}\}$, $B = \bigcup_{i \in J} V(F_i)$, $B_1 = \{v \mid v \in B \setminus \{v_{10}, v_{20}, v_{30}\}\}$ and $|N_H(v)| \geq 2\ell - 1$ and $B_2 = B \setminus (B_1 \cup \{v_{10}, v_{20}, v_{30}\})$. Clearly, $|A_1| = \ell |I|$, $|B_2| = |J| - |B_1|$ and $|I| + |J| = 3$.

**Claim 7.** If $|N_H(v_0)| = 0$ for some $i \in \{1, 2, 3\}$, and $|N_H(v_j)| \geq 2\ell - 1$ for some $j \in \{1, \ldots, \ell\}$, then $|N_H(v_{ij'})| \leq \ell - 2$ for all $j' \in \{1, \ldots, \ell\} \setminus \{j\}$.

**Proof.** If $|N_H(v_{ij'})| \geq \ell - 1$ for some $j' \in \{1, \ldots, \ell\} \setminus \{j\}$, let $\{u_1, \ldots, u_{\ell-1}\} \subseteq N_H(v_{ij'})$, then we can find an $S_\ell$ in $G[\{u_1, \ldots, u_{\ell-1}\} \cup \{v_0, v_{ij'}\}]$ whose center is $v_{ij'}$. By $|N_H(v_{ij}) \setminus \{u_1, \ldots, u_{\ell-1}\}| \geq 2\ell - 1 - (\ell - 1) = \ell$, we can find another $S_\ell$ in $G[N_H(v_{ij}) \setminus \{u_1, \ldots, u_{\ell-1}\}]$ whose center is $v_{ij}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 7.

**Claim 8.** $|B_1| \leq |J|$.

**Proof.** Let $i \in J$. If $1 \leq |N_H(v_0)| \leq \ell$, by Claim 3, then $|N_H(v_j)| \leq \ell$ for all $j \in \{1, \ldots, \ell\}$, implying that $|N_H(v)| < 2\ell - 1$ for all $v \in V(F_i)$. If $|N_H(v_0)| = 0$, by Claim 7, then $F_i$ contains at most one vertex, say $v$, with $|N_H(v)| \geq 2\ell - 1$. Thus $|B_1| \leq |J|$. This proves Claim 8.
Now by $|N_H(v_{i0})| \leq |N_G(v_{i0}) \setminus \{v_{1i}, \ldots, v_{\ell i}\}| \leq d_1 - \ell \leq 3\ell + 2$ for $i \in I$, $\ell \geq 3$ and Claims 4 and 8, we have

$$e(H) = e(G) - e(H') - \frac{3}{\ell} \sum_{i=1}^{\ell} |N_H(v_{ij})|$$

$$= e(G) - \frac{\sum_{i=1}^{\ell} d_H(v_{i0}) + \sum_{v \in A_1} d_H(v) + \sum_{v \in B_1} d_H(v) + \sum_{v \in B_2} d_H(v)}{2}$$

$$- \sum_{i \in I} |N_H(v_{i0})| - \sum_{v \in A_1} |N_H(v)| - \sum_{i \in J} |N_H(v_{i0})| - \sum_{v \in B_1} |N_H(v)| - \sum_{v \in B_2} |N_H(v)|$$

$$\geq e(G) - \frac{1}{2} \left( \sum_{i \in I} (d_1 - |N_H(v_{i0})|) + \sum_{v \in A_1} (d_G(v) - |N_H(v)|) + (3\ell + 2)|J| \right.$$

$$+ \sum_{v \in B_1} (d_G(v) - |N_H(v)|) + \sum_{v \in B_2} (d_G(v) - |N_H(v)|) \bigg)$$

$$- \sum_{i \in I} |N_H(v_{i0})| - \sum_{v \in A_1} |N_H(v)| - \ell|J| - \sum_{v \in B_1} |N_H(v)| - \sum_{v \in B_2} |N_H(v)|$$

$$= e(G) - \frac{1}{2} \left( \sum_{i \in I} (d_1 + |N_H(v_{i0})|) + \sum_{v \in A_1} (d_G(v) + |N_H(v)|) + (5\ell + 2)|J| \right.$$

$$+ \sum_{v \in B_1} (d_G(v) + |N_H(v)|) + \sum_{v \in B_2} (d_G(v) + |N_H(v)|) \bigg)$$

$$\geq e(G) - \frac{1}{2} \left( (4\ell^2 + 2 + 3\ell + 2)|I| + (3\ell + 2 + \ell - 1)|A_1| + (5\ell + 2)|J| \right.$$

$$+ (3\ell + 2 + 3\ell + 1)|B_1| + (3\ell + 2 + 2\ell - 2)|B_2| \bigg)$$

$$= e(G) - \frac{(4\ell^2 + 8\ell + 4)|I| + (5\ell^2 + 5\ell + 2)|J| + (\ell + 3)|B_1|}{2}$$

$$\geq e(G) - \frac{(5\ell^2 + 7\ell + 2)|I| + (5\ell^2 + 7\ell + 2)|J|}{2} \geq e(G) - \frac{(5\ell^2 + 11\ell + 6)}{2}.$$

If $5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3$, then $e(H) \geq \frac{(\ell - 1)n - 3\ell^2 + 4}{2} + 1 - \frac{1}{2}(15\ell^2 + 21\ell + 6) \geq \frac{(\ell - 1)n - 3\ell^2 + 4}{2}$. However, $e(H) \leq e_x(n - 3\ell - 3, S_\ell) = \frac{(\ell - 1)n - 3\ell^2 + 3}{2}$, a contradiction. If $n \geq 2\ell^2 + 4\ell + 3$, then $e(H) \geq \frac{(\ell + 5)n - 3(\ell^2 + 3)}{2} + 1 - \frac{1}{2}(15\ell^2 + 21\ell + 6) \geq \frac{(\ell - 1)n - 3\ell^2 + 4}{2}$. However, $e(H) \leq e_x(n - 3\ell - 3, S_\ell) = \frac{(\ell - 1)n - 3\ell^2 + 3}{2}$, a contradiction.

Thus, we have proved that every graph $G$ on $n \geq 5(\ell + 1)$ vertices with $e(G) \geq f(\ell, n) + 1$ contains $4 \cdot S_\ell$ as a subgraph. In other words, $ex(n, 4 \cdot S_\ell) \leq f(\ell, n)$. The proof of Theorem 7 is completed.

**Remark.** The general case $ex(n, k \cdot S_\ell)$ seems to be much more challenging. The method presented here cannot be used to determine $ex(n, k \cdot S_\ell)$ for all positive
integers $k$, $\ell$ and $n$. The proofs of Claims 2–4 can be adapted to the general $k$, but the proofs of the remaining parts cannot be extended to the general case $k$.

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