DECOMPOSING 10-REGULAR GRAPHS INTO PATHS OF LENGTH 5

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Abstract

Let $G$ be a 10-regular graph which does not contain any 4-cycles. In this paper, we prove that $G$ can be decomposed into paths of length 5, such that every vertex is a terminal of exactly two paths.

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1. Introduction

Graphs in this paper are simple. Let $G$ and $H$ be graphs. We say that $G$ has an $H$-decomposition $D = \{H_1, H_2, \ldots, H_n\}$, if any two elements of $D$ are edge-disjoint subgraphs of $G$, $H_i$ ($1 \leq i \leq n$) is isomorphic to $H$ and $E(G) = \bigcup_{i=1}^{n} H_i$.

For convenience, we use $P_m$ and $C_m$ to denote the path and cycle with $m$ edges, respectively. For a positive integer $r$, an $r$-factor of $G$ is a spanning subgraph $F$ of $G$ such that $d_F(v) = r$ for each vertex $v$ of $G$. A decomposition $F$ of $G$ is an $r$-factorization if every element of $F$ is an $r$-factor, any two elements of $F$ are edge-disjoint subgraphs of $G$, and $E(G)$ can be covered by $F$.

Graham and Hägkkvist [6] posed the following conjecture.

Conjecture 1 (Graham-Hägkkvist [6]). Let $T$ be a tree with $l$ edges. If $G$ is a $2l$-regular graph, then $G$ admits a $T$-decomposition.
In the same paper, Häggkvist proved that Conjecture 1 is true when the girth of $G$ is at least the diameter of $T$. In the past several decades, this conjecture interested many researchers and many related results were presented. Fink [5] stated that if $T$ is any tree having $n$ edges ($n \geq 1$), then the $n$-cube $Q_n$ can be decomposed into $2^{n-1}$ edge-disjoint induced subgraphs, each of which is isomorphic to $T$. Erde [4] confirmed that if $n$ is odd and $k \leq n$ such that $k|n2^{n-1}$, then $Q_n$ can be decomposed into paths of length $k$. In [7], Jacobson, Truszczyński and Tuza proved that: (1) there is a wide class of $r$-regular bipartite graphs that can be decomposed into any tree of size $r$; (2) every $r$-regular bipartite graph can be decomposed into any double star of size $r$; (3) every 4-regular bipartite graph can be decomposed into paths of length 4. As one corollary of main result in [8], Jao, Kostochka and West confirmed Conjecture 1 for a $2l$-regular graph which has a 2-factorization such that every cycle consisting of edges from distinct 2-factors has length greater than the diameter of $T$. El-Zanati et al. [3] verified Conjecture 1 when $T$ is a double-star, and further they proved that the double-star $S_{k,k-1}$ can decompose every $2k$-regular graph which contains a perfect matching.

It is natural to ask whether Conjecture 1 holds if $T$ is a path. Unfortunately, there is no definitive answer for general graphs. Kouider and Lonc [9] proposed a strengthening of Conjecture 1 in the case where $T$ is a path, and it is still open. We say a path decomposition $D$ is balanced if each vertex is a terminal of exactly two paths of $D$.

**Conjecture 2** (Kouider and Lonc [9]). Let $l$ be a positive integer. If $G$ is a $2l$-regular graph, then $G$ admits a balanced $P_l$-decomposition.

By Petersen’s Factorization Theorem (see Theorem 3.1), Botler et al. [1] proposed an equivalent form of Conjecture 2.

**Conjecture 3** (Botler et al. [1]). Let $m$ and $l$ be positive integers. Then every $2ml$-regular graph admits a balanced $P_l$-decomposition.

In the same paper, they proved that if $m \geq \lceil (l - 2)/(g - 2) \rceil$, then every $2ml$-regular graph with girth at least $g$ admits a $P_l$-decomposition. Furthermore, every $2ml$-regular graph with girth at least $l - 1$ admits a $P_l$-decomposition for $m \geq 1$. By controlling the girth, Kouider and Lonc [9] confirmed Conjecture 2 for a $2l$-regular graph $G$ with girth at least $(l + 3)/2$.

**Theorem 4** (Kouider and Lonc [9]). If $l \leq 2g - 3$, then every $2l$-regular graph $G$ of girth $g$ has a balanced $P_l$-decomposition.

By Theorem 4, Conjecture 2 is true for $l = 1, 2$ and 3. When $l = 4$ or 5, every $2l$-regular graph $G$ without triangles has a balanced $P_l$-decomposition. For later use, we will present a short proof when $l = 3$ in Conjecture 2 in Section 3. Based on analysis of the structure of the graph, Botler and Talon [2] used a different method from that in [9] to confirm the conjecture for $l = 4$. 
Theorem 5 (Botler and Talon [2]). If $G$ is an 8-regular graph, then $G$ admits a balanced $P_4$-decomposition.

Motivated by Theorem 5, we want to solve the case $l = 5$. However, the structure of a $P_5$-decomposition in a 10-regular graph is more complex than the structure of a $P_4$-decomposition in an 8-regular graph. Thus we consider $P_5$-decompositions of 10-regular graphs which contain no 4-cycles, and get the main result of this paper.

Theorem 6. Let $G$ be a 10-regular graph. If $G$ does not contain any 4-cycles, then $G$ admits a balanced $P_5$-decomposition.

2. Notations and Terminologies

A trail $T = x_0x_1 \cdots x_l$ is a graph for whose $V(T) = \{x_i | 0 \leq i \leq l\}$, $E(T) = \{x_ix_{i+1} | 0 \leq i \leq l-1\}$ and $x_ix_{i+1} \neq x_jx_{j+1}$, for every $i \neq j$. Denote the vertices $x_0$ and $x_l$ as the terminal vertices of $T$, $x_1$ and $x_{l-1}$ as the preterminal vertices of $T$. If a trail has $l$ edges, then we call it an $l$-trail. If a set of edge-disjoint trails $B$ of a graph $G$ is such that $\bigcup_{B \in B} E(B) = E(G)$, then $B$ is a decomposition of $G$ into trails. If every trail of $B$ has length $l$, then $B$ is a decomposition into $l$-trails (or an $l$-trail decomposition). For a trail decomposition $B$ of $G$, if every vertex of $G$ is a terminal of exactly two trails of $B$, then $B$ is called balanced. If every trail of $B$ is a path, then $B$ is a decomposition into paths (or a path decomposition). We use $\tau(B)$ to denote the number of elements of $B$ that are cycles.

A tour of a connected graph $G$ is a closed walk that traverses each edge of $G$ at least once, and an Eulerian tour one that traverses each edge exactly once. A graph is Eulerian if it admits an Eulerian tour. Since an Eulerian tour traverses each edge exactly once, $d(v)$ is even for every vertex $v \in V(G)$. On the other hand, if $G$ is a connected graph and every vertex has even degree, then $G$ has an Eulerian tour by Fleury’s Algorithm. A graph in which each vertex has even degree is called an even graph. Therefore, a graph is Eulerian if and only if it is even and connected. An orientation $O$ of a subset $E'$ of $E(G)$ is an attribution of a direction to each edge of $E'$. If an edge $xy$ is directed from $x$ to $y$ in $O$, we say that $xy$ leaves $x$ and enters $y$. For a vertex $v$ of $G$, let $d^+_O(v)$ (respectively, $d^-_O(v)$) be the number of edges leaving (respectively, entering) $v$ with respect to $O$. If $O$ is an orientation of $G$ and every vertex $v$ has $d^+_O(v) = d^-_O(v)$, then $O$ is an Eulerian orientation of $G$. It is easy to see that $G$ is even if it has an Eulerian orientation. If $G$ is even, then each of its components has an Eulerian tour. We can get an Eulerian orientation of $G$ by assigning each edge of $G$ an orientation in such a way that the Eulerian tour of each component of $G$ is a directed Eulerian tour. Thus a graph has an Eulerian orientation if and only if it is even.
3. Proof of Main Theorem

First of all, we will present Petersen’s Factorization Theorem [10].

**Theorem 7** (Petersen’s 2-Factorization Theorem [10]). Every 2k-regular graph admits a 2-factorization.

Nextly, we will introduce a approach used in [2] to get a trial decomposition. By adjusting this decomposition, we finally get the desired result.

Let \( G \) be an \( r \)-regular graph (\( r \geq 6 \) and is even), \( F \) be a 2-factorization of \( G \) given by Theorem 7. By combining the elements of \( F \), we obtain a decomposition of \( G \) into an \((r - 4)\)-factor and a 4-factor, say \( F_1 \) and \( F_2 \), respectively. Let \( O \) be an Eulerian orientation of \( F_2 \). Suppose \( F_1 \) has a balanced \( P_{(r-4)/2} \)-decomposition \( D \). So every vertex \( v \) of \( G \) is a terminal of exactly two paths in \( D \). Note that \( d_O^+ (v) = 2 \) for every vertex \( v \) of \( F_2 \). Thus, we can extend every path \( P = x_1x_2 \cdots x_{(r-4)/2+1} \) in \( D \) to a \((D, O)\)-extension \( Q_P = x_0x_1 \cdots x_{(r-4)/2+2} \) such that \( x_0x_1 \) and \( x_{(r-4)/2+1}x_{(r-4)/2+2} \) are two edges in \( F_2 \) leaving \( x_1 \) and \( x_{(r-4)/2+1} \), respectively, and further every edge of \( F_2 \) is used exactly once. Therefore, \( \{Q_P \mid P \in D \} \) is a decomposition into \((D, O)\)-extensions of \( G \), which may not be a decomposition into paths, just into trails. Obviously, each decomposition into \((D, O)\)-extensions is balanced.

In this paper, we focus on the path decompositions of a 10-regular graph which does not contain any 4-cycles. Let \( F_1 \) be a 6-factor of \( G \), \( F_2 \) be a 4-factor such that \( F_1 \cup F_2 = G \). \( O \) be an Eulerian orientation of \( F_2 \). By Theorem 4, \( F_1 \) has a balanced \( P_3 \)-decomposition \( D \). Following the method above, we first obtain a decomposition into \((D, O)\)-extensions of \( G \) from \( D \), and then adjust this trail decomposition to a path decomposition of \( G \).

Let \( G \) be a 6-regular graph. We present a brief proof that \( G \) has a balanced \( P_3 \)-decomposition.

**Lemma 8.** If \( G \) is a 6-regular graph, then \( G \) admits a balanced \( P_3 \)-decomposition \( D \) and every vertex of \( G \) is a preterminal of exactly two paths in \( D \).

**Proof.** Let \( F \) be a 2-factorization of \( G \) given by Theorem 7. By combining the elements of \( F \), we obtain a decomposition of \( G \) into a 2-factor and a 4-factor, say \( F_3 \) and \( F_4 \), respectively. Obviously, \( F_3 \) has a balanced \( P_1 \)-decomposition, denoted by \( D_1 \). Because every vertex of \( F_4 \) has even degree, there is an Eulerian orientation \( O \) on \( F_4 \). Let \( D \) be a decomposition of \( G \) into \((D_1, O)\)-extensions which minimizes \( \tau(D) \). If every element in \( D \) is a \( P_3 \), then we are done. Suppose there is a triangle \( C = x_0x_1x_2x_3 \) in \( D \), \( x_0 = x_3 \), \( x_1x_2 \in D_1 \). There is an element \( T = y_0y_1y_2y_3 \) of \( D \) such that \( y_1 = x_1 \), \( y_1y_2 \in D_1 \), \( y_1y_2 \neq x_1x_2 \) and \( T \neq C \). Let \( C' = y_0x_1x_2x_3 \) and \( T' = x_0y_1y_2y_3 \). Obviously, \( C' \) is a path of length 3. Because, \( G \) is simple, \( y_1, y_2 \) and \( y_3 \) are distinct vertices, \( x_0 \neq y_1 \) and \( x_0 \neq y_2 \). If \( T' \)}
is a triangle, then \( y_3 = x_0 = x_3 \) and \( d_O(x_0) = 3 \), which is a contradiction to the assumption before that \( O \) is an Eulerian orientation on \( F_1 \). Hence, \( T' \) is a path of length 3. Let \( D' = (D - \{T,C\}) \cup \{T',C'\} \). \( D' \) is a decomposition of \( G \) into \((D_1,O)\)-extensions and \( \tau(D') \leq \tau(D) - 1 \), which is a contradiction to the minimality of \( \tau(D) \). Therefore, \( D \) is a balanced \( P_3 \)-decomposition of \( G \). By the construction of \( D \), we can find that every vertex of \( G \) is a preterminal of exactly two paths in \( D \).

![Figure 1. Extensions.](image)

Now, let \( G \) be a 10-regular graph without a \( C_4 \), \( F_1 \) be a 6-factor of \( G \), \( F_2 \) be a 4-factor such that \( F_1 \cup F_2 = G \). Let \( O \) be an Eulerian orientation of \( F_2 \) and \( D_1 \) be a balanced \( P_3 \)-decomposition of \( F_1 \), and further, \( T = \{Q_P \mid P \in D_1\} \) be a decomposition into \((D_1,O)\)-extensions of \( G \). Let \( T = x_0x_1x_2x_3x_4x_5 \in T \).

Because \( D_1 \) is a balanced \( P_3 \)-decomposition of \( F_1 \) and \( G \) does not contain any \( C_4 \), we have \( x_1, x_2, x_3, x_4 \) are distinct vertices, \( x_0 \neq x_4, x_5 \neq x_1 \) and it is impossible that both \( x_0 = x_3 \) and \( x_5 = x_2 \) hold. Hence, if \( T \) is a trail of \( T \), then exactly one of the following holds: (a) \( T \) is a path of length 5; (b) \( T \) is a trail of length 5 which contains a triangle; (c) \( T \) is a cycle of length 5 (see Figure 1). In the figures throughout this section, we illustrate the edges of \( F_1 \) as straight edges, and the edges of \( F_2 \) as dashed edges. The next result shows that every 10-regular graph admits a decomposition into \((D_1,O)\)-extensions which are not cycles.

**Lemma 9.** Let \( G \) be a 2l-regular graph, \( F_1 \) be a 2\((l-2)\)-factor of \( G \), \( F_2 = G \setminus E(F_1) \) and \( O \) be an Eulerian orientation of \( F_2 \). If there is a balanced \( P_{(l-2)} \)-decomposition \( D_1 \) of \( F_1 \), then \( G \) admits a decomposition into \((D_1,O)\)-extensions which are not cycles.

**Proof.** Let \( G, F_1, F_2, D_1, \) and \( O \) be as in the statement above. Now, let \( D \) be a decomposition of \( G \) into \((D_1,O)\)-extensions which minimizes \( \tau(D) \).

Suppose, for contradiction, that \( \tau(D) > 0 \). Let \( T = x_0x_1x_2 \cdots x_{l-1}x_l \) be a cycle of length \( l \) in \( D \), where \( L_1 = x_1x_2 \cdots x_{l-1} \in D_1 \) and \( x_0 = x_l \). Note that \( D_1 \) is balanced. Let \( L_2 = y_1y_2 \cdots y_{l-1} \) be the element of \( D_1 \) such that \( L_2 \neq L_1 \) and \( y_1 = x_1 \). Suppose \( Q = y_0y_1y_2 \cdots y_{l-1}y_l \) is the \((D_1,O)\)-extension of \( L_2 \) in \( D \). Let \( T' = y_0x_1x_2 \cdots x_{l-1}x_l \) and \( Q' = x_0y_1y_2 \cdots y_{l-1}y_l \). Clearly, \( T' \) and \( Q' \) are \((D_1,O)\)-extensions. Because \( G \) is simple, \( y_0 \neq x_l \). Hence, \( T' \) is not a cycle. Moreover, if \( Q' \) is a cycle, then the edges \( x_0x_1, x_{l-1}x_l \), and \( y_{l-1}y_l \) are directed
towards $x_0$, which implies $d^+_G(x_0) \geq 3$, contrary to the fact that $O$ is an Eulerian orientation of $F_2$. Therefore, $\mathcal{D} = (\mathcal{D} - \{T, Q\}) \cup \{T', Q'\}$ is a decomposition into $(\mathcal{D}_1, O)$-extensions of $G$ such that $\tau(\mathcal{D}') \leq \tau(\mathcal{D}) - 1$, which is a contradiction to the minimality of $\tau(\mathcal{D})$. This completes the proof of Lemma 9.

In the following, we will define a special Eulerian orientation, which is important for the proof of Theorem 6.

Definition 10. Let $G$ be a 10-regular graph, $F$ be a 6-factor of $G$, $\mathcal{D}$ be a balanced $P_3$-decomposition of $F$, $H = G \setminus E(F)$. We say that an Eulerian orientation $O$ on $H$ is good if the following holds. For each path $U = v_1v_2v_3$ of $H$ and distinct vertices $x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2, v_3$, if there exists three elements $T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}$ and $x_1 = v_1 = y_1, y_3 = v_2 = z_3, x_4 = v_3 = z_1$, then $U$ is a directed path under orientation $O$, no matter which direction it goes (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A good orientation on $U = v_1v_2v_3$.}
\end{figure}

Lemma 11. Let $G$ be a 10-regular graph without $C_4$, $F$ be a 6-factor of $G$, and $H = G \setminus E(F)$. Then, there is a good Eulerian orientation on the edges of $H$.

Proof. By Lemma 8, we assume that $\mathcal{D}$ is a balanced $P_3$-decomposition of $F$ such that every vertex of $G$ is a preterminal of exactly two paths in $\mathcal{D}$. Let $U = v_1v_2v_3$ be a path of length 2 in $H$. Because $G$ is simple and does not contain any 4-cycles, if there exists three elements $T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}$ and $x_1 = v_1 = y_1, y_3 = v_2 = z_3, x_4 = v_3 = z_1$, then $x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2$ and $v_3$ are distinct vertices. Hence, $U$ and $T_1, T_2, T_3$ form a structure defined in Definition 10. In order to obtain a good Eulerian orientation on $H$, we need to construct a new even graph $H'$ from $H$.

Let $\mathcal{U} = \{v_i^1v_i^2v_i^3 | 1 \leq i \leq k\}$ be the set of all the paths of length 2 in $H$ which are contained in the structure defined in Definition 10. Note that these paths in $\mathcal{U}$ are not necessarily edge-disjoint. We claim that $v_i^1 \neq v_j^1$ for $U_i = v_i^1v_i^2v_i^3, U_j = v_i^1v_j^2v_j^3 \in \mathcal{U}$ and $i \neq j$. If not, suppose that $v_i^1 = v_j^1$. Without loss of generality, let $U_i$ and three elements $T_1, T_2, T_3$ of $\mathcal{D}$ be contained in the structure depicted in Definition 10 such that $v_i^1, v_i^2 \in V(T_1), v_i^3, v_j^3 \in V(T_2), v_i^1, v_j^3 \in V(T_3)$. If $|E(U_i) \cap E(U_j)| = 1$, then without loss of generality let $v_i^1 = v_j^1$. This implies that there
are two paths $T_4$ and $T_5$ of $D$ (because $G$ is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ $(1 \leq m \leq 5)$, $T_k \neq T_q, k \in \{4, 5\}, q \in \{1, 2, 3\}$) together with $U_j$ and $T_1$ form another structure defined in Definition 10, such that $v_1^2, v_2^1 \in V(T_1)$, $v_3^2, v_2^1 \in V(T_4), v_1^1, v_3^3 \in V(T_5)$. If $|E(U_i) \cap E(U_j)| = 0$, then this implies that there are three paths $T_4, T_5$ and $T_6$ of $D$ (because $G$ is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ $(1 \leq m \leq 6)$, $T_k \neq T_q, k \in \{4, 5, 6\}, q \in \{1, 2, 3\}$) together with $U_j$ form another structure defined in Definition 10, such that $v_1^2, v_2^1 \in V(T_4)$, $v_3^2, v_2^1 \in V(T_5), v_1^1, v_3^3 \in V(T_6)$. In the two cases, $v_2^1$ appears in at least three paths in $D$ as their preterminal vertex, contrary to that $v_2^1$ is the preterminal vertex of exactly two paths in $D$. Thus, $v_2^1 \neq v_2^1$ when $i \neq j$, as claimed. This means that for every vertex $v$ of $G$, there is at most one $U_i \in U$ such that edges incident with $U_i$ is contained in the subgraph induced by $E_H(v)$ which is the set of $v$ in $H$.

Now we can split edges of $U_i$ in the following way: delete edges $v_1^2v_2^1$ and $v_2^1v_3^3$, add a new vertex $z_1$ and two edges $v_1^2z_1$ and $z_1v_3^3$. By operating on all elements in $U$ as described above, we can get a new graph $H'$ from $H$. Let $O'$ be an Eulerian orientation of $H'$ on $H$. By identifying $z_1$ and $v_2^1 (1 \leq i \leq k)$ in $H'$ and preserving the orientation of $O'$ on all edges after identifying, we get an Eulerian orientation $O$ on $H$. It is obviously that $O$ is good.

Now we are able to prove Theorem 6. For a 5-trail decomposition $B$ of a 10-regular graph $G$, we use $\tau'(B)$ to denote the number of elements of $B$ that are paths.

**Proof of Theorem 6.** Let $G$ be a 10-regular graph without $C_4$, $F$ be a 6-factor of $G$, $D$ be a balanced $P_3$-decomposition of $F$, $H = G \setminus E(F)$, and $O$ be a good Eulerian orientation of $H$. By Lemma 9, $G$ has a decomposition $\mathcal{B}$ into $(D, O)$-extensions which are not cycles. Further, we may assume that $\tau'(\mathcal{B})$ is maximum. If $\tau'(\mathcal{B}) = |\mathcal{B}|$, then we are done. Suppose that $\tau'(\mathcal{B}) < |\mathcal{B}|$. Let $T \in \mathcal{B}$ be a trail containing a triangle.

Let $T = x_0x_1x_2x_3x_4x_5$, where $x_1x_2x_3x_4 \in D$, $x_0 = x_3$. There is a trail $Q = y_0y_1y_2y_3y_4y_5 \in \mathcal{B}$ with $Q \neq T$ such that $y_1y_2y_3y_4 \in D$ and $y_1 = x_1$. We put $T' = y_0x_1x_2x_3x_4x_5$, $Q' = x_0y_1y_2y_3y_4y_5$. Because $G$ is simple and does not contain $C_4$, $y_0 \notin V(T)$, which implies that $T'$ is a path. Moreover, $x_0 \neq y_3$ which follows from the fact that $G$ does not contain $C_4$. Hence, if $Q'$ contains a triangle only if $Q$ contains the triangle $y_2y_3y_4y_5$. If $Q'$ is not a cycle, then $\mathcal{B}' = (\mathcal{B} \setminus \{T, Q\}) \cup \{T', Q'\}$ is a decomposition of $G$ into $(D, O)$-extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. In the following, we assume $Q'$ is a cycle.

Now, $y_5 = x_0 = x_3$. Note that $G$ is simple and does not contain any 4-cycles. We have that $y_1 \neq y_4, y_2$ and $y_3$ are not equal to any one of $\{x_1, x_2, x_3, x_4, x_5\}$, $y_4$ is not equal to any one of $\{x_1, x_2, x_3, x_4\}$. In this case, $y_4$ and $x_5$ may be the same one. Let $R = z_0z_1z_2z_3z_4z_5$ be an element in $\mathcal{B}$ different from $T$ and
If $z \in V$, then $G$ has a cycle of length 4, a contradiction. If $y \in V$, then $G$ has a cycle of length 4, a contradiction.

Hence, $R'$ is not a cycle. If $R$ contains a triangle, then $B' = (G - \{T, Q, R\}) \cup \{T', Q', R'\}$ is a decomposition of $G$ into $(D, O)$-extensions with $\tau'(B') \geq \tau'(B) + 1$, which is a contradiction to the maximality of $\tau'(B)$. In the following, we assume $R$ is a path. Because $G$ is simple and does not contain $C_4$, $y_5 \neq z_1, z_3$.

If $y_5 = z_2$, then let $U = x_1 x_0 y_4$, $T_1 = y_1 y_2 y_3 y_4$, $T_2 = x_1 x_2 x_3 x_4$ and $T_3 = z_4 z_3 z_2 z_1$. Now, we want to prove that $U, T_1, T_2$ and $T_3$ form the structure defined in the Definition 10. Note that $x_0 = x_3 = y_5 = z_2$, $x_1 = y_1$ and $y_4 = z_4$. Therefore, we should check $x_0, x_1, x_2, x_4, y_2, y_3, y_4, z_1$ and $z_3$ are distinct vertices of $G$. Because $G$ is simple, $x_0, x_1, x_2, x_4, y_4, z_1$ and $z_3$ are distinct vertices, $x_0, x_1, x_2, y_2$ and $y_4$ are distinct vertices, $x_2 \neq y_2$ and $y_3 \neq z_3$. What remains is the following cases. If $z_1 = y_2$ (respectively, $y_3$), then $z_1 x_1 x_2 x_3 z_1$ (respectively, $z_1 z_4 z_3 z_2 z_1$) is a cycle of length 4, a contradiction. If $z_4 = y_2$ (respectively, $y_3$), then $y_2 x_1 x_2 x_3 y_2$ (respectively, $y_3 y_2 y_1 x_3 y_3$) is a cycle of length 4, a contradiction. If $y_2 = z_3$, then $y_2 x_0 x_2 x_1 y_2$ is a cycle of length 4, a contradiction. If $y_3 = x_2$ (respectively, $z_1$), then $y_3 z_4 z_3 z_2 z_2$ (respectively, $z_1 z_4 z_3 z_2 z_1$) is a cycle of length 4, also a contradiction. Hence, $x_0, x_1, x_2, x_4, y_2, y_3, y_4, z_1$ and $z_3$ are distinct vertices of $G$, and $U, T_1, T_2$ and $T_3$ form the structure defined in the Definition 10. But the orientation of $E(U)$ implies that $O$ is not a good Eulerian orientation of $H$, a contradiction to our assumption. Hence, $R'$ is a path and $B' = (G - \{T, Q, R\}) \cup \{T', Q', R'\}$ is a decomposition of $G$ into $(D, O)$-extensions with $\tau'(B') \geq \tau'(B) + 1$, which is a contradiction to the maximality of $\tau'(B)$. This completes the proof of Theorem 6.

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References


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