ROMAN \(\{2\}\)-DOMINATION PROBLEM IN GRAPHS\(^1\)

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Abstract

For a graph \(G = (V,E)\), a Roman \(\{2\}\)-dominating function (R2DF) \(f : V \rightarrow \{0, 1, 2\}\) has the property that for every vertex \(v \in V\) with \(f(v) = 0\), either there exists a neighbor \(u \in N(v)\), with \(f(u) = 2\), or at least two neighbors \(x, y \in N(v)\) having \(f(x) = f(y) = 1\). The weight of an R2DF \(f\) is the sum \(f(V) = \sum_{v \in V} f(v)\), and the minimum weight of an R2DF on \(G\) is the Roman \(\{2\}\)-domination number \(\gamma_{\{R2\}}(G)\). An R2DF is independent if the set of vertices having positive function values is an independent set. The independent Roman \(\{2\}\)-domination number \(i_{\{R2\}}(G)\) is the minimum weight of an independent Roman \(\{2\}\)-dominating function on \(G\). In this paper, we show that the decision problem associated with \(\gamma_{\{R2\}}(G)\) is NP-complete even when restricted to split graphs. We design a linear time algorithm for computing the value of \(i_{\{R2\}}(T)\) in any tree \(T\), which answers an open problem raised by Rahmouni and Chellali [Independent Roman \(\{2\}\)-domination in graphs, Discrete Appl. Math. 236 (2018) 408–414]. Moreover, we present a linear time algorithm for computing the value of \(\gamma_{\{R2\}}(G)\) in any block graph \(G\), which is a generalization of trees.

Keywords: Roman \(\{2\}\)-domination, domination, algorithms.

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1. Introduction

Let $G = (V, E)$ be a simple graph. The open neighborhood $N_G(v)$ of a vertex $v$ consists of the vertices adjacent to $v$ and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. $N^2_G[v] = \{u : d_G(u, v) \leq 2\}$, where $d_G(u, v)$ is the distance between $u$ and $v$ in graph $G$. For an edge $e = uv$, it is said that $u$ (respectively, $v$) is incident to $e$, denoted by $u \in e$ (respectively, $v \in e$). A Roman dominating function (RDF) on graph $G$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function $f$ is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on a graph $G$ is called the Roman domination number $\gamma(R)(G)$ of $G$. Roman domination and its variations have been studied in a number of recent papers (see, for example, [1, 6, 9]).

Chellali, Haynes, Hedetniemi and McRae [4] introduced a variant of Roman dominating functions. For a graph $G = (V, E)$, a Roman $\{2\}$-dominating function (R2DF) $f : V \rightarrow \{0, 1, 2\}$ has the slightly different property that only for every vertex $v \in V$ with $f(v) = 0$, $f(N(v)) \geq 2$, that is, either there exists a neighbor $u \in N(v)$, with $f(u) = 2$, or at least two neighbors $x, y \in N(u)$ have $f(x) = f(y) = 1$. The weight of a Roman $\{2\}$-dominating function is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{2\}$-dominating function $f$ is the Roman $\{2\}$-domination number, denoted $\gamma_{\{2\}}(G)$. Roman $\{2\}$-domination is also called Italian domination by some scholars ([8]). Suppose that $f : V \rightarrow \{0, 1, 2\}$ is an R2DF on a graph $G = (V, E)$. Let $V_i = \{v : f(v) = i\}$, for $i \in \{0, 1, 2\}$. If $V_1 \cup V_2$ is an independent set, then $f$ is called an independent Roman $\{2\}$-dominating function (IR2DF), which was introduced by Rahmouni and Chellali [11] in a recent paper. The minimum weight of an independent Roman $\{2\}$-dominating function $f$ is the independent Roman $\{2\}$-domination number, denoted $\iota_{\{2\}}(G)$. The authors in [4, 11] have showed that the associated decision problems for Roman $\{2\}$-domination and independent Roman $\{2\}$-domination are NP-complete for bipartite graphs. The authors in [4] have showed that $\gamma_{\{2\}}(T)$ can be computed by a linear time algorithm for any tree $T$. In [11], the authors raised some interesting open problems, one of which is whether there is a linear time algorithm for computing $\iota_{\{2\}}(T)$ for any tree $T$.

A graph $G = (V, E)$ is a split graph if $V$ can be partitioned into $C$ and $I$, where $C$ is a clique and $I$ is an independent set of $G$. Split graph is an important subclass of chordal graphs, and it turns out to be very important in the domination theory (see [2, 7]). A maximal connected induced subgraph without a cut-vertex is called a block of $G$. We use $K_n$ to denote the complete graph of order $n$. A graph $G$ is a block graph if every block in $G$ is a complete graph. If every block of $G$ is a $K_2$, then $G$ is a tree. Hence, block graphs contain trees.
as its subclass. There are widely research on variations of domination in block graphs (see, for example, [3, 5, 10, 14]).

In this paper, we first show that the decision problem associated with $\gamma_{(R2)}(G)$ is NP-complete for split graphs. Then, we give a linear time algorithm for computing $i_{(R2)}(T)$ in any tree $T$. Moreover, we present a linear time algorithm for computing $\gamma_{(R2)}(G)$ in any block graph $G$.

2. Complexity Result

In this section, we consider the decision problem associated with Roman $\{2\}$-dominating functions.

**ROMAN $\{2\}$-DOMINATING FUNCTION (R2D)**

**INSTANCE:** A graph $G = (V,E)$ and a positive integer $k \leq |V|$.

**QUESTION:** Does $G$ have a Roman $\{2\}$-dominating function of weight at most $k$?

A vertex cover of $G$ is a subset $V' \subseteq V$ such that for each edge $uv \in E$, at least one of $u$ and $v$ belongs to $V'$. Vertex Cover (VC) problem is a well-known NP-complete problem. We show R2D problem is NP-complete by reducing the Vertex Cover (VC) to R2D.

**VERTEX COVER (VC)**

**INSTANCE:** A graph $G = (V,E)$ and a positive integer $k \leq |V|$.

**QUESTION:** Is there a vertex cover of size $k$ or less for $G$?

**Theorem 1.** R2D is NP-complete for split graphs.

**Proof.** R2D is a member of NP, since we can check in polynomial time that a function $f : V \rightarrow \{0,1,2\}$ has weight at most $k$ and is a Roman $\{2\}$-dominating function. The proof is given by reducing the VC problem in general graphs to the R2D problem in split graphs.

Let $G = (V,E)$ be a graph with $V = \{v_1,v_2,\ldots,v_n\}$ and $E = \{e_1,e_2,\ldots,e_m\}$. Let $V^1 = \{v'_1,v'_2,\ldots,v'_n\}$. We construct the graph $G' = (V',E')$ with $V' = V^1 \cup V \cup E$, $E' = \{v_iv_j : v_i \neq v_j, v_i \in V, v_j \in V\} \cup \{v_iv'_i : i = 1,\ldots,n\} \cup \{ve : v \in e, e \in E\}$.

Notice that $G'$ is a split graph whose vertex set $V'$ is the disjoint union of the clique $V$ and the independent set $V^1 \cup E$. It is clear that $G'$ can be constructed in polynomial time from $G$.

If $G$ has a vertex cover $C$ of size at most $k$, let $f : V' \rightarrow \{0,1,2\}$ be a function
defined as follows.

\[
f(v) = \begin{cases} 
2, & \text{if } v \in C, \\
1, & \text{if } v \in V^1 \text{ and let } v' \text{ be a neighbor of } v \text{ such that } v' \in V \setminus C, \\
0, & \text{otherwise.}
\end{cases}
\]

It is clear that \( f \) is a Roman \( \{2\} \)-dominating function of \( G' \) with weight at most \( 2k + (n - k) \).

On the other hand, suppose that \( G' \) has a Roman \( \{2\} \)-dominating function of weight at most \( 2k + (n - k) \). Among all such functions, let \( g = (V_0, V_1, V_2) \) be one chosen so that:

(C1) \( |V^1 \cap V_2| \) is minimized;

(C2) subject to condition (C1): \( |E \cap V_0| \) is maximized;

(C3) subject to conditions (C1) and (C2): \( |V \cap V_4| \) is minimized;

(C4) subject to conditions (C1), (C2) and (C3): the weight of \( g \) is minimized.

We make the following remarks.

(i) No vertex in \( V^1 \) belongs to \( V_2 \). Indeed, suppose to the contrary that \( g(v'_i) = 2 \) for some \( i \). We reassign 0 to \( v'_i \) instead of 2 and reassign 2 to \( v_i \). Then it provides an R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with less vertices of \( V^1 \) assigned 2, contradicting the condition (C1) in the choice of \( g \).

(ii) No vertex in \( E \) belongs to \( V_2 \). Indeed, suppose that \( g(e) = 2 \) for some \( e \in E \) and \( v_j, v_k \in e \). By reassigning 0 to \( e \) instead of 2 and reassigning 2 to \( v_j \) instead of \( g(v_j) \), we obtain an R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with more vertices of \( E \) assigned 0, contradicting the condition (C2) in the choice of \( g \).

(iii) No vertex in \( E \) belongs to \( V_1 \). Assume that \( g(e) = 1 \) for some \( e \in E \) and \( v_j, v_k \in e \). If \( g(v'_j) = 0 \), then \( g(v_j) = 2 \) (by the definition of R2DF). By reassigning 0 to \( e \) instead of 1, we obtain an R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with more vertices of \( E \) assigned 0, contradicting the condition (C2) in the choice of \( g \). Hence we may assume that \( g(v'_j) = 1 \) (by (i)). Clearly we can reassign 2 to \( v_j \) instead of 0, 0 to \( v'_j \) instead of 1 and 0 to \( e \) instead of 1. We also obtain a R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with more vertices of \( E \) assigned 0, contradicting the condition (C2) in the choice of \( g \).

(iv) No vertex in \( V \) belongs to \( V_1 \). Suppose to the contrary that \( g(v_i) = 1 \) for some \( i \), then \( g(v'_i) = 1 \) (by (i) and the definition of R2DF). We reassign 0 to \( v'_i \) instead of 1 and 2 to \( v_i \) instead of 1. It provides a R2DF on \( G' \) of weight at most
2k + (n − k) but with less vertices of V assigned 1, contradicting the condition (C3) in the choice of g.

(v) If a vertex in V is assigned 2, then its neighbor in V₁ is assigned 0 by the condition (C4) in the choice of g.

(vi) If a vertex in V is assigned 0, then its neighbor in V₁ is assigned 1 by the definition of R₂DF and (i).

Therefore, according to the previous items, we conclude that V₁ ∩ V₂ = ∅, E ⊆ V₀, and V ∩ V₂ = ∅. Hence V₂ ⊆ V. Let C = \{v : g(v) = 2\}. Since each vertex in E ∪ (V \ C) belongs to V₀ in G', it is clear that C is a vertex cover of G by the definition of R₂DF. Then \(g(V₁) + g(V) + g(E) = 2|C| + (n − |C|) ≤ 2k + (n − k)\), implying that |C| ≤ k. Consequently, C is a vertex cover for G of size at most k.

Since the vertex cover problem is NP-complete, the Roman \{2\}-domination problem is NP-complete for split graphs.

\[\square\]

3. Independent Roman \{2\}-Domination in Trees

In this section, a linear time dynamic programming style algorithm is given to compute the exact value of the independent Roman \{2\}-dominating number in any tree. This algorithm is constructed using the methodology of Wimer [13].

A rooted tree is a pair \((T, r)\) with T is a tree and r is a vertex of T. We call r is the root of tree T. A rooted tree \((T, r)\) is trivial if \(V(T) = r\). Given two rooted trees \((T₁, r₁)\) and \((T₂, r₂)\) with \(V(T₁) ∩ V(T₂) = ∅\), the composition of them is \((T₁, r₁) ∪ (T₂, r₂) = (T, r₁)\) with \(V(T) = V(T₁) ∪ V(T₂)\) and \(E(T) = E(T₁) ∪ E(T₂) ∪ \{r₁r₂\}\). It is clear that any rooted tree can be constructed recursively from trivial rooted trees using the defined composition.

Let \(f : V(T) → \{0, 1, 2\}\) be a function on T. Then f splits two functions \(f₁\) and \(f₂\) according to this decomposition. We express this as follows: \((T, f, r) = (T₁, f₁, r₁) ∪ (T₂, f₂, r₂)\), where \(r = r₁\), \(f_i = f|T_i\) is the function f restricted to the vertices of \(T_i\), \(i = 1, 2\). On the other hand, let \(f_i : V(T_i) → \{0, 1, 2\}\) be a function on \(T_i\) \((i = 1, 2)\). We can define the composition as follows: \((T₁, f₁, r₁) ∩ (T₂, f₂, r₂) = (T, f, r)\), where \(V(T) = V(T₁) ∪ V(T₂)\), \(E(T) = E(T₁) ∪ E(T₂) ∪ \{r₁r₂\}\), \(r = r₁\) and \(f = f₁ ∪ f₂ : V(T) → \{0, 1, 2\}\) with \(f(v) = f_i(v)\) if \(v ∈ V(T_i)\), \(i = 1, 2\). Before presenting the algorithm, let us give the following observation.

**Observation 2.** Let f be an IR₂DF of \(T = T₁ ∩ T₂\) and \(f_i = f|T_i\) \((i = 1, 2)\). If \(f_i(r₁) \neq 0\), then \(f_i\) is an IR₂DF of \(T_i\). If \(f_i(r₁) = 0\), then \(f_i\) restricted to the vertices of \(T_i − r₁\) is an IR₂DF of \(T_i − r₁\).

In order to construct an algorithm for computing the independent Roman \{2\}-domination number, we must characterize the possible tree-subset tuples \((T, f, r)\). For this purpose, we introduce some additional notations as follows:
IR2DF(T) = \{ f : f is an IR2DF of T \},
IR2DF_r(T) = \{ f : f \notin IR2DF(T), but f|_{T-r} is an IR2DF of T - r \}.

Then we consider the following five classes:
A = \{(T, f, r) : f \in IR2DF(T) and f(r) = 2 \},
B = \{(T, f, r) : f \in IR2DF(T) and f(r) = 1 \},
C = \{(T, f, r) : f \in IR2DF(T) and f(r) = 0 \},
D = \{(T, f, r) : f \in IR2DF_r(T) and f(N[r]) = 1 \},
F = \{(T, f, r) : f \in IR2DF_r(T) and f(N[r]) = 0 \}.

Let M, N \in \{ A, B, C, D, F \}. If (T_1, f_1, r_1) \in M and (T_2, f_2, r_2) \in N, we use M \circ N to denote the set of (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2). Let (T, r) = (T_1, r_1) \circ (T_2, r_2) and r = r_1. Suppose that f_1 (respectively, f_2) is a function on T_1 (respectively, T_2). Define f as the function on T with f|_{T_1} = f_1 and f|_{T_2} = f_2. Next, we provide some lemmas.

Lemma 3. A = (A \circ C) \cup (A \circ D) \cup (A \circ F).

Proof. It is clear that the following items are true.

(i) If (T_1, f_1, r_1) \in A and (T_2, f_2, r_2) \in C, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A.

(ii) If (T_1, f_1, r_1) \in A and (T_2, f_2, r_2) \in D, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A.

(iii) If (T_1, f_1, r_1) \in A and (T_2, f_2, r_2) \in F, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A. Thus, (A \circ C) \cup (A \circ D) \cup (A \circ F) \subseteq A.

Now we prove that A \subseteq (A \circ C) \cup (A \circ D) \cup (A \circ F). Let (T, f, r) \in A and (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2), then f_1(r_1) = f(r) = 2. Since f \in IR2DF(T), then f_1 \in IR2DF(T_1). So (T_1, f_1, r_1) \in A. From the independence of V_1 \cup V_2, we have f_2(r_2) = f(r_2) = 0. If f_2 \in IR2DF(T_2), then we obtain (T_2, f_2, r_2) \in C. If f_2 \notin IR2DF(T_2), then (T_2, f_2, r_2) \in D or F. Hence, we conclude that A \subseteq (A \circ C) \cup (A \circ D) \cup (A \circ F). ■

Lemma 4. B = (B \circ C) \cup (B \circ D).

Proof. It is easy to check the following items.

(i) If (T_1, f_1, r_1) \in B and (T_2, f_2, r_2) \in C, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B.

(ii) If (T_1, f_1, r_1) \in B and (T_2, f_2, r_2) \in D, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B. So, (B \circ C) \cup (B \circ D) \subseteq B.

Next we need to show B \subseteq (B \circ C) \cup (B \circ D). Let (T, f, r) \in B and (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2), then f_1(r_1) = f(r) = 1. It is clear that f_1 \in IR2DF(T_1). So we conclude that (T_1, f_1, r_1) \in B. From the definition of IR2DF, we must have f_2(r_2) = f(r_2) = 0. If f_2 \in IR2DF(T_2), then we obtain (T_2, f_2, r_2) \in C. If f_2 \notin IR2DF(T_2), then f_2(N[r_2]) = 1 and f_2|_{T_2-r_2} \in IR2DF(T_2 - r_2) using the fact that (T, f, r) \in B. Therefore, we have f_2 \in IR2DF_r(T_2), implying that (T_2, f_2, r_2) \in D. Hence, we deduce that B \subseteq (B \circ C) \cup (B \circ D). ■
Lemma 5. \( C = (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A) \).

**Proof.** It is easy to check the following remarks by definitions.

(i) If \((T_1, f_1, r_1) \in C \) and \((T_2, f_2, r_2) \in A\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C\).

(ii) If \((T_1, f_1, r_1) \in C \) and \((T_2, f_2, r_2) \in B\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C\).

(iii) If \((T_1, f_1, r_1) \in C \) and \((T_2, f_2, r_2) \in C\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C\).

(iv) If \((T_1, f_1, r_1) \in D \) and \((T_2, f_2, r_2) \in A\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C\).

(v) If \((T_1, f_1, r_1) \in D \) and \((T_2, f_2, r_2) \in B\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C\).

(vi) If \((T_1, f_1, r_1) \in F \) and \((T_2, f_2, r_2) \in A\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C\).

Hence, we deduce that \((C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A) \subseteq C\).

Therefore, we need to prove \( C \subseteq (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)\). Let \((T, f, r) \in C \) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\), then \(f \in IR2DF(T)\) and \(f_1(r_1) = f(r) = 0\). Consider the following cases.

**Case 1.** \( f(r_2) = 2\). Since \(f \in IR2DF(T)\), \(f_2 \in IR2DF(T_2)\). Hence, \((T_2, f_2, r_2) \in A\). If \(f_1 \in IR2DF(T_1)\), then we obtain that \((T_1, f_1, r_1) \subseteq C\). If \(f_1 \notin IR2DF(T_1)\), we have \((T_1, f_1, r_1) \subseteq D\) or \(F\).

**Case 2.** \( f(r_2) = 1\). Since \(f \in IR2DF(T)\), \(f_2 \in IR2DF(T_2)\). So \((T_2, f_2, r_2) \in B\). If \(f_1 \in IR2DF(T_1)\), then we deduce \((T_1, f_1, r_1) \subseteq C\). If \(f_1 \notin IR2DF(T_1)\), therefore, it implies that \((T_1, f_1, r_1) \subseteq D\).

**Case 3.** \( f(r_2) = 0\). It is clear that \(f_1\) and \(f_2\) are both \(IR2DF\). Then we obtain that \((T_1, f_1, r_1) \subseteq C\) and \((T_2, f_2, r_2) \subseteq C\).

Hence, \(C \subseteq (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)\).

Lemma 6. \( D = (D \circ C) \cup (F \circ B)\).

**Proof.** It is easy to check the following remarks by definitions.

(i) If \((T_1, f_1, r_1) \in D \) and \((T_2, f_2, r_2) \in C\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \subseteq D\).

(ii) If \((T_1, f_1, r_1) \in F \) and \((T_2, f_2, r_2) \in B\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \subseteq D\).

Thus, \((D \circ C) \cup (F \circ B) \subseteq D\).

On the other hand, we show \(D \subseteq (D \circ C) \cup (F \circ B)\). Let \((T, f, r) \in D\) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\). Then \(f_1(r_1) = f(r) = 0\). By the definition of \(D\), \(f_2 \in IR2DF(T_2)\). Using the fact that \(f(N_T[r_1]) = 1\), we deduce that \(f(r_2) < 2\). Consider the following cases.

**Case 1.** \( f(r_2) = 1\). It is clear that \((T_2, f_2, r_2) \subseteq B\) because \(f_2\) is an \(IR2DF\) of \(T_2\). Since \(f_1(N_{T_1}[r_1]) = 0\), we obtain \(f_1[r_1 - r_1] \subseteq IR2DF(T_1 - r_1)\). Hence, we have \(f_1 \in IR2DF_{r_1}(T_1)\), implying that \((T_1, f_1, r_1) \subseteq F\).

**Case 2.** \( f(r_2) = 0\). Then \(f_2\) is an \(IR2DF\) of \(T_2\), implying that \((T_2, f_2, r_2) \subseteq C\). Using the fact that \(f(N_T[r_1]) = 1\) and \(f(r_2) = 0\), we know \(f_1(N_{T_1}[r_1]) = 1\). So \(f_1 \in IR2DF_{r_1}(T_1)\). It implies that \((T_1, f_1, r_1) \subseteq D\). ■
Lemma 7. \( F = F \circ C \).

**Proof.** If \((T_1, f_1, r_1) \in F\) and \((T_2, f_2, r_2) \in C\), then it is clear that \((T, f, r) \in F\). Hence, \((F \circ C) \subseteq F\).

On the other hand, let \((T, f, r) \in F\) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\). Then \(f_1(r_1) = f(r) = 0\). By the definition of \(F\), we deduce that \(f(r_2) = 0\). Using the fact that \((T, f, r) \in F\), we have that \(f_2 \in \text{IR2DF}(T_2)\). So \((T_2, f_2, r_2) \in C\).

Notice that \((T, f, r) \in F\), we have \(f_1(N_{T_1}[r_1]) = 0\), implying that \((T_1, f_1, r_1) \notin D\). We can easily check that \(f_1 \in \text{IR2DF}_{r_1}(T_1)\). Hence, we have \((T_1, f_1, r_1) \in F\), implying that \(F \subseteq (F \circ C)\). \(\blacksquare\)

Let \(T = (V, E)\) be a tree with \(n\) vertices. It is well known that the vertices of \(T\) have an ordering \(v_1, v_2, \ldots, v_n\) such that for each \(1 \leq i \leq n-1\), \(v_i\) is adjacent to exactly one vertex \(v_j\) with \(j > i\) (see [12]). The ordering is called a tree ordering where the only neighbor \(v_j\) with \(j > i\) is called the father of \(v_i\) and \(v_i\) is a child of \(v_j\). For each \(1 \leq i \leq n-1\), the father of \(v_i\) is denoted by \(F(v_i) = v_j\).

For each vertex \(v_i\) (\(1 \leq i \leq n\)), define a vector \(l[i,1..5]\). Let \(T_{v_i}\) be a tree such that \(v_i\) is the root of \(T_{v_i}\). For each rooted tree \((T_{v_i}, v_i)\), let \(f_{v_i} : V(T_{v_i}) \to \{0, 1, 2\}\) be a function on \(T_{v_i}\) and define \(w(f_{v_i}) = f_{v_i}(V(T_{v_i}))\). In this case, for a tree, the only basis graph is a single vertex. Then, the vector \(l[i,1..5]\) is initialized by \(\min_{(T_{v_i}, f_{v_i}, v_i) \in A} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in B} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in C} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in D} w(f_{v_i})\), \(\min_{(T_{v_i}, f_{v_i}, v_i) \in F} w(f_{v_i})\).

It means \(l[i,1..5] = [2, 1, \infty, \infty, 0]\), where \(\infty\) means undefined. Now, we are ready to present the algorithm.

**Algorithm 1:** INDEPENDENT-ROMAN \(\{2\}\)-DOM-IN-TREE

**Input:** A tree \(T = (V, E)\) with a tree ordering \(v_1, v_2, \ldots, v_n\).

**Output:** The independent Roman \(\{2\}\)-domination number \(i_{\{R2\}}(T)\).

1. **if** \(T = K_1\) **then**
2.  \(\text{return } i_{\{R2\}}(T) = 1\);
3. **for** \(i := 1\) **to** \(n\) **do**
4.  \(_{\text{initialize }l[i,1..5]\text{ to }[2,1,\infty,\infty,0]}\);
5. **for** \(j := 1\) **to** \(n-1\) **do**
6.  \(v_k = F(v_j)\);
7.  \(l[k, 1] = \min\{l[k, 1] + l[j, 3], l[k, 1] + l[j, 4], l[k, 1] + l[j, 5]\}\);
8.  \(l[k, 2] = \min\{l[k, 2] + l[j, 3], l[k, 2] + l[j, 4]\}\);
9.  \(l[k, 3] = \min\{l[k, 3] + l[j, 1], l[k, 3] + l[j, 2], l[k, 3] + l[j, 3], l[k, 4] + l[j, 1]\}\);
10. \(l[k, 4] = \min\{l[k, 4] + l[j, 2], l[k, 5] + l[j, 1]\}\);
11. \(l[k, 5] = \min\{l[k, 5] + l[j, 3]\}\);
12. **return** \(i_{\{R2\}}(T) = \min\{l[n, 1], l[n, 2], l[n, 3]\}\);
From the above argument, we can obtain the following theorem.

**Theorem 8.** Algorithm INDEPENDENT-ROMAN \{2\}-DOM-IN-TREE can output the independent Roman \{2\}-domination number of any tree \( T = (V, E) \) in linear time \( O(n) \), where \( n = |V| \).

**Proof.** It is clear that the running time of Algorithm 1 is linear. We only need to show \( i_{\{R2\}}(T) = \min\{l[n,1], l[n,2], l[n,3]\} \). Suppose that \( f \in \text{IR2DF}(T) \). Then, \( (T,f,r) \in A \cup B \cup C \). By the Algorithm 1 and Lemmas 3–7, we have \( l[n,1] = \min_{(T,f,r) \in A} f(V), \ l[n,2] = \min_{(T,f,r) \in B} f(V), \) and \( l[n,3] = \min_{(T,f,r) \in C} f(V) \). By the definition of \( i_{\{R2\}}(T) \), we deduce that

\[
i_{\{R2\}}(T) = \min_{(T,f,r) \in A \cup B \cup C} f(V) = \min\{l[n,1], l[n,2], l[n,3]\}.
\]

\( \square \)

4. Roman \{2\}-Domination in Block Graph

Let \( G(\neq K_n) \) be a connected block graph. The block-cutpoint graph of \( G \) is a bipartite graph \( T_G = (C \cup B, E) \) in which one partite set \( C \) consists of the cutvertices of \( G \), and the other \( B \) has a vertex \( h \) for each block \( H \) of \( G \). Let \( v \in C \) and \( h \in B \). We include \( vh \) as an edge of \( T_G \) if and only if \( v \) is in \( H \), where \( H \) is the block of \( G \) represented by \( h \). Obviously, \( T_G \) is a tree and can be constructed from \( G \) in linear time (see [12]). In this section, we call each vertex in \( C \) a \( C \)-vertex and each vertex in \( B \) a \( B \)-vertex.

Let \( H \) be a block of \( G \). Suppose that \( S = \{v : v \in H \text{ and } v \text{ is a cut-vertex of } G\} \). We say \( H \) is a block of type 0 if \( |H| = |S| \) and \( H \) is a block of type 1 if \( |H| = |S| + 1 \). If \( |H| \geq |S| + 2 \), we say \( H \) is a block of type 2. Let \( f : V(G) \rightarrow \{0,1,2\} \) be a function of a block graph \( G(\neq K_n) \). \( f_* : V(T_G) \rightarrow \mathbb{Z} \) is defined as follows:

\[
f_*(v) = \begin{cases} f(v), & \text{if } v \text{ is a } C \text{-vertex,} \\ f(H) - f(S), & \text{if } v \text{ is a } B \text{-vertex representing the block } H.\end{cases}
\]

We say that \( f_* \) is the function induced by \( f \). Now we present a key result on the relationship between \( f \) and \( f_* \).

**Theorem 9.** Let \( f : V(G) \rightarrow \{0,1,2\} \) be a function of a connected block graph \( G(G \neq K_n) \) and \( f_* \) be the function induced by \( f \). Then, \( f \) satisfies the following properties:

1. \( f(v) = 0 \) or 1 if \( v \in H \) is not a cut-vertex of \( G \), where \( H \) is a block of type 1 of \( G \).
(2) \( f(v) = 0 \) if \( v \in H \) is not a cut-vertex of \( G \), where \( H \) is a block of type 2 of \( G \).

(3) \( f \) is an R2DF of \( G \).

if and only if \( f_\ast \) satisfies the following properties:

(a) \( f_\ast(v) = 0 \) or 1 if \( v \) is a \( B \)-vertex and the block \( H \) represented by \( v \) is type 1.

(b) \( f_\ast(v) = 0 \) if \( v \) is a \( B \)-vertex and the block \( H \) represented by \( v \) is not type 1.

(c) If \( v \) is a \( C \)-vertex with \( f_\ast(v) = 0 \), then there exists either \( u \in N^2_{T_G}(v) \) with \( f_\ast(u) = 2 \) or \( u_1, u_2 \in N^2_{T_G}(v) \) with \( f_\ast(u_1) = f_\ast(u_2) = 1 \).

(d) If \( v \) is a \( B \)-vertex with \( f_\ast(v) = 0 \) and the block \( H \) represented by \( v \) is not type 0, then there exists either \( u \in N_{T_G}(v) \) with \( f_\ast(u) = 2 \) or \( u_1, u_2 \in N_{T_G}(v) \) with \( f_\ast(u_1) = f_\ast(u_2) = 1 \).

Proof. If \( f \) satisfies the above properties, it is clear that \( f_\ast \) satisfies the above items (a), (b). Suppose that \( v \) is a \( C \)-vertex with \( f_\ast(v) = 0 \). By the definition of \( f_\ast \), \( f(v) = 0 \). If there exists a vertex \( u \in N_G(v) \) with \( f(u) = 2 \), then \( u \) is a cut-vertex of \( G \), and hence \( u \in N^2_{T_G}(v) \) with \( f_\ast(u) = 2 \). Otherwise, there exists at least two vertices \( x, y \in N_G(v) \) having \( f(x) = f(y) = 1 \). If \( x \) and \( y \) are both cut-vertices of \( G \), then we obtain \( x, y \in N^2_{T_G}(v) \) having \( f_\ast(x) = f_\ast(y) = 1 \). If \( x \) is not a cut-vertex of \( G \) and \( H \) is the block containing \( x \), we deduce that \( H \) is type 1 by the second property of \( f \). It implies that \( f_\ast(h) = 1 \) and \( vh \in E(T_G) \), where \( h \) is the \( B \)-vertex representing the block \( H \). In this case, \( f_\ast \) also satisfies item (c). Suppose that \( v \) is a \( B \)-vertex with \( f_\ast(v) = 0 \) and the block \( H \) represented by \( v \) is not type 0. Let \( S = \{ u : u \in H \text{ and } u \text{ is a cut-vertex of } G \} \). By the definition of \( f_\ast \), we know that \( f(x) = 0 \) for each \( x \in H \setminus S \). Since \( f \) is an R2DF of \( G \), then there exists either \( u \in N_G(v) \) with \( f(u) = 2 \) or \( u_1, u_2 \in N_G(v) \) such that \( f(u_1) = f(u_2) = 1 \). It is clear that \( u, u_1, u_2 \) are cut-vertices. It means that \( f_\ast(u) = 2 \) and \( f_\ast(u_1) = f_\ast(u_2) = 1 \). So \( f_\ast \) satisfies item (d).

On the other hand, if \( f_\ast \) satisfies the above properties, by the definition of \( f_\ast \), it is easy to know that \( f \) satisfies items (1) and (2).

We now need to show that \( f \) is an R2DF of \( G \). Suppose that \( v \) is a cut-vertex with \( f(v) = 0 \). Hence, \( f_\ast(v) = f(v) = 0 \). If there exists \( u \in N^2_{T_G}(v) \) such that \( f_\ast(u) = 2 \), we deduce that \( u \) is a cut-vertex of \( G \), \( f(u) = 2 \) and \( u \in N_G(v) \). Otherwise, there exists \( h_1, h_2 \in N^2_{T_G}(v) \) such that \( f_\ast(h_1) = f_\ast(h_2) = 1 \). If \( h_1 \) and \( h_2 \) are both \( C \)-vertices, then we have \( h_1, h_2 \in N_G(v) \) and \( f(h_1) = f(h_2) = 1 \). If \( h_1 \) is a \( B \)-vertex and \( h_1 \) represents block \( H_1 \) in \( T_G \). We deduce that \( H_1 \) is a block of type 1. Hence, there exists \( v_1 \in H_1 \) and \( v_1 \) is not a cut-vertex of \( G \) such that \( f(v_1) = f(h_1) = 1 \). Therefore, we obtain \( f(N(v)) \geq 2 \). Suppose that \( H \) is a block containing \( v \) and \( v \) is not a cut-vertex with \( f(v) = 0 \). Then \( f_\ast(h) = f(v) = 0 \), where \( h \) is the \( B \)-vertex representing the block \( H \). As \( H \) is not type 0, there either exists \( u \in N_{T_G}(h) \) such that \( f_\ast(u) = 2 \) or exists \( u_1, u_2 \in N_{T_G}(h) \).
such that \( f_*(u_1) = f_*(u_2) = 1 \). It is clear that \( u, u_1, u_2 \) are cut-vertices and \( u, u_1, u_2 \in N_G(v) \). We also obtain \( f(u) = f_*(u) = 2 \) and \( f(u_1) = f(u_2) = 1 \). Therefore, we deduce \( f(N(v)) \geq 2 \).

**Lemma 10.** There exists an R2DF \( f \) of \( G \) with weight \( \gamma_{\{R2\}}(G) \), which satisfies the following properties:

1. \( f(v) = 0 \) or 1 if \( v \in H \) is not a cut-vertex of \( G \), where \( H \) is a block of type 1 of \( G \).
2. \( f(v) = 0 \) if \( v \in H \) is not a cut-vertex of \( G \), where \( H \) is a block of type 2 of \( G \).

**Proof.** Let \( f \) be an R2DF of weight \( \gamma_{\{R2\}}(G) \) and \( u \in H \) be a cut-vertex of \( G \), where \( H \) is not a block of type 0, \( S = \{v : v \in H \text{ and } v \text{ is a cut-vertex of } G\} \) and \( f(u) = \max_{v \in S} f(v_0) \). Suppose \( v \in H \) is not a cut-vertex of \( G \). If \( f(v) = 2 \), we can reassign 0 to \( v \) and 2 to \( u \). Hence, \( f(v) = 0 \) or 1. Further, if \( H \) is a block of type 2, we suppose that there exists a vertex \( v \in H \) such that \( f(v) = 1 \). If \( f(u) \geq 1 \), then we can reassign 2 to \( u \) and 0 to \( v \), a contradiction. Suppose that \( f(u) = 0 \), then there exists a vertex \( w \in H \), such that \( w \) is not a cut-vertex and \( f(w) \geq 1 \). We reassign 2 to \( u \) and 0 to \( v, w \), a contradiction.

Let \( f \) be an R2DF of block graph \( G(\neq K_n) \) and \( f_* \) be the function induced by \( f \). We say \( f_* \) is an induced Roman \( \{2\} \)-domination function (R2DF\(_*\)) of \( T_G \) if it satisfies the four properties in Theorem 9. By Theorem 9 and Lemma 10, we can transform the Roman \( \{2\} \)-domination problem on block graph \( G \) into the induced Roman \( \{2\} \)-domination problem on tree \( T_G \). Then, we can also use the method of tree composition and decomposition in Section 3. For convenience, \( T_G = (C \cup B, E) \) is denoted by \( T \) and \( v \in C \) (respectively, \( v \in B \)) is used to represent that \( v \) is a \( C \)-vertex (respectively, \( B \)-vertex) of \( T_G \) if there is no ambiguity.

Suppose that \( T \) is a tree rooted at \( r \) and \( f : V(T) \to \{0, 1, 2\} \) is a function on \( T \). \( T' \) is defined as a new tree rooted at \( r' \) and \( f' : V(T') \to \{0, 1, 2\} \) is a function on \( T' \), where \( V(T') = V(T) \cup \{r'\} \) and \( E(T') = E(T) \cup \{rr'\}, f'_{|T} = f \).

In order to construct an algorithm for computing the Roman \( \{2\} \)-domination number, we must characterize the possible tree-subset tuples \( (T, f, r) \). For this purpose, we introduce some additional notations as follows:

- \( \text{R2DF}_*(T) = \{f : f \text{ is an R2DF}_* \text{ of } T\} \),
- \( \text{F}_1(T) = \{f : f \in \text{R2DF}_*(T) \text{ with } f(r) = 1\} \),
- \( \text{F}_2(T) = \{f : f \in \text{R2DF}_*(T) \text{ with } f(r) = 2\} \),
- \( \text{R2DF}_*(T+1) = \{f : f \notin \text{R2DF}_*(T), f' \in \text{F}_1(T') \text{ and } f'|_T = f\} \),
- \( \text{R2DF}_*(T+2) = \{f : f \notin \text{R2DF}_*(T), f' \in \text{F}_2(T') \text{ and } f'|_T = f\} - \text{R2DF}_*(T+1) \).
Then we consider the following eleven classes:

- \( A_1 = \{(T,f,r) : f \in \text{R2DF}_s(T), r \in C \text{ and } f(r) = 2\}, \)
- \( A_2 = \{(T,f,r) : f \in \text{R2DF}_s(T), r \in C \text{ and } f(r) = 1\}, \)
- \( A_3 = \{(T,f,r) : f \in \text{R2DF}_s(T), r \in C \text{ and } f(r) = 0\}, \)
- \( A_4 = \{(T,f,r) : f \in \text{R2DF}_s(T^{+1}), r \in C\}, \)
- \( A_5 = \{(T,f,r) : f \in \text{R2DF}_s(T^{+2}), r \in C\}, \)
- \( B_1 = \{(T,f,r) : f \in \text{R2DF}_s(T), r \in B \text{ and } f(N[r]) \geq 2\}, \)
- \( B_2 = \{(T,f,r) : f \in \text{R2DF}_s(T), r \in B \text{ and } f(N[r]) = 1\}, \)
- \( B_3 = \{(T,f,r) : f \in \text{R2DF}_s(T), r \in B \text{ and } f(N[r]) = 0\}, \)
- \( B_4 = \{(T,f,r) : f \in \text{R2DF}_s(T^{+1}), r \in B \text{ and } f(N[r]) = 1\}, \)
- \( B_5 = \{(T,f,r) : f \in \text{R2DF}_s(T^{+1}), r \in B \text{ and } f(N[r]) = 0\}, \)
- \( B_6 = \{(T,f,r) : f \in \text{R2DF}_s(T^{+2}), r \in B\}. \)

Let \((T,r) = (T_1,r_1) \circ (T_2,r_2)\) and \(r = r_1\). Suppose that \(f_1\) (respectively, \(f_2\)) is a function on \(T_1\) (respectively, \(T_2\)). Define \(f\) as the function on \(T\) with \(f|_{T_1} = f_1\) and \(f|_{T_2} = f_2\). In order to give the algorithm, we present the following lemmas.

**Lemma 11.** \(A_1 = (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6). \)

**Proof.** For each \(1 \leq i \leq 6\), if \((T_1,f_1,r_1) \in A_1\) and \((T_2,f_2,r_2) \in B_i\), it is clear that \(f\) is an R2DF\(_s\) of \(T\), \(r \in C\) and \(f(r) = 2\). We deduce that \((T_1,f_1,r_1) \circ (T_2,f_2,r_2) \in A_1\). So \((A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6) \subseteq A_1\).

Now we prove that \(A_1 \subseteq (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6)\). Let \((T,f,r) \in A_1\) and \((T,f,r) = (T_1,f_1,r_1) \circ (T_2,f_2,r_2)\), then \(f_1(r_1) = f(r) = 2\). Since \(f \in \text{R2DF}_s(T)\), \(f_1 \in \text{R2DF}_s(T_1)\) and \(r_1 \in C\). So \((T_1,f_1,r_1) \in A_1\) and \(r_2 \in B\). If \(f_2 \in \text{R2DF}_s(T_2)\), then we obtain \((T_2,f_2,r_2) \in B_1\), \(B_2\) or \(B_3\). If \(f_2 \notin \text{R2DF}_s(T_2)\), then \((T_2,f_2,r_2) \in B_4\), \(B_5\) or \(B_6\). Hence, we conclude that \(A_1 \subseteq (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6)\). 

**Lemma 12.** \(A_2 = (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5). \)

**Proof.** For each \(1 \leq i \leq 5\), if \((T_1,f_1,r_1) \in A_2\) and \((T_2,f_2,r_2) \in B_i\), it is clear that \(f\) is an R2DF\(_s\) of \(T\), \(r \in C\) and \(f(r) = f(r_1) = 1\). We conclude that \((T_1,f_1,r_1) \circ (T_2,f_2,r_2) \in A_2\), implying that \((A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5) \subseteq A_2\).

Then we need to show \(A_2 \subseteq (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5)\). Let \((T,f,r) \in A_2\) and \((T,f,r) = (T_1,f_1,r_1) \circ (T_2,f_2,r_2)\), then \(f_1(r_1) = f(r) = 1\). It is clear that \((T_1,f_1,r_1) \in A_2\) and \(r_2 \in B\). If \(f_2\) is an R2DF\(_s\) of \(T_2\), then \(f_2(N_{T_2}[r_2]) \leq 1\) and \(f_2 \in \text{R2DF}_s(T^{+1})\) by using the fact
that \((T, f, r) \in A_2\). Therefore, we have \((T_2, f_2, r_2) \in B_4 \) or \(B_5\). Hence, \(A_2 \subseteq (A_2 \circ B_4) \cup (A_2 \circ B_5) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5)\).

**Lemma 13.** \(A_3 = (A_3 \circ B_4) \cup (A_3 \circ B_5) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1)\).

**Proof.** We make some remarks.

(i) For each \(1 \leq i \leq 3\), if \((T_1, f_1, r_1) \in A_3\) and \((T_2, f_2, r_2) \in B_i\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3\). Indeed, if \((T_1, f_1, r_1) \in A_3\) and \((T_2, f_2, r_2) \in B_i\), then \(f_1\) is an \(R_{2DF}^*)\) of \(T_1\) and \(f_2\) is an \(R_{2DF}^*)\) of \(T_2\). Hence, \(f\) is an \(R_{2DF}^*)\) of \(T\), \(r \in C\) and \(f(r) = 0\). So \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3\).

(ii) For each \(1 \leq i \leq 2\), if \((T_1, f_1, r_1) \in A_4\) and \((T_2, f_2, r_2) \in B_i\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4\). Indeed, if \((T_1, f_1, r_1) \in A_4\), then we have that \(f_1\) is an \(R_{2DF}^*)\) of \(T_1\). Since \((T, f, r) \in A_4\) and \((T, f, r) \in A_4\), we obtain \(f_1(N^1_{T^i}[r]) = 1\). By the definition of \(B_4\), we obtain \(f_1(N^1_{T^i}[r]) \geq 2\) and \(f \in R_{2DF}^*(T)\). Hence, \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3\).

(iii) If \((T_1, f_1, r_1) \in A_5\) and \((T_2, f_2, r_2) \in B_i\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3\). Indeed, if \((T_1, f_1, r_1) \in A_5\), then we have that \(f_1\) is an \(R_{2DF}^*)\) of \(T_1\). Since \((T, f, r) \in A_4\) and \((T, f, r) \in A_4\), we obtain \(f_1(N^1_{T^i}[r]) = 1\). By the definition of \(B_4\), we obtain \(f_1(N^1_{T^i}[r]) \geq 2\) and \(f \in R_{2DF}^*(T)\). Hence, \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3\).

Therefore, we need to prove \(A_3 \subseteq (A_3 \circ B_4) \cup (A_3 \circ B_5) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1) \subseteq A_3\).

**Case 1.** \(f_1(N^1_{T^i}[r_1]) = 1\). Then we obtain \(f_1 \in R_{2DF}^*(T_1^i)\), implying that \((T_1, f_1, r_1) \in A_4\). Since \((T, f, r) \in A_3\), we have \(f_2(N^1_{T^i}[r_2]) \geq 1\). So \((T_2, f_2, r_2) \in B_1\) or \(B_2\).

**Case 2.** \(f_1(N^1_{T^i}[r_1]) = 0\). So we have \(f_1 \in R_{2DF}^*(T_1^i)\). Then \((T_1, f_1, r_1) \in A_5\). Since \((T, f, r) \in A_3\), we obtain \(f_2(N^1_{T^i}[r_2]) \geq 2\). Hence, \((T_2, f_2, r_2) \in B_1\).

**Lemma 14.** \(A_4 = (A_4 \circ B_3) \cup (A_5 \circ B_2)\).

**Proof.** It is easy to check the following remarks by definitions.

(i) If \((T_1, f_1, r_1) \in A_4\) and \((T_2, f_2, r_2) \in B_3\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4\).

(ii) If \((T_1, f_1, r_1) \in A_5\) and \((T_2, f_2, r_2) \in B_2\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4\). Therefore, \((A_4 \circ B_3) \cup (A_5 \circ B_2) \subseteq A_4\).

On the other hand, we show \(A_4 \subseteq (A_4 \circ B_3) \cup (A_5 \circ B_2)\). Let \((T, f, r) \in A_4\) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\). Then we have that \(f \in R_{2DF}^*(T_1^i)\) and \(r_1 \in C\), implying that \(f(N^1_{T^i}[r_1]) = 1\). It means that \(r_2 \in B\). By the definition
of $A_4$, $f_2 \in \text{R2DF}_*(T_2)$. Using the fact that $f(N^2_1[r_1]) = 1$, we deduce that $f_2(N^2[r_2]) < 2$. Consider the following cases.

Case 1. $f_2(N[r_2]) = 1$. It is clear that $(T_2, f_2, r_2) \in B_2$. Since $f_1(N^2_1[r_1]) = f(N^2_1[r_1]) - f_2(N[r_2]) = 0$, we obtain $(T_1, f_1, r_1) \in A_5$.

Case 2. $f_2(N[r_2]) = 0$. Then $(T_2, f_2, r_2) \in B_3$. Since $f_1(N^2_1[r_1]) = f(N^2_1[r_1]) - f_2(N[r_2]) = 1$, we have $(T_1, f_1, r_1) \in A_4$.

Consequently, we deduce that $A_4 \subseteq (A_4 \circ B_3) \cup (A_5 \circ B_2)$.

Lemma 15. $A_5 = A_5 \circ B_3$.

Proof. It is easy to check that $(A_3 \circ B_3) \subseteq A_5$ by the definitions. On the other hand, let $(T, f, r) \in A_5$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we obtain $f \in \text{R2DF}_*(T^{+2})$, $r_1 \in C$ and $f_1(N^2_1[r_1]) = f(N^2_1[r]) = 0$. It implies that $(T_1, f_1, r_1) \in A_5$ and $r_2 \in B$. Using the fact that $(T, f, r) \in A_5$, we deduce $f_2(N[r_2]) = 0$ and $f_2 \in \text{R2DF}_*(T_2)$. Therefore, $(T_2, f_2, r_2) \in B_3$. Then we obtain $A_5 \subseteq (A_5 \circ B_3)$.

Lemma 16. $B_1 = (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$.

Proof. We make some remarks.

(i) For each $1 \leq i \leq 5$, if $(T_1, f_1, r_1) \in B_1$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. It is easy to check it by the definitions of $B_1$ and $A_i$.

(ii) For each $2 \leq i \leq 6$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. We can easily check it by definitions too.

(iii) For each $i \in \{2, 4\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. Indeed, it is clear that $f \in \text{R2DF}_*(T)$, $r \in B$ and $f(N[r]) = f_1(N[r_1]) + f_2(r_2)$ = 2. Hence, $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$.

Therefore, we need to prove $B_1 \subseteq (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$.

Let $(T, f, r) \in B_1$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \text{R2DF}_*(T)$, $r_1 \in B$ and $f(N[r]) \geq 2$. It means that $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 2$. Then we have $f_2 \in \text{R2DF}_*(T_2)$, implying that $(T_2, f_2, r_2) \in A_1$. If $f_1 \in \text{R2DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_1, B_2$ or $B_3$. Suppose that $f_1 \notin \text{R2DF}_*(T_1)$, then $f_1 \in \text{R2DF}_*(T_1^{+1})$ or $f_1 \in \text{R2DF}_*(T_1^{+2})$. Hence, $(T_1, f_1, r_1) \in B_4, B_5$ or $B_6$.

Case 2. $f(r_2) = 1$. It is clear that $(T_2, f_2, r_2) \in A_2$. We also have $f_1(N[r_1]) = f(N[r]) - f_2(r_2) \geq 2 - 1 \geq 1$. If $f_1 \in \text{R2DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_1$ or $B_2$. 

Suppose that $f_1 \notin R2DF_*(T_1)$, then $f_1 \in R2DF_*(T_1+1)$. Therefore, $(T_1, f_1, r_1) \in B_4$.

**Case 3.** $f(r_2) = 0$. Then we obtain $f_1(N[r_1]) = f(N[r]) - f_2(r_2) \geq 2$ and $f_1 \in R2DF_*(T_1)$, implying that $(T_1, f_1, r_1) \in B_1$. If $f_2 \in R2DF_*(T_2)$, we deduce $(T_1, f_1, r_1) \in A_3$. Suppose that $f_2 \notin R2DF_*(T_2)$, then $f_2 \in R2DF_*(T_2+1)$ or $f_2 \in R2DF_*(T_2+2)$. Therefore, $(T_2, f_2, r_2) \in A_3$ or $A_5$.

Hence, $B_1 \subseteq (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$. ■

**Lemma 17.** $B_2 = (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$.

**Proof.** We make some remarks.

(i) For each $3 \leq i \leq 4$, if $(T_1, f_1, r_1) \in B_2$ and $(T_2, f_2, r_2) \in A_4$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_3$. It is easy to check it by the definitions.

(ii) For each $i \in \{3, 5\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. Indeed, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, we obtain that $f \in R2DF_*(T)$, $r \in B$ and $f(N[r]) = f_1(N[r_1]) + f_2(r_2) = 1$. Hence, we deduce $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. Thus, $(B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2) \subseteq B_2$.

Now we need to prove $B_2 \subseteq (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$. Let $(T, f, r) \in B_2$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have that $f \in R2DF_*(T)$, $r_1 \in B$ and $f(N[r]) = 1$. It implies $r_2 \in C$. Consider the following cases.

**Case 1.** $f(r_2) = 1$. Then we have $f_1(N[r_1]) = f(N[r]) - f_2(r_2) = 0$ and $f_2(r_2) = 1$, implying that $f_2 \in R2DF_*(T_2)$. So $(T_2, f_2, r_2) \in A_2$. If $f_1 \in R2DF_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_3$. Suppose that $f_1 \notin R2DF_*(T_1)$, then $f_1(r_1) = 0$ because $f \in R2DF_*(T)$. Since $f_1(N[r_1]) = 0$, we have that $(T_1, f_1, r_1) \in B_5$.

**Case 2.** $f(r_2) = 0$. It is clear that $f_1(N[r_1]) = f(N[r]) - f(r_2) = 1$. Since $f_1 = f|_{T_1}$ and $f \in R2DF_*(T)$, we have $f_1 \in R2DF_*(T_1)$. Hence, $(T_1, f_1, r_1) \in B_2$. If $f_2 \in R2DF_*(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_3$. Suppose that $f_2 \notin R2DF_*(T_2)$, then $f_2(N^2[r_2]) = 1$. It implies $f_2 \in R2DF_*(T_2+1)$. Therefore, $(T_2, f_2, r_2) \in A_4$.

Hence, $B_2 \subseteq (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$.

**Lemma 18.** $B_3 = B_3 \circ A_3$.

**Proof.** It is easy to check that $(B_3 \circ A_3) \subseteq B_3$ by the definitions. On the other hand, let $(T, f, r) \in B_3$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we obtain $f_1(N[r_1]) = f(N[r]) = 0$, $r_1 \in B$ and $f_2(r_2) = 0$. It means that $r_2 \in C$. Since $f \in R2DF_*(T)$ and $f_2(r_2) = 0$, we obtain that $f_1 \in R2DF_*(T_1)$, implying that $(T_1, f_1, r_1) \in B_3$. Using the fact that $f_1(N[r_1]) = 0$ and $f_2(r_2) = 0$, we deduce that $f_2 \in R2DF_*(T_2)$. Therefore, $(T_2, f_2, r_2) \in A_3$. Then $B_3 \subseteq (B_3 \circ A_3)$.
Lemma 19. \( B_4 = (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2) \).

Proof. It is easy to check the following remarks by definitions.

(i) If \((T_1, f_1, r_1) \in B_2\) and \((T_2, f_2, r_2) \in A_5\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4\).

(ii) For each \(3 \leq i \leq 5\), if \((T_1, f_1, r_1) \in B_4\) and \((T_2, f_2, r_2) \in A_i\), then 
\[(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4.\]

(iii) If \((T_1, f_1, r_1) \in B_6\) and \((T_2, f_2, r_2) \in A_2\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4.\)

Therefore, we need to prove \( B_4 \subseteq (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2).\) Let \((T, f, r) \in B_4\) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\), then we have \(f \in R2DF_*(T+1)\), \(r_1 \in B\) and \(f(N[r]) = 1\). It implies \(r_2 \in C\). Consider the following cases.

Case 1. \( f(r_2) = 1\). Then we have \(f_1(N[r_1]) = f(N[r]) - f(r_2) = 0\) and \(f_2(r_2) = 1\), implying that \(f_2 \in R2DF_*(T_2)\). So \((T_2, f_2, r_2) \in A_2\) and \(f_1 \notin R2DF_*(T_1)\). Since \(f_1(N[r_1]) = 0\) and \((T, f, r) \in B_4\), we obtain \((T_1, f_1, r_1) \in B_6.\)

Case 2. \( f(r_2) = 0\). It is clear that \(f_1(N[r_1]) = f(N[r]) - f(r_2) = 1\). If \(f_2 \in R2DF_*(T_2)\), we deduce that \((T_2, f_2, r_2) \in A_3\), implying \((T_1, f_1, r_1) \in B_4\). Suppose that \(f_2 \notin R2DF_*(T_2)\), then \(f_2(N^2[r_2]) = 0\) or \(1\). If \(f_2(N^2[r_2]) = 0\), we obtain \((T_2, f_2, r_2) \in A_5\). Then, we have \((T_1, f_1, r_1) \in B_2\) or \(B_4\). If \(f_2(N^2[r_2]) = 1\), we obtain \((T_2, f_2, r_2) \in A_4\). Then, we have \((T_1, f_1, r_1) \in B_4\).

Hence, \(B_4 \subseteq (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2).\) \(\blacksquare\)

Lemma 20. \( B_5 = (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4) \cup (B_5 \circ A_5) \cup (B_6 \circ A_2) \).

Proof. It is easy to check the following remarks by definitions.

(i) If \((T_1, f_1, r_1) \in B_3\) and \((T_2, f_2, r_2) \in A_4\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_5\).

(ii) For each \(3 \leq i \leq 4\), if \((T_1, f_1, r_1) \in B_5\) and \((T_2, f_2, r_2) \in A_i\), then 
\[(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_5.\]

Therefore, we need to prove \( B_5 \subseteq (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4) \cup (B_5 \circ A_5) \). Let \((T, f, r) \in B_5\) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\), then we have \(f \in R2DF_*(T+1)\), \(r_1 \in B\) and \(f(N[r]) = 0\). It implies \(r_2 \in C\) and \(f_2(r_2) = f(r_2) = 0\). Consider the following cases.

Case 1. If \(f_2 \in R2DF_*(T_2)\), then we have \((T_2, f_2, r_2) \in A_3\) and \(f_1 \notin R2DF_*(T_1)\). Since \(f_1(N[r_1]) = 0\) and \((T, f, r) \in B_5\), we obtain \((T_1, f_1, r_1) \in B_5.\)

Case 2. If \(f_2 \notin R2DF_*(T_2)\), we deduce that \((T_2, f_2, r_2) \in A_4\). It is clear that \((T_1, f_1, r_1) \in B_3\) or \(B_5\).

Hence, \(B_5 \subseteq (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4).\) \(\blacksquare\)

Lemma 21. \( B_6 = (B_3 \circ A_5) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5).\)
\textbf{Proof.} It is easy to check the following remarks by definitions.

(i) For each \(i \in \{3, 5\}\), if \((T_1, f_1, r_1) \in B_i\) and \((T_2, f_2, r_2) \in A_5\), then
\((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_6\).

(ii) For each \(3 \leq i \leq 5\), if \((T_1, f_1, r_1) \in B_6\) and \((T_2, f_2, r_2) \in A_i\), then
\((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_6\).

Therefore, we need to prove \(B_6 \subseteq (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)\).

Let \((T, f, r) \in B_6\) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\), then we have \(f \in R2DF_s(T + 2), r_1 \in B\) and \(f(N[r]) = 0\). It implies \(r_2 \in C\). Consider the following cases.

\textit{Case 1.} \(f_1 \in R2DF_s(T_1)\). Since \(f_1(N[r_1]) = f(N[r]) = 0\), we have \((T_1, f_1, r_1) \in B_3\). It implies \((T_2, f_2, r_2) \in A_5\).

\textit{Case 2.} \(f_1 \notin R2DF_s(T_1)\). Since \(f_1(N[r_1]) = f(N[r]) = 0\), then we obtain \((T_1, f_1, r_1) \in B_5\) or \(B_6\). If \((T_1, f_1, r_1) \in B_5\), we have \(f_1 \in R2DF_s(T_1 + 1)\). Since \(f \in R2DF_s(T + 2)\), it means that \(f_2 \in R2DF_s(T + 2)\). Then we deduce \((T_2, f_2, r_2) \in A_5\).

Hence, \(B_6 \subseteq (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)\).

The final step is to define the initial vector. In this case, for block-cutpoint graphs, the only basis graph is a single vertex. We can use the similar method in Section 3 to initialize the vector. It is clear that if \(v\) is a C-vertex, then the initial vector is \([2, 1, \infty, \infty, 0, \infty]\); if \(v\) is a B-vertex and \(v\) represents a block of type 0, then the initial vector is \([\infty, \infty, 0, \infty, \infty, \infty]\); if \(v\) is a B-vertex and \(v\) represents a block of type 1, then the initial vector is \([\infty, 1, \infty, \infty, 0, 0]\); if \(v\) is a B-vertex and \(v\) represents a block of type 2, then the initial vector is \([\infty, \infty, \infty, \infty, 0, 0]\). Among them, \(\infty\) means undefined. From the above argument, we can obtain the following theorem.

\textbf{Theorem 22.} Algorithm \(ROMAN\{2\}\)-\textit{DOM-IN-BLOCK} can output the Roman \(\{2\}\)-domination number of any block graphs \(G = (V, E)\) in linear time \(O(n)\), where \(n = |V|\).

\textbf{Proof.} One can prove Theorem 22 by the similar argument as in the proof of Theorem 8.
Algorithm 2: ROMAN $\{2\}$-DOM-IN-BLOCK

Input: A connected block graph $G$ ($G \not\cong K_n$) and its corresponding block-cutpoint graph $T = (V, E)$ with a tree ordering $v_1, v_2, \ldots, v_n$.

Output: The Roman $\{2\}$-domination number $\gamma_{\{2\}}(G)$.

for $i := 1$ to $n$ do
  if $v_i$ is a $C$-vertex then
    initialize $h[i, 1.6]$ to $[2, 1, \infty, \infty, 0, \infty]$;
  else if $v_i$ is a $B$-vertex representing a block of type 0 then
    initialize $h[i, 1.6]$ to $[\infty, \infty, 0, \infty, \infty, \infty]$;
  else if $v_i$ is a $B$-vertex representing a block of type 1 then
    initialize $h[i, 1.6]$ to $[\infty, 1, \infty, \infty, 0, 0]$;
  else
    initialize $h[i, 1.6]$ to $[\infty, \infty, \infty, \infty, 0, 0]$;
  for $j := 1$ to $n - 1$ do
    $v_k = F(v_j)$;
    if $v_k$ is a $C$-vertex then
      $h[k, 1] = \min\{h[k, 1] + h[j, 1], h[k, 1] + h[j, 2], h[k, 1] + h[j, 3], h[k, 1] + h[j, 4], h[k, 1] + h[j, 5], h[k, 1] + h[j, 6]\};$
      $h[k, 2] = \min\{h[k, 2] + h[j, 1], h[k, 2] + h[j, 2], h[k, 2] + h[j, 3], h[k, 2] + h[j, 4], h[k, 2] + h[j, 5], h[k, 2] + h[j, 6]\};$
      $h[k, 3] = \min\{h[k, 3] + h[j, 1], h[k, 3] + h[j, 2], h[k, 3] + h[j, 3], h[k, 3] + h[j, 4], h[k, 3] + h[j, 5], h[k, 3] + h[j, 6]\};$
      $h[k, 4] = \min\{h[k, 4] + h[j, 1], h[k, 4] + h[j, 2], h[k, 4] + h[j, 3], h[k, 4] + h[j, 4], h[k, 4] + h[j, 5], h[k, 4] + h[j, 6]\};$
      $h[k, 5] = \min\{h[k, 5] + h[j, 1], h[k, 5] + h[j, 2], h[k, 5] + h[j, 3], h[k, 5] + h[j, 4], h[k, 5] + h[j, 5], h[k, 5] + h[j, 6]\};$
    else
      $S_1 = h[k, 2] + h[j, 1];$
      $S_2 = h[k, 3] + h[j, 2];$
      $S_3 = h[k, 4] + h[j, 3];$
      $h[k, 1] = \min\{h[k, 1] + h[j, 1], h[k, 1] + h[j, 2], h[k, 1] + h[j, 3], h[k, 1] + h[j, 4], h[k, 1] + h[j, 5], h[k, 1] + h[j, 6]\};$
      $h[k, 2] = \min\{h[k, 2] + h[j, 1], h[k, 2] + h[j, 2], h[k, 2] + h[j, 3], h[k, 2] + h[j, 4], h[k, 2] + h[j, 5], h[k, 2] + h[j, 6]\};$
      $h[k, 3] = \min\{h[k, 3] + h[j, 1], h[k, 3] + h[j, 2], h[k, 3] + h[j, 3], h[k, 3] + h[j, 4], h[k, 3] + h[j, 5], h[k, 3] + h[j, 6]\};$
      $h[k, 4] = \min\{h[k, 4] + h[j, 1], h[k, 4] + h[j, 2], h[k, 4] + h[j, 3], h[k, 4] + h[j, 4], h[k, 4] + h[j, 5], h[k, 4] + h[j, 6]\};$
      $h[k, 5] = \min\{h[k, 5] + h[j, 1], h[k, 5] + h[j, 2], h[k, 5] + h[j, 3], h[k, 5] + h[j, 4], h[k, 5] + h[j, 5], h[k, 5] + h[j, 6]\};$
  end for

end for

return $\gamma_{\{2\}}(G) = \min\{h[n, 1], h[n, 2], h[n, 3]\};$
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References


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