A NEW UPPER BOUND FOR THE PERFECT ITALIAN DOMINATION NUMBER OF A TREE

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Abstract

A perfect Italian dominating function (PIDF) on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for every vertex $u$ with $f(u) = 0$, the total weight of $f$ assigned to the neighbors of $u$ is exactly two. The weight of a PIDF is the sum of its functions values over all vertices. The perfect Italian domination number of $G$, denoted $\gamma^*_p(G)$, is the minimum weight of a PIDF of $G$. In this paper, we show that for every tree $T$ of order $n \geq 3$, with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma^*_p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5}$, improving a previous bound given by T.W. Haynes and M.A. Henning in [Perfect Italian domination in trees, Discrete Appl. Math. 260 (2019) 164–177].

Keywords: Italian domination, Roman domination, perfect Italian domination.

2010 Mathematics Subject Classification: 05C69.
1. Introduction

Throughout this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V, E$). The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its closed neighborhood is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A leaf of $G$ is a vertex of degree one and a support vertex is a vertex adjacent to a leaf. An end support vertex is a support vertex having at most one non-leaf neighbor. For every vertex $v \in V$, the set of all leaves adjacent to $v$ is denoted by $L(v)$ and $L[v] = L(v) \cup \{v\}$.

We denote the set of leaves of a graph $G$ by $L(G)$ and the set of support vertices by $S(G)$. We also let $|S(G)| = s(G)$ and $|L(G)| = \ell(T)$. A double star $DS_{q,p}$, with $q \geq p \geq 1$, is a graph consisting of the union of two stars $K_{1,q}$ and $K_{1,p}$ together with an edge joining their centers. The subdivision graph $S_b(G)$ of a graph $G$ is that graph obtained from $G$ by replacing each edge $uv$ of $G$ by a vertex $w$ and edges $uw$ and $vw$. A healthy spider $S_k(G)$ is the subdivision graph of a star $K_{1,k}$ for $k \geq 2$. A wounded spider $S_{k,t}$ is a graph obtained from a star $K_{1,k}$ by subdividing $t$ edges exactly once, where $1 \leq t \leq k - 1$. We denote by $P_n$ the path on $n$ vertices. The distance $d_G(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the greatest distance between two vertices of $G$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denote the set of children of $v$, $D(v)$ denotes the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. Also, the depth of $v$, $\text{depth}(v)$, is the largest distance from $v$ to a vertex in $D(v)$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

For a real-valued function $f : V \rightarrow \mathbb{R}$, the weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$.

A Roman dominating function on $G$, abbreviated RDF, is a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. Roman domination was introduced by Cockayne et al. in [7] and was inspired by the work of ReVelle and Rosing [12] and Stewart [13]. Several new varieties of Roman domination have been introduced since 2004, among them, we quote the Italian domination originally published in [1] and called Roman $\{2\}$-domination. Further results on Roman domination and its variant can be found in [2–6].

An Italian dominating function on $G$, abbreviated IDF, is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that for every vertex $v \in V$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$, that is either $v$ is adjacent to a vertex $u$ with $f(u) = 2$, or to at least two vertices $x$ and $y$ with $f(x) = f(y) = 1$. The Italian domination number, denoted $\gamma_I(G)$, is the minimum weight of an IDF in $G$.

The concept of perfect dominating sets introduced by Livingston and Stout...
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in [11] has been extended to Roman and Italian dominating functions in [10] and [9], respectively. An RDF $f$ is called perfect if for every vertex $v$ with $f(v) = 0$, there is exactly one vertex $u \in N(v)$ with $f(u) = 2$, while a IDF $g$ is perfect if for every vertex $w$ with $g(w) = 0$, $g(N(v)) = 2$. The perfect Roman domination number (respectively, perfect Italian domination number) of $G$, denoted $\gamma^R_p(G)$ (respectively, $\gamma^p_I(G)$), is the minimum weight of a perfect RDF (respectively, perfect IDF) in $G$. A perfect IDF on $G$ will be abbreviated PIDF. A PIDF $f$ is called a $\gamma^p_I(G)$-function if $\omega(f) = \gamma^p_I(G)$.

It was shown in [10] that every tree $T$ of order $n \geq 3$ satisfies $\gamma^R_p(T) \leq \frac{4}{5}n$. However, this upper bound has recently been improved by Darko et al. [8] for trees $T$ with $\ell(T) \geq 2s(T) - 2$, by showing that for any tree $T$ of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma^p_R(T) \leq (4n - \ell(T) + 2s(T) - 2)/5$. Moreover, Henning and Haynes showed in [9] that $\frac{2}{5}n$ is also an upper bound of the perfect Italian domination number for any tree of order $n \geq 3$.

In this paper, we shall show that for any tree $T$ of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma^p_R(T) \leq (4n - \ell(T) + 2s(T) - 1)/5$. But first let us point out that for both parameters $\gamma^p_R(G)$ and $\gamma^p_I(G)$, one may be larger or smaller than the other even for trees. Indeed, for the path $P_3$ we have $\gamma^p_R(P_3) = 4$ and $\gamma^p_I(P_3) = 3$ while for the double star $DS_{3,1}$ we have $\gamma^p_R(DS_{3,1}) = 3$ and $\gamma^p_I(DS_{3,1}) = 4$. The next result shows that the differences $\gamma^I_p(G) - \gamma^p_R(G)$ and $\gamma^p_R(G) - \gamma^p_I(G)$ can be arbitrarily large.

**Observation 1.** For any integer $k \geq 1$, there exist trees $T_k$ and $H_k$ such that $\gamma^p_I(T_k) - \gamma^p_R(T_k) = k$ and $\gamma^p_R(H_k) - \gamma^p_I(H_k) = k$.

**Proof.** Let $T_k$ be the tree formed by $k$ double stars $DS_{3,1}$ by adding a new vertex attached to every support vertex of degree four. One can easily see that $\gamma^I_p(T_k) = 4k + 1$ while $\gamma^p_R(T_k) = 3k + 1$.

Now, let $H_k$ be the tree formed by $k$ paths $P_3$ by adding a new vertex attached to all center vertices of the paths. Then $\gamma^p_I(H_k) = 3k + 1$ while $\gamma^p_R(H_k) = 4k + 1$. ■

### 2. New Upper Bound

In this section, we present our main result which is an upper bound on the perfect Italian domination number of a tree.

**Theorem 2.** If $T$ is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then

$$\gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$ 

**Proof.** We proceed by induction on the order $n$. If $n \in \{3, 4\}$, then clearly $\gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$, establishing the base case. Let $n \geq 5$ and assume that
any tree $T'$ of order $n'$, with $3 \leq n' < n$ satisfies $\gamma_f^p(T') \leq \frac{4n - \ell(T') + 2s(T') - 1}{5}$. Let $T$ be a tree of order $n$. If $\text{diam}(T) = 2$, then $T$ is a star, where $\gamma_f^p(T) = 2 < \frac{4n - \ell(T) + 2s(T) - 1}{5}$. If $\text{diam}(T) = 3$, then $T$ is a double star, and since $n \geq 5$ we have $\gamma_f^p(T) = 4 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence, we may assume that $T$ has diameter at least 4. If $n = 5$, then $T$ is a path $P_5$, where $\gamma_f^p(P_5) = 3 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence let $n \geq 6$.

Suppose $v_1v_2 \cdots v_k (k \geq 5)$ is a diametral path in $T$ such that $\text{deg}_T(v_2)$ is as large as possible. Root $T$ at $v_k$. First, assume that $T$ has an end support vertex $y$ of degree three. Without loss of generality, assume that $y = v_2$. Let $T' = T - T_{v_2}$ and $f'$ be a $\gamma_f^p(T')$-function. If $f'(v_3) = 0$, then $f'$ can be extended to a PIDF of $T$ by assigning a 0 to $v_2$ and a 1 to the two leaves of $v_2$. If $f'(v_3) \geq 1$, then $f'$ can be extended to a PIDF of $T$ by assigning a 2 to $v_2$ and a 0 to the leaves of $v_2$. In either case, $\gamma_f^p(T) \leq \gamma_f^p(T') + 2$, and by the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_f^p(T') + 2 \leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + 2 \\
\leq \frac{4(n - 3) - \ell(T) + 2 + 2s(T) - 1}{5} + 2 \\
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Hence we can assume that $T$ has no end support vertex of degree three, in particular we have $\text{deg}_T(v_2) \neq 3$. Next, suppose that $\text{deg}_T(v_3) = 2$. If $\text{deg}_T(v_2) = 2$, then let $T'' = T - T_{v_3}$ and $f'$ be a $\gamma_f^p(T')$-function. Note that $n' = n - 3$, $s(T') \leq s(T)$ and $\ell(T') \geq \ell(T) - 1$. Now if $f'(v_4) = 0$, then the function $f$ defined by $f(v_2) = 2$, $f(v_1) = f(v_3) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus \{v_1, v_2, v_3\}$ is a PIDF of $T$. If $f'(v_4) \geq 1$, then the function $f$ defined by $f(v_1) = f(v_3) = 1$, $f(v_2) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus \{v_1, v_2, v_3\}$ is a PIDF of $T$. In either case, $\gamma_f^p(T) \leq \gamma_f^p(T') + 2$, and by the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_f^p(T') + 2 \leq \frac{4(n - 3) - \ell(T) + 1 + 2s(T) - 1}{5} + 2 \\
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Suppose now that $\text{deg}_T(v_2) \geq 4$. Let $T' = T - T_{v_3}$ and $f'$ be a $\gamma_f^p$-function of $T'$. Note that $T'$ has order $n' \geq 2$. Clearly if $n' = 2$, then $\gamma_f^p(T') = 4 < \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence we assume that $n' \geq 3$. If $f'(v_4) = 0$, then we can extend $f'$ to a PIDF of $T$ by assigning a 2 to $v_2$ and a 0 to every neighbor of $v_2$. If $f'(v_4) \geq 1$, then we can extend $f'$ to a PIDF $f$ of $T$ by assigning a 2 to $v_2$, a 1 to $v_3$, and a 0 to all leaves of $v_2$. In either case, $\gamma_f^p(T) \leq \gamma_f^p(T') + 3$ and by the induction hypothesis we obtain
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\[ \gamma_p^I(T) \leq \gamma_p^I(T') + 3 \leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + 3 \]
\[ \leq \frac{4(n - |L(v_2)| - 2) - (\ell(T) - |L(v_2)|) + 2s(T) - 1}{5} + 3 \]
\[ = \frac{4n - \ell(T) + 2s(T) - 1 - 3L(v_2) - 8}{5} + 3 < \frac{4n - \ell(T) + 2s(T) - 1}{5}. \]

From now on, we can assume that \( \deg_T(v_3) \geq 3 \) and \( \deg_T(v_2) \neq 3 \). Note that often in our proof a subtree \( T' \) of \( T \) is considered, and so in either case, let \( f' \) be a \( \gamma_p^I(T') \)-function. Consider the following cases.

Case 1. \( \deg_T(v_2) \geq 4 \) and \( T_{v_3} \neq DS_{3,1} \). Let us examine the following situations.

Subcase 1.1. \( v_3 \) has at least two leaves. Let \( T' \) be the tree of order \( n' \) obtained from \( T \) by removing all leaves of \( v_2 \). Note that \( n' = n - |L(v_2)| \), \( s(T') = s(T) - 1 \) and \( \ell(T') = \ell(T) - |L(v_2)| + 1 \). Since \( v_3 \) has at least three leaves in \( T' \), we conclude that \( f'(v_3) \geq 1 \). Hence the function \( f \) defined by \( f(v_2) = 2, f(x) = 0 \) for all \( x \in L(v_2) \) and \( f(x) = f'(x) \) for \( x \in V(T) \setminus L[v_2] \) is a PIDF of \( T \). It follows that \( \gamma_p^I(T) \leq \gamma_p^I(T') + 2 \), and by the induction hypothesis we obtain

\[ \gamma_p^I(T) \leq \gamma_p^I(T') + 2 \leq \frac{4(n - |L(v_2)|) - \ell(T) + |L(v_2)| - 1 + 2s(T) - 3}{5} + 2 \]
\[ < \frac{4n - \ell(T) + 2s(T) - 1}{5}. \]

Subcase 1.2. \( v_3 \) has exactly one leaf, say \( v' \). If \( v_2 \) is the unique child of \( v_3 \) with depth 1, then let \( T' \) be the tree of order \( n' \) obtained from \( T \) by removing all vertices in \( T_{v_2} \) and adding two new vertices \( x_1, x_2 \) attached at \( v_3 \). Since \( v_3 \) has at least three leaves, we have \( f'(v_3) \geq 1 \), and thus the function \( f \) defined by \( f(v_2) = 2, f(x) = 0 \) for \( x \in L(v_2) \) and \( f(x) = f'(x) \) for \( x \in V(T) \setminus L(v_2) \) is a PIDF of \( T \). Hence \( \gamma_p^I(T) \leq \gamma_p^I(T') + 2 \), and since \( T_{v_3} \neq DS_{3,1} \), we must have \( |L(v_2)| \geq 4 \). It follows from the induction hypothesis that

\[ \gamma_p^I(T) \leq \gamma_p^I(T') + 2 \leq \frac{4(n + 1 - |L(v_2)|) - \ell(T) + |L(v_2)| - 2 + 2s(T) - 3}{5} + 2 \]
\[ < \frac{4n - \ell(T) + 2s(T) - 1}{5}. \]

Suppose that \( v_3 \) has (at least) two children with depth 1, say \( a \) and \( b \) such that \( \deg_T(a) \geq 4 \) and \( \deg_T(b) \geq 4 \). Let \( T' \) be the tree formed from \( T \) by deleting all leaves of \( a \) and \( b \). Note that \( n' = n - |L(a)| - |L(b)| \), \( s(T') = s(T) - 2 \) and \( \ell(T') = \ell(T) - |L(a)| - |L(b)| + 2 \). Clearly, \( f'(v_3) \geq 1 \) since \( v_3 \) has three leaves in \( T' \). Thus the function \( f \) defined by \( f(a) = f(b) = 2, f(x) = 0 \) for all
$x \in L(a) \cup L(b)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus (L[a] \cup L[b])$ is a PIDF of $T$, and so $\gamma_T^p(T) \leq \gamma_T^p(T') + 4$. Using the fact $|L(a)| \geq 3$ and $|L(b)| \geq 3$ and the induction hypothesis we obtain

$$\gamma_T^p(T) \leq \gamma_T^p(T') + 4$$

$$\leq \frac{4(n - |L(a)| - |L(b)|) - \ell(T) + |L(a)| + |L(b)| - 2 + 2s(T) - 5}{5} + 4$$

$$< \frac{5}{5}.$$ 

Hence we can assume now that $v_2$ is the unique child of $v_3$ with depth one and degree at least 4. Recall that since $\deg_T(v_2) \neq 3$, we may assume that every child of $v_3$ with depth 1 that is different from $v_2$ has degree two. Note that $|C(v_3)| \geq 3$.

Assume first that $|C(v_3)| \geq 4$, and let $T'$ be the tree of order $n'$ obtained from $T - T_{v_3}$ by adding three new vertices $x_1, x_2, x_3$ attached at $v_4$. Note that $n' = n - |C(v_3)| - |L(T_{v_3})| + 3$, $\ell(T') = \ell(T) - L(T_{v_3}) + 3$, $s(T') \leq s(T) - |C(v_3)| + 1$. Now, since $v_3$ has three leaves in $T'$, we must have $f'(v_4) \geq 1$, and thus the function $f$ defined by $f(v_2) = 2$, $f(x) = 1$ for $x \in \{v', v_3\} \cup (L(T_{v_3}) \setminus L(v_2))$, $f(x) = 0$ for all $x \in (C(v_3) \setminus \{v_2, v'\}) \cup L(v_2)$ and $f(x) = f'(x)$ for otherwise, is a PIDF of $T$. Hence $\gamma_T^p(T) \leq \gamma_T^p(T') + |C(v_3)| + 2$, and by the induction hypothesis it follows that

$$\gamma_T^p(T) \leq \gamma_T^p(T') + |C(v_3)| + 2$$

$$\leq \frac{4(n - |C(v_3)| + 3 - |L(T_{v_3})|) - \ell(T) + |L(T_{v_3})| - 3 + 2s(T) - 2|C(v_3)| + 1}{5} + |C(v_3)| + 2 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{|C(v_3)| - 3|L(T_{v_3})| + 1}{5}.$$ 

Moreover, since $|L(T_{v_3})| \geq |C(v_3)| + 2$, we have $\gamma_T^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-4|C(v_3)| + 15}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}$ because of $|C(v_3)| \geq 4$. Next, we can assume that $|C(v_3)| = 3$, that is $T_{v_3}$ is isomorphic to $H_1$ in Figure 1. In this case, let $T'$ be the tree formed from $T$ by removing all vertices of $T_{v_3}$ except $v_3$. Clearly $v_3$ is a leaf in $T'$. If $f'(v_3) = 0$, then $f(v_4) = 2$ and so the function $f$ defined by $f(v_3) = f(v') = f(u_1) = 1$, $f(v_2) = 2$, $f(x) = 0$ for all $x \in L(v_2) \cup \{u_2\}$ and $f(x) = f'(x)$ for otherwise is a PIDF of $T$. If $f'(v_3) = 1$, then we can extend $f'$ to be a PIDF of $T$ as above when $f'(v_3) = 0$, except that we do not assign a 1 to $v_3$. In either case, $\gamma_T^p(T) \leq \gamma_T^p(T') + 5$. It follows from the induction hypothesis that

$$\gamma_T^p(T) \leq \gamma_T^p(T') + 5 \leq \frac{4(n - 4 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 5}{5} + 5$$

$$< \frac{5}{5}.$$
Finally, if $f'(v_3) = 2$, then the function $f$ defined by $f(v_2) = f(u_2) = 2$, $f(x) = 0$ for all $x \in L(v_2) \cup \{u_1, v'\}$ and $f(x) = f'(x)$ for otherwise is a PIDF of $T$. Using the induction hypothesis we obtain

$$\gamma^p_I(T) \leq \gamma^p_I(T') + 4 \leq \frac{4(n - 4 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 5}{5} + 4$$

$$< \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$ 

Figure 1. The trees.

Subcase 1.3. $v_3$ is not a support vertex. Suppose that $v_3$ has at least three children of degree at least 4, say $a, b$ and $c$. Let $T'$ be the tree obtained from $T$ by removing all leaves of $a, b$ and $c$. Note that $n' = n - |L(a)| - |L(b)| - |L(c)|$, $s(T') = s(T) - 2$ and $\ell(T') = \ell(T) - |L(a)| - |L(b)| - |L(c)| + 3$. Clearly, since $v_3$ has three leaves in $T'$, $f'(v_3) \geq 1$, and thus the function $f$ defined by $f(a) = f(b) = f(c) = 2$, $f(x) = 0$ for all $x \in L(a) \cup L(b) \cup L(c)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus (L[a] \cup L[b] \cup L[c])$ is a PIDF of $T$. By the induction hypothesis, it follows that

$$\gamma^p_I(T) \leq \gamma^p_I(T') + 6$$

$$\leq \frac{4(n - |L(a)| - |L(b)| - |L(c)|) - \ell(T) + |L(a)| + |L(b)| + |L(c)| - 3 + 2s(T) - 5}{5} + 6$$

$$< \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$
Hence, $v_3$ has at most two children of degree at least 4, say $v_3$ and $u$ (if any). Let $T'$ be the tree of order $n'$ obtained from $T - T_{v_3}$ by adding three new vertices attached at $v_4$. Note that $n' = n - |C(v_3)| - |L(T_{v_3})| + 2$, $s(T') \leq s(T) - |C(v_3)| + 1$ and $\ell(T') = \ell(T) - |L(T_{v_3})| + 3$. Clearly, $\gamma_f(3) \geq 1$. Hence the function $f$ defined by $f(x) = 2$ for $x \in \{v_2, u\}$, $f(x) = 1$ for $x \in (L(T_{v_3}) \cup \{v_3\}) \setminus (L(v_2) \cup L(u))$, $f(x) = 0$ for $x \in (C(v_3) \setminus \{v_2, u\}) \cup (L(v_2) \cup L(u))$ and $f(x) = f'(x)$ for otherwise is a PIDF of $T$. By the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_f^p(T') + |C(v_3)| + 3
\]

\[
\leq 4(n - |C(v_3)| - |L(T_{v_3})| + 2) - \ell(T) + |L(T_{v_3})| - 3 + 2s(T) - 2|C(v_3)| + 1
\]

\[
+ |C(v_3)| + 3 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-|C(v_3)| - 3|\ell(T_{v_3})| + 22}{5}.
\]

Since $|L(T_{v_3})| \geq |C(v_3)| + 2$, we have $\gamma_f^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-4|C(v_3)| + 16}{5}$. If $|C(v_3)| \geq 4$, then $\gamma_f^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence, $2 \leq |C(v_3)| \leq 3$. If $|C(v_3)| = 3$ and $v_3$ has two children of degree at least 4, then one can easily see that $\gamma_f^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$ (since $|L(T_{v_3})| \geq |C(v_3)| + 4$). In the sequel, we can assume that $T_{v_3}$ is isomorphic to one of $H_2, H_3, H_4$ depicted in Figure 1. In that case, let $T''$ be the tree formed from $T$ by removing all vertices of $T_{v_3}$ except $v_3$. Clearly $v_3$ is a leaf in $T''$. Let $f''$ be a $\gamma_f^p(T'')$-function. If $f''(v_3) = 0$, then $f''(v_3) = 2$ and so let $f$ be a PIDF of $T$ defined as follows: $f(x) = f''(x)$ for all $x \in V(T) \setminus \{v_3\}$ and $f(v_3) = 1$. Moreover, every child of $v_3$ of degree 2 is assigned a 0 and its unique leaf a 1; every child of $v_3$ of degree at least 4 is assigned a 2 and its leaves a 0. If $f''(v_3) = 1$, then $f''$ will be extended to a PIDF of $T$ as above when $f'(x) = 0$, except we do not assign a 1 to $v_3$. Finally, if $f''(v_3) = 2$, then we use the following assignment for vertices of $T_{v_3}$: assign a 2 to each child of $v_3$ and a 0 to each of their leaves. Now, if $T_{v_3} = H_2$, then in either case described above, we have $\gamma_f^p(T) \leq \gamma_f^p(T'') + 4$. By the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_f^p(T'') + 4 \leq \frac{4n - 3 - |L(v_2)| - \ell(T) + |L(v_2)| + 1 + 2s(T) - 3}{5} + 4
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

If $T_{v_3} = H_3$, then $\gamma_f^p(T) \leq \gamma_f^p(T'') + 5$, and by the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_f^p(T'') + 5
\]

\[
< \frac{4n - 2 - |L(v_2)| - |L(u)| - \ell(T) + |L(v_2)| + |L(u)| + 2s(T) - 3}{5} + 5
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]
Moreover, if \( T_{v_3} = H_4 \), then \( \gamma^p_I(T) \leq \gamma^p_I(T'') + 6 \), and by the induction hypothesis it follows that
\[
\gamma^p_I(T) \leq \gamma^p_I(T'') + 6 \leq \frac{4(n - 5 - |L(v_2)|) - \ell(T) + 2 + |L(v_2)| + 2s(T) - 5}{5} + 6 \\
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Before discussing Case 2, we will need the following claim.

**Claim.** Let \( T \) be a wounded spider of order \( n \) different from \( DS_{2,1} \), with \( s(T) \) support vertices and \( \ell(T) \) leaves. Then we have the following.

(i) If \( 6s(T) - 2\ell(T) \geq 11 \), then \( \gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 6}{5} \).

(ii) If \( 6s(T) - 2\ell(T) \leq 11 \), then \( \gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 3}{5} \).

**Proof.** Let \( v \) be the center vertex of \( T \).

(i) If \( 6s(T) - 2\ell(T) \geq 11 \), then the function \( f \) defined by assigning a 1 to \( v \) and every leaf of \( T \), and a 0 to remaining vertices of \( T \), is a PIDF of \( T \) and so
\[
\gamma^p_I(T) \leq \omega(f) = \ell(T) + 1 \leq \frac{4n - \ell(T) + 2s(T) - 6}{5}.
\]

(ii) Let \( t = |L(v)| - 1 \). Clearly, \( \ell(T) = s(T) + t \) and since \( 6s(T) - 2\ell(T) \leq 11 \), then \( T \) is a double star and since \( T \) is not a \( DS_{2,1} \), we can see that we have
\[
4s(T) - 2t \leq 11 \quad \text{and thus} \quad t \geq 2s(T) - \frac{11}{2}.
\]
Now if \( s(T) = 2 \), then \( T \) is a double star and since \( T \) is not a \( DS_{2,1} \), we can see that \( \gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 3}{5} \). Hence, let \( s(T) \geq 3 \). Then the function \( f \) defined by assigning a 2 to the support vertices of \( T \) and a 0 to remaining vertices of \( T \) is a PIDF of \( T \) of weight \( 2s(T) \). Since, \( n = s(T) + \ell(T) \) and \( \ell(T) = s(T) + t \), it follows that
\[
\frac{4n - \ell(T) + 2s(T) - 3}{5} = \frac{9s(T) + 3t - 3}{5}.
\]
Moreover, since \( t \geq 2s(T) - \frac{11}{2} \) we obtain
\[
\frac{9s(T) + 3t - 3}{5} \geq \frac{9s(T) + 6s(T) - \frac{33}{2} - 3}{5} = 3s(T) - \frac{39}{10}.
\]
Now, if \( s(T) \geq 4 \), then \( 3s(T) - \frac{39}{10} \geq 2s(T) \geq \gamma^p_I(T) \) and so the desired result follows. Thus we assume that \( s(T) = 3 \). If \( t \geq 2s(T) - \frac{7}{2} \), then as above we have \( \frac{9s(T) + 3t - 3}{5} \geq 3s(T) - \frac{27}{10} \geq 2s(T) \geq \gamma^p_I(T) \). Hence, let \( t \leq 2s(T) - \frac{7}{2} = 2.5 \). Note that in this case \( \ell(T) \in \{3, 4, 5\} \). Then assigning a 1 to \( v \) and the leaves of \( T \) and a 0 to remaining vertices of \( T \) provides a PIDF of \( T \) of weight \( \ell(T) + 1 \leq \frac{4n - \ell(T) + 2s(T) - 3}{5} \), which completes the proof of the claim. \( \square \)

We note from the proof of the claim that there exist PIDFs of \( T \) of weight at most \( \frac{4|V(T_{v_3})| - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \) that assign to the center vertex a 1 or 2.
Now we are ready to examine the next case.

**Case 2.** $\text{deg}_T(v_2) = 2$ or $T_{v_3} = DS_{3,1}$. From Case 1 and since $v_2$ was chosen having a maximum degree, we conclude that $T_{v_3}$ is a spider. Assume first that $T_{v_3}$ is a healthy spider. If $|C(v_3)| \geq 3$, then let $T'$ be the tree obtained by removing $T_{v_3}$ and adding three new vertices attached at $v_4$. Note that $n' = n - 2|C(v_3)| + 2$, $s(T') \leq s(T) - |C(v_3)| + 1$ and $\ell(T') = \ell(T) - |C(v_3)| + 3$. Clearly, $f'(v_4) \geq 1$ (since $v_4$ has three leaves in $T'$). Thus the function $f$ defined by $f(x) = 1$ for $x \in L(T_{v_3}) \cup \{v_3\}$, $f(x) = 0$ for $x \in C(v_3)$ and $f(x) = f'(x)$ for $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of $T$. Hence $\gamma^p_i(T) \leq \gamma^p_i(T') + |C(v_3)| + 1$, and by the induction hypothesis we obtain

\[
\gamma^p_i(T) \leq \gamma^p_i(T') + |C(v_3)| + 1
\]

\[
\leq \frac{4(n - |C(v_3)| + 2) - \ell(T) + |C(v_3)| - 3 + 2s(T) - 2|C(v_3)| + 1}{5} + |C(v_3)| + 1
\]

\[
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Now, assume that $|C(v_3)| = 2$, and let $T' = T - T_{v_3}$. If $f'(v_4) \geq 1$, then the function $f$ defined by $f(x) = 1$ for $x \in L(T_{v_3}) \cup \{v_3\}$, $f(x) = 0$ for every $x \in C(v_3)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of $T$ of weight $\gamma^p_i(T') + 3$. If $f'(v_4) = 0$, then the function $f$ defined by $f(x) = 1$ for $x \in V(T_{v_3}) \setminus \{v_3\}$, $f(v_3) = 0$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of $T$ of weight $\gamma^p_i(T') + 4$. In either case, $\gamma^p_i(T) \leq \gamma^p_i(T') + 4$ and by the induction hypothesis we obtain

\[
\gamma^p_i(T) \leq \gamma^p_i(T') + 4 \leq \frac{4(n - 5) - \ell(T) + 2s(T) - 3}{5} + 4
\]

\[
= \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Suppose now that $T_{v_3}$ is a wounded spider $S_{k,k}$. If $T_{v_3} = DS_{2,1}$, then let $T' = T - T_{v_3}$. Clearly we assume that $n' \geq 2$. If $n' = 2$, then $\gamma^p_i(T') = 5 < \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence we assume that $n' \geq 3$. If $f'(v_4) \geq 1$, then the function $f$ defined by $f(v_2) = f(v_3) = 2, f(x) = 0$ for $x \in L(T_{v_3})$ and $f(x) = f'(x)$ for $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of $T$. If $f'(v_4) = 0$, then the function $f$ defined by $f(v_1) = 2, f(x) = 1$ for $x \in L(v_3), f(v_2) = f(v_3) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of $T$. In either case, $\gamma^p_i(T) \leq \gamma^p_i(T') + 4$. If $\text{deg}_T(v_4) \geq 3$, then $s(T') = s(T) - 2$ and $\ell(T') = \ell(T) - 3$ and by the induction hypothesis we obtain

\[
\gamma^p_i(T) \leq \gamma^p_i(T') + 4 \leq \frac{4(n - 5) - \ell(T) + 3 + 2s(T) - 5}{5} + 4
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]
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If $\text{deg}_T(v_4) = 2$, then $s(T') \leq s(T) - 1$ and $\ell(T') = \ell(T) - 2$ and by the induction hypothesis we obtain

$$\gamma^P_I(T) \leq \gamma^P_I(T') + 4 \leq \frac{4(n - 5) - \ell(T) + 2 + 2s(T) - 3}{5} + 4 = \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$ 

From now on we may assume that $v_4$ has no child $x$ such that $T_x = DS_{2, 1}$.

Let $s_1$ be the number of children of $v_4$ that are leaves and for $i \geq 2$, let $s_i$ be the number of children of $v_4$ of degree $i$ whose children are all leaves. As we assumed at the beginning of the proof, $T$ has no end support vertex with degree three, it follows that $s_3 = 0$. Let $s_{\geq 4}$ be the number of children of $v_4$ of degree at least 4 having no grandchild. Thus

$$s_{\geq 4} = \sum_{i \geq 4} s_i.$$ 

Adopting our earlier notation, for each child $v$ of $v_4$ with depth 2, let $n_v$ denote the number of children in the subtree $T_v$ of $T$. Furthermore, let $n^*$ denote the sum of the number of vertices in all such trees $T_v$. Also, let $s^*$ and $\ell^*$ denote the sum of the number of support vertices and leaves vertices in all such trees $T_v$, respectively. Note that every child of $v_4$ is one of the following four types: (1) a leaf; (2) a support vertex of degree 2; (3) a vertex with depth 2; (4) a support vertex of degree at least 4 whose children are all leaves. For ease of discussion, we sometimes refer to these children as Type-1, Type-2, Type-3, or Type-4, respectively. Moreover, let $m$ be the number of leaves of all Type-4 children. Consider now the following subcases.

Subcase 2.1. $s_1 + s_{\geq 4} \geq 3$. Let $T' = T - T_{v_3}$ be a tree of order $n'$. We claim that $f'(v_4) \geq 1$. Suppose to the contrary that $f'(v_4) = 0$. This implies that at most two children of $v_4$ in $T'$ are assigned positive values under $f'$. But since every Type-1 and Type-4 child of $v_4$ must be assigned a positive value by $f'$ when $f'(v_4) = 0$, this implies that $s_1 + s_{\geq 4} \leq 2$, a contradiction. Hence, $f'(v_4) \geq 1$. Consequently, we can extend $f'$ to a PIDF $f$ by adding to it any PIDF of $T_{v_3}$ of weight at most $\frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5}$ assigning a 1 or 2 to $v_3$ (as claimed above). By the induction hypothesis we obtain

$$\gamma^P_I(T) \leq \gamma^P_I(T') + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \leq \frac{4(n - n_{v_3}) - \ell(T) + \ell(T_{v_3}) + 2s(T) - 2s(T_{v_3}) - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$
In the sequel, we may assume that \( s_1 + s_{\geq 4} \leq 2 \).

**Subcase 2.2.** \( s_1 = 2 \). Since \( s_1 + s_{\geq 4} \leq 2 \), we deduce that \( s_{\geq 4} = 0 \). Let \( F \) be the forest formed by the Type-3 children of \( v_4 \) and their descendants. We note any component of \( F \) is a wounded spider including \( T_{v_3} \) and different from \( DS_{2,1} \). Let \( T' \) be the tree obtained from \( T \) by deleting all vertices in \( V(F) \) and adding a new vertex \( a \) attached at \( v_4 \). Since \( v_4 \) has three leaf neighbors in \( T' \), we have \( f'(v_4) \geq 1 \). Let \( f \) be the PIDF of \( T \) defined as follows: \( f(x) = f'(x) \) for all \( x \in V(T') \setminus \{a\} \) and let the restriction of \( f \) to each component, say \( T_v \), in \( F \) be any PIDF of that component of weight at most \( \frac{4n_s - \ell(T_v) + 2s(T_v) - 3}{5} \). By our earlier observations, the total weight assigned to \( F \) is at most \( \frac{4n_s - \ell^* + 2s^* - 3}{5} \). Now, by the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_{f'}^p(T') + \frac{4n_s - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n - n^*) - \ell(T) + \ell^* - 1 + 2s(T) - 2s^* - 1 + 4n_s - \ell^* + 2s^* - 3}{5}
\]

\[
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Hence, in the next we may assume that \( s_1 \in \{0, 1\} \).

**Subcase 2.3.** \( s_2 \geq 3 \). Let \( T' \) be the tree of order \( n' \) obtained from \( T - T_{v_4} \) by adding three new vertices \( x_1, x_2, x_3 \) attached at \( v_5 \). Note that \( n' = n - n^* - s_1 - 2s_2 - s_{\geq 4} - m + 2 \), \( \ell(T') = \ell(T) - \ell^* - s_1 - s_2 - m + 3 \) and \( s(T') \leq s(T) - s^* - s_1 - s_2 - s_{\geq 4} + 1 \). Clearly, \( f'(v_5) \geq 1 \) (since \( v_5 \) has three leaves in \( T' \)). Let \( f \) be the PIDF of \( T \) defined by \( f(x) = f'(x) \) for all \( x \in V(T') \setminus \{x_1, x_2, x_3\} \) and let \( f(v_4) = 1 \). Then assign the weights to the descendants of \( v_4 \) in \( T \) as follows: assign a 1 to each Type-1 (leaf) child of \( v_4 \) (recall that \( s_1 \in \{0, 1\} \)); assign a 0 to each Type-2 child of \( v_4 \) and a 1 to its leaf neighbor; assign a 2 to each Type-4 child of \( v_4 \) and a 0 to each of its leaves. Finally, for each Type-3 child \( v \), assign a PIDF to the subtree \( T_v \) rooted at \( v \) of weight at most \( \frac{4n_v - \ell(T_v) + 2s(T_v) - 3}{5} \) so that \( f(v) \geq 1 \). By our earlier observations, the total weight assigned to all Type-3 children of \( v \) and their descendants is at most \( \frac{4n_v - \ell^* + 2s^* - 3}{5} \). It follows from the induction hypothesis that

\[
\gamma_f^p(T) \leq \gamma_{f'}^p(T') + \frac{4n_s - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n - n^*) - \ell(T') + 2s(T') - 1 + 4n_s - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4} + 2) - \ell(T) + \ell^* + 1 + s_1 + s_2 + m - 3}{5}
\]

\[
= \frac{4n - \ell(T) + 2s(T) - 1 + 9 - 3m - 4s_2 + 4s_{\geq 4}}{5}.
\]
Using the fact that \( m \geq 3s \geq 4 \), it follows that \( \gamma_p^I(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{9} + \frac{9 - 4s - 5s \geq 4}{5} \).

Now since \( s_2 \geq 3 \), we deduce that \( \gamma_p^I(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} \).

By Subcase 2.3, we can assume that \( s_2 \leq 2 \).

**Subcase 2.4.** \( s_2 + s \geq 4 \geq 1 \). Let \( T' \) be the tree of order \( n' \) obtained by deleting all vertices of \( T_{v_4} \) except \( v_4 \). Note that \( n' = n - n^* - s_1 - 2s_2 - s \geq 4 - m \), \( s(T') \leq s(T) - s^* - s_1 - s_2 - s \geq 4 + 1 \) and \( \ell(T') = \ell(T) - \ell^* - s_1 - s_2 - m + 1 \) (since \( v_4 \) is a leaf vertex in \( T' \)). First, let \( f'(v_4) = 2 \) and \( f \) be a PIDF of \( T \) defined by \( f(x) = f'(x) \) for all \( x \in V(T') \); and then assign the weights to the descendants of \( v_4 \) in \( T \) as follows: assign a 0 to each Type-1 (leaf) child of \( v_4 \), assign a 2 to each Type-2 child of \( v_4 \) and a 0 to its leaf, and assign a 2 to each Type-4 child of \( v_4 \) and a 0 to its leaves. Finally, for each Type-3 child \( v \), assign a PIDF to the subtree \( T_v \) rooted at \( v \). By our earlier observations, the total weight assigned to all Type-3 children of \( v \) and their descendants is at most \( \frac{4n^* - \ell^* + 2s^* - 3}{5} \). By the induction hypothesis it follows that

\[
\gamma_p^I(T) \leq \gamma_p^I(T') + \frac{4n - \ell(T') + 2s(T') - 1}{5} + 2s_2 + 2s \geq 4
\]

\[
\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s \geq 4
\]

\[
\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s \geq 4) - \ell(T) + \ell^* + s_1 + s_2 + m - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s \geq 4
\]

\[
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-5s_1 + s_2 - 3m + 4s \geq 4 - 2}{5}.
\]

Now since \( m \geq 3s \geq 4 \) and \( s_2 \leq 2 \), we get

\[
\gamma_p^I(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-5s_1 + s_2 - 5s \geq 4 - 2}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Suppose now that \( f'(v_4) \in \{0, 1\} \), and let \( f \) be a PIDF of \( T \) defined by \( f(x) = f'(x) \) for all \( x \in V(T') \) and let \( f(v_4) = 1 \). Then assign the weights to the descendants of \( v_4 \) in \( T \) as follows: assign a 1 to each Type-1 (leaf) child of \( v_4 \); assign a 0 to each Type-2 child of \( v_4 \) and a 1 to its leaf neighbor and assign a 2 to each Type-4 child of \( v_4 \) and 0 to its leaves. Finally, for each Type-3 child \( v \), assign a PIDF of weight at most \( \frac{4n - \ell(T_v) + 2s(T_v) - 3}{5} \) to vertices of \( T_v \) rooted at \( v \) so that \( f(v) \geq 1 \). By our earlier observations, the total weight assigned to all Type-3 children of \( v \) and their descendants is at most \( \frac{4n^* - \ell^* + 2s^* - 3}{5} \). By the induction hypothesis we obtain
\[
\gamma_p^f(T) \leq \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]
\[
\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]
\[
\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4}) - \ell(T) + \ell^* + s_1 + s_2 + m - 1}{5}
\]
\[
\quad + \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]
\[
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5} - 4s_2 - 3m + 4s_{\geq 4} + 3.
\]

Now since \( m \geq 3s_{\geq 4} \), it follows that \( \gamma_p^f(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} - \frac{4s_2 - 5s_{\geq 4} + 3}{5} \), and since \( s_2 + s_{\geq 4} \geq 1 \), the result follows.

Subcase 2.5. \( s_2 + s_{\geq 4} = 0 \). Recall that \( s_1 \in \{0, 1\} \). Let \( v' \) be the leaf neighbor of \( v_4 \) (if any). First, let \( v_4 \) has at least two children of Type-3. Let \( T' \) be the tree of order \( n' \) obtained by deleting all vertices of \( T_{v_4} \) except \( v_4 \). Note that \( n' = n - n^* - s_1 \), \( s(T') \leq s(T) - s^* - s_1 + 1 \) and \( \ell(T') = \ell(T) - \ell^* - s_1 + 1 \) (since \( v_4 \) is a leaf vertex in \( T' \)). We also note that if \( f'(v_4) = 0 \), then since \( v_4 \) is a leaf in \( T' \), we must write \( f'(v_4) = 2 \). Now, we define a PIDF \( f \) of \( T \) by \( f(x) = f'(x) \) for all \( x \in V(T') \setminus \{v_4\} \). Moreover, \( f(v') = 1 \), \( f(1) = 1 \) if \( f(v_4) = 0 \) and \( f(v_4) = f'(v) \) if \( f'(v_4) \geq 1 \). Also, for each other child \( v \) of \( v_4 \), assign a PIDF to the subtree \( T_v \) of weight at most \( \frac{4n - \ell(T_v) + 2s(T_v) - 1}{5} \). Since there are at least two Type-3 children of \( v_4 \), the total weight assigned to such subtree \( T_v \) is \( \frac{4n^* - \ell^* + 2s^* - 2s_{\geq 4}}{5} + s_1 + 1 \). Using the induction hypothesis we obtain

\[
\gamma_p^f(T) \leq \gamma_p^f(T') + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1
\]
\[
\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1
\]
\[
\leq \frac{4(n - n^* - s_1) - \ell(T) + \ell^* + s_1 - 1 + 2s(T) - 2s^* - 2s_1 + 1}{5}
\]
\[
\quad + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

In the sequel, \( v_3 \) is the only child of \( v_4 \) of Type-3. We distinguish the following.

(i) \( T_{v_3} = DS_{1.3} \). Consider two situations depending on whether \( s_1 = 0 \) or \( s_1 = 1 \).

(a) \( s_1 = 0 \). Hence \( \deg_T(v_4) = 2 \). Let \( T' = T - T_{v_4} \). Clearly, \( n' \geq 1 \). If \( n' = 1 \), then \( T \) is a wounded spider and by the claim the result follows, and if \( n' = 2 \), then
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one can easily see that $\gamma_p^I(T) = 6 < \frac{4n-\ell(T)+2s(T)-1}{5} = 7.2$. So let $n' \geq 3$. Note that $n' = n-7$, $\ell(T') \geq \ell(T) - 4$ and $s(T') \leq s(T) - 1$. Any $\gamma_p^I(T')$-function can be extended to a PIDF of $T$ by assigning a 2 to $v_2, v_3$ and a 0 to remaining vertices of $T_{v_4}$ except $v_4$ which will be assigned a 0 if $f'(v_3) = 0$ and a 1 if $f'(v_3) \geq 1$. In either case, $\gamma_p^I(T) \leq \gamma_p^I(T') + 5$. By the induction hypothesis we obtain

$$\gamma_p^I(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + 5 \leq \frac{4(n - 7) - \ell(T) + 4 + 2s(T) - 3}{5} + 5$$

(b) $s_1 = 1$. Let $T'$ be the tree obtained from $T$ by removing all vertices $T_{v_3}$ except $v_3$. If $f'(v_3) = 0$, then $f'(v_4) = 2$, and so $f'$ can be extended to a PIDF of $T$ by assigning a 2 to $v_2, v_3$ and a 0 to remaining vertices of $T_{v_4}$. Hence $\gamma_p^I(T) \leq \gamma_p^I(T') + 4$. If $f'(v_3) = 2$, then $f'(v_4) = 0$ and so the other leaf neighbor of $v_4$ is assigned a 1, which is a contradiction. Hence, $f'(v_3) = 1$. Now, if $|L(v_3)| = 1$, then we extend $f'$ to a PIDF of $T$ by assigning a 2 to $v_2, v_3$ and a 0 to remaining vertices of $T_{v_4}$. If $|L(v_3)| = 3$, then we extend $f'$ to a PID-function of $T$ by assigning a 1 to $L(T_{v_3})$ and a 0 to $v_2$. In either case, $\gamma_p^I(T) \leq \gamma_p^I(T') + 4$. By the induction hypothesis we obtain

$$\gamma_p^I(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + 4 \leq \frac{4(n - 5) - \ell(T) + 3 + 2s(T) - 5}{5} + 4$$

(ii) $T_{v_3} = S_{k,t} \neq DS_{3,1}$. We recall that $T_{v_3}$ is different from $DS_{2,1}$. First let $6s(T_{v_3}) - 2\ell(T_{v_3}) \geq 11$. By our Claim, $\gamma_p^I(T_{v_3}) \leq \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5}$. Let $T'$ be the tree obtained from $T$ by removing all vertices of $T_{v_4}$ except $v_4$. Note that $n' \geq 2$. Moreover, if $n' = 2$, then one can see that $\gamma_p^I(T) \leq \gamma_p^I(T_{v_3}) + 2 < \frac{4n' - \ell(T') + 2s(T') - 1}{5}$. Hence let $n' \geq 3$. Note that $n' = n - n_{v_3} - s_1$, $\ell(T') = \ell(T) - \ell(T_{v_3}) - s_1 + 1$ and $s(T') \leq s(T) - s(T_{v_3}) - s_1 + 1$. Then any $\gamma_p^I(T')$-function $f'$ can be extended to a PIDF of $T$ by adding to it a PIDF of $T_{v_3}$ of weight $\frac{4n' - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5}$ that assigns a 1 to $v_3$. Moreover, the leaf neighbor of $v_4$ (if any) is assigned a 1, while $v_4$ will be assigned a 1 if $f'(v_4) = 0$ (note that in that case $f'(v_3) = 2$) or $v_4$ will keep the same assignment under $f'$ if $f'(v_3) \geq 1$. In either case, $\gamma_p^I(T) \leq \gamma_p^I(T') + \gamma_p^I(T_{v_3}) + s_1 + 1$. Using the induction, we obtain

$$\gamma_p^I(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5} + s_1 + 1$$

$$\leq \frac{4(n - n_{v_3} - s_1) - \ell(T) + \ell(T_{v_3}) + s_1 - 1 + 2s(T) - 2s(T_{v_3}) - 2s_1 + 1}{5}$$

$$+ \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5} + s_1 + 1 = \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$
Therefore, we can now assume that \( 6s(T_{v_3}) - 2\ell(T_{v_3}) \leq 11 \). Recall that (by the proof of the Claim) there exists PIDF, say \( g \), of \( T_{v_3} \) of weight at most \( \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \) assigning a 2 to \( v_3 \). We now consider two situations depending on whether \( s_1 = 0 \) or \( s_1 = 1 \).

(a) \( s_1 = 0 \). Then \( \deg_T(v_4) = 2 \). Let \( T' = T - T_{v_4} \). If \( n' = 1 \), then \( T \) is a wounded spider and by the claim the result follows, and if \( n' = 2 \), then one can easily see that \( g \) can be extended to a PIDF of \( T \) by assigning a 2 to \( v_5 \) and a 0 to both \( v_4 \) and \( v_5 \), and thus \( \gamma_1^p(T) \leq \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 2 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} \). So let \( n' \geq 3 \). In this case, any \( \gamma_1^p(T') \)-function can be extended to a PIDF of \( T \) by adding to it the PIDF \( g \) of \( T_{v_3} \). Moreover, \( v_4 \) will be assigned a 0 if \( f'(v_5) = 0 \) and a 1 if \( f'(v_5) \geq 1 \). In either case, \( \gamma_1^p(T) \leq \gamma_1^p(T') + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1 \). Using the fact that \( n' = n - n_{v_3} - 1, \ell(T') \geq \ell(T) - \ell(T_{v_3}), s(T') \leq s(T) - s(T_{v_3}) + 1 \), it follows from the induction hypothesis that

\[
\gamma_1^p(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1
\leq \frac{4(n - n_{v_3} - 1) - \ell(T) + \ell(T_{v_3}) + 2s(T) - 2s(T_{v_3}) + 1}{5} + \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

(b) \( s_1 = 1 \). Assume first that \( v_3 \) has at least four leaves, and let \( T' = T \setminus \{w, v_1, v_2\} \), where \( w \in L(v_3) \). Since \( v_3 \) has at least three leaves we have \( f'(v_3) \geq 1 \). If \( f'(v_3) = 2 \), then \( f' \) is extended to a PIDF of \( T \) by assigning a 2 to \( v_3 \) and a 0 to \( w, v_1 \). If \( f'(v_3) = 1 \), then \( f' \) to a PIDF of \( T \) by assigning a 1 to \( v_1, w \) and 0 to \( v_2 \). In either case, \( \gamma_1^p(T) \leq \gamma_1^p(T') + 2 \). By the induction hypothesis we get

\[
\gamma_1^p(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + 2 \leq \frac{4(n - 3) - \ell(T) + 2 + 2s(T) - 3}{5} + 2
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Hence, we can assume that \( v_3 \) has at most three leaves and thus \( \ell(T_{v_3}) \leq s(T_{v_3}) + 2 \). Let \( T' \) be the tree obtained from \( T \) by removing all vertices of \( T_{v_3} \) except \( v_3 \). Then \( n' = n - n_{v_3} + 1, \ell(T') = \ell(T) - \ell(T_{v_3}) + 1 \) and \( s(T') = s(T) - s(T_{v_3}) \). If \( f'(v_3) = 0 \), then \( f'(v_4) = 2 \), and \( f' \) can be extended to a PIDF of \( T \) by adding to it the PIDF \( g \) of \( T_{v_3} \), where \( v_3 \) is reassigned \( g(v_3) \) instead of \( f'(v_3) \). Applying our induction hypothesis, we obtain
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\[ \gamma_{pI}(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \]

\[ \leq \frac{4(n - n_{v_3} + 1) - \ell(T) + \ell(T_{v_3}) - 1 + 2s(T) - 2s(T_{v_3}) - 1}{5} \]

\[ + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} = \frac{4n - \ell(T) + 2s(T) - 1}{5} + \ell(T_{v_3}) \]

If \( f'(v_3) = 2 \), then \( f'(v_4) = 0 \) and the other leaf neighbor of \( v_4 \) in \( T' \) is assigned a 1, which provides a contradiction. Hence let \( f'(v_3) = 1 \). Then we extend \( f' \) to a PIDF of \( T \) by assigning a 1 to all leaves vertices of \( T_{v_3} \) and a 0 to remaining vertices of \( T_{v_3} \) but \( v_3 \). Using the fact that \( \ell(T_{v_3}) \leq s(T_{v_3}) + 2 \), \( n_{v_3} = \ell(T_{v_3}) + s(T_{v_3}) \) and the induction hypothesis, we obtain

\[ \gamma_{pI}(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \ell(T_{v_3}) \]

\[ \leq \frac{4(n - n_{v_3} + 1) - \ell(T) + \ell(T_{v_3}) - 1 + 2s(T) - 2s(T_{v_3}) - 1}{5} + \ell(T_{v_3}) \]

\[ \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} \]

This completes the proof. \( \Box \)

References


Received 4 September 2019
Revised 8 April 2020
Accepted 10 April 2020