A NEW UPPER BOUND FOR THE PERFECT ITALIAN DOMINATION NUMBER OF A TREE

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Abstract

A perfect Italian dominating function (PIDF) on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for every vertex $u$ with $f(u) = 0$, the total weight of $f$ assigned to the neighbors of $u$ is exactly two. The weight of a PIDF is the sum of its functions values over all vertices. The perfect Italian domination number of $G$, denoted $\gamma_p^I(G)$, is the minimum weight of a PIDF of $G$. In this paper, we show that for every tree $T$ of order $n \geq 3$, with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_p^I(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$, improving a previous bound given by T.W. Haynes and M.A. Henning in [Perfect Italian domination in trees, Discrete Appl. Math. 260 (2019) 164–177].

Keywords: Italian domination, Roman domination, perfect Italian domination.

2010 Mathematics Subject Classification: 05C69.
1. Introduction

Throughout this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V, E$). The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its closed neighborhood is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A leaf of $G$ is a vertex of degree one and a support vertex is a vertex adjacent to a leaf. An end support vertex is a support vertex having at most one non-leaf neighbor. For every vertex $v \in V$, the set of all leaves adjacent to $v$ is denoted by $L(v)$ and $L[v] = L(v) \cup \{v\}$. We denote the set of leaves of a graph $G$ by $L(G)$ and the set of support vertices by $S(G)$. We also let $|S(G)| = s(G)$ and $|L(G)| = \ell(T)$. A double star $DS_{q,p}$, with $q \geq p \geq 1$, is a graph consisting of the union of two stars $K_{1,q}$ and $K_{1,p}$ together with an edge joining their centers. The subdivision graph $S_b(G)$ of a graph $G$ is that graph obtained from $G$ by replacing each edge $uv$ of $G$ by a vertex $w$ and edges $uw$ and $vw$. A healthy spider $S_k(G)$ is the subdivision graph of a star $K_{1,k}$ for $k \geq 2$. A wounded spider $S_{k,t}$ is a graph obtained from a star $K_{1,k}$ by subdividing $t$ edges exactly once, where $1 \leq t \leq k - 1$. We denote by $P_n$ the path on $n$ vertices. The distance $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the greatest distance between two vertices of $G$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denote the set of children of $v$, $D(v)$ denotes the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. Also, the depth of $v$, $\text{depth}(v)$, is the largest distance from $v$ to a vertex in $D(v)$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

For a real-valued function $f : V \to \mathbb{R}$, the weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$.

A Roman dominating function on $G$, abbreviated RDF, is a function $f : V \to \{0, 1, 2\}$ such that every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. Roman domination was introduced by Cockayne et al. in [7] and was inspired by the work of ReVelle and Rosing [12] and Stewart [13]. Several new varieties of Roman domination have been introduced since 2004, among them, we quote the Italian domination originally published in [1] and called Roman $\{2\}$-domination. Further results on Roman domination and its variant can be found in [2–6].

An Italian dominating function on $G$, abbreviated IDF, is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that for every vertex $v \in V$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$, that is either $v$ is adjacent to a vertex $u$ with $f(u) = 2$, or to at least two vertices $x$ and $y$ with $f(x) = f(y) = 1$. The Italian domination number, denoted $\gamma_I(G)$, is the minimum weight of an IDF in $G$.

The concept of perfect dominating sets introduced by Livingston and Stout...
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in [11] has been extended to Roman and Italian dominating functions in [10] and [9], respectively. An RDF \( f \) is called perfect if for every vertex \( v \) with \( f(v) = 0 \), there is exactly one vertex \( u \in N(v) \) with \( f(u) = 2 \), while a IDF \( g \) is perfect if for every vertex \( w \) with \( g(w) = 0 \), \( g(N(w)) = 2 \). The perfect Roman domination number (respectively, perfect Italian domination number) of \( G \), denoted \( \gamma_R^p(G) \) (respectively, \( \gamma_I^p(G) \)), is the minimum weight of a perfect RDF (respectively, perfect IDF) in \( G \). A perfect IDF on \( G \) will be abbreviated PIDF.

A PIDF \( f \) is called a \( \gamma_I^p(G) \)-function if \( \omega(f) = \gamma_I^p(G) \).

It was shown in [10] that every tree \( T \) of order \( n \geq 3 \) satisfies \( \gamma_R^p(T) \leq \frac{4}{5}n \). However, this upper bound has recently been improved by Darkooti et al. [8] for trees \( T \) with \( \ell(T) \geq 2s(T) - 2 \), by showing that for any tree \( T \) of order \( n \geq 3 \) with \( \ell(T) \) leaves and \( s(T) \) support vertices, \( \gamma_R^p(T) \leq (4n - \ell(T) + 2s(T) - 2)/5 \). Moreover, Henning and Haynes showed in [9] that \( \frac{4}{5}n \) is also an upper bound of the perfect Italian domination number for any tree of order \( n \geq 3 \).

In this paper, we shall show that for any tree \( T \) of order \( n \geq 3 \) with \( \ell(T) \) leaves and \( s(T) \) support vertices, \( \gamma_R^p(T) \leq (4n - \ell(T) + 2s(T) - 1)/5 \). But first let us point out that for both parameters \( \gamma_R^p(G) \) and \( \gamma_I^p(G) \), one may be larger or smaller than the other even for trees. Indeed, for the path \( P_5 \) we have \( \gamma_R^p(P_5) = 3 \) and \( \gamma_I^p(P_5) = 2 \) while for the double star \( DS_{3,1} \) we have \( \gamma_R^p(DS_{3,1}) = 3 \) and \( \gamma_I^p(DS_{3,1}) = 4 \). The next result shows that the differences \( \gamma_I^p(G) - \gamma_R^p(G) \) and \( \gamma_R^p(G) - \gamma_I^p(G) \) can be arbitrarily large.

**Observation 1.** For any integer \( k \geq 1 \), there exist trees \( T_k \) and \( H_k \) such that \( \gamma_I^p(T_k) - \gamma_R^p(T_k) = k \) and \( \gamma_R^p(H_k) - \gamma_I^p(H_k) = k \).

**Proof.** Let \( T_k \) be the tree formed by \( k \) double stars \( DS_{3,1} \) by adding a new vertex attached to every support vertex of degree four. One can easily see that \( \gamma_I^p(T_k) = 4k + 1 \) while \( \gamma_R^p(T_k) = 3k + 1 \).

Now, let \( H_k \) be the tree formed by \( k \) paths \( P_5 \) by adding a new vertex attached to all center vertices of the paths. Then \( \gamma_R^p(H_k) = 3k + 1 \) while \( \gamma_I^p(H_k) = 4k + 1 \).

2. New Upper Bound

In this section, we present our main result which is an upper bound on the perfect Italian domination number of a tree.

**Theorem 2.** If \( T \) is a tree of order \( n \geq 3 \) with \( \ell(T) \) leaves and \( s(T) \) support vertices, then

\[
\gamma_I^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

**Proof.** We proceed by induction on the order \( n \). If \( n \in \{3,4\} \), then clearly \( \gamma_I^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} \), establishing the base case. Let \( n \geq 5 \) and assume that
any tree $T'$ of order $n'$, with $3 \leq n' < n$ satisfies $\gamma^p_I(T') \leq 4n - \ell(T') + 2s(T') - 1$. Let $T$ be a tree of order $n$. If $\text{diam}(T) = 2$, then $T$ is a star, where $\gamma^p_I(T) = 2 < \frac{4n - \ell(T) + 2s(T) - 1}{5}$. If $\text{diam}(T) = 3$, then $T$ is a double star, and since $n \geq 5$ we have $\gamma^p_I(T) = 4 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence, we may assume that $T$ has diameter at least 4. If $n = 5$, then $T$ is a path $P_5$, where $\gamma^p_I(P_5) = 3 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence let $n \geq 6$.

Suppose $v_1v_2 \cdots v_k (k \geq 5)$ is a diametral path in $T$ such that $\deg_T(v_2)$ is as large as possible. Root $T$ at $v_k$. First, assume that $T$ has an end support vertex $y$ of degree three. Without loss of generality, assume that $y = v_2$. Let $T' = T - T_{v_2}$ and $f'$ be a $\gamma^p_I(T')$-function. If $f'(v_3) = 0$, then $f'$ can be extended to a PIDF of $T$ by assigning a 0 to $v_2$ and a 1 to the two leaves of $v_2$. If $f'(v_3) \geq 1$, then $f'$ can be extended to a PIDF of $T$ by assigning a 2 to $v_2$ and a 0 to the leaves of $v_2$. In either case, $\gamma^p_I(T) \leq \gamma^p_I(T') + 2$, and by the induction hypothesis we obtain

$$\gamma^p_I(T) \leq \gamma^p_I(T') + 2 \leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + 2$$

$$\leq \frac{4(n - 3) - \ell(T) + 2 + 2s(T) - 1}{5} + 2$$

$$\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Hence we can assume that $T$ has no end support vertex of degree three, in particular we have $\deg_T(v_2) \neq 3$. Next, suppose that $\deg_T(v_3) = 2$. If $\deg_T(v_2) = 2$, then let $T' = T - T_{v_2}$ and $f'$ be a $\gamma^p_I(T')$-function. Note that $n' = n - 3$, $s(T') \leq s(T)$ and $\ell(T') \geq \ell(T) - 1$. Now if $f'(v_4) = 0$, then the function $f$ defined by $f'(v_2) = 2$, $f(v_1) = f(v_3) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus \{v_1, v_2, v_3\}$ is a PIDF of $T$. If $f'(v_4) \geq 1$, then the function $f$ defined by $f(v_1) = f(v_3) = 1$, $f(v_2) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus \{v_1, v_2, v_3\}$ is a PIDF of $T$. In either case, $\gamma^p_I(T) \leq \gamma^p_I(T') + 2$, and by the induction hypothesis we obtain

$$\gamma^p_I(T) \leq \gamma^p_I(T') + 2 \leq \frac{4(n - 3) - \ell(T') + 1 + 2s(T') - 1}{5} + 2$$

$$< \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Suppose now that $\deg_T(v_2) \geq 4$. Let $T' = T - T_{v_3}$ and $f'$ be a $\gamma^p_I$-function of $T'$. Note that $T'$ has order $n' \geq 2$. Clearly if $n' = 2$, then $\gamma^p_I(T) = 4 < \frac{4n - \ell(T) + 2s(T) - 1}{5}$. Hence we assume that $n' \geq 3$. If $f'(v_4) = 0$, then we can extend $f'$ to a PIDF of $T$ by assigning a 2 to $v_2$ and a 0 to every neighbor of $v_2$. If $f'(v_4) \geq 1$, then we can extend $f'$ to a PIDF $f$ of $T$ by assigning a 2 to $v_2$, a 1 to $v_3$, and a 0 to all leaves of $v_2$. In either case, $\gamma^p_I(T) \leq \gamma^p_I(T') + 3$ and by the induction hypothesis we obtain


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\[
\gamma_p^p(T) \leq \gamma_p^p(T') + 3 \leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + 3 \\
\leq \frac{4(n - |L(v_2)| - 2) - (\ell(T) - |L(v_2)|) + 2s(T) - 1}{5} + 3 \\
= \frac{4n - \ell(T) + 2s(T) - 1 - 3L(v_2) - 8}{5} + 3 < \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

From now on, we can assume that deg\(_T(v_3) \geq 3\) and deg\(_T(v_2) \neq 3\). Note that often in our proof a subtree \(T'\) of \(T\) is considered, and so in either case, let \(f'\) be a \(\gamma_p^p(T')\)-function. Consider the following cases.

**Case 1.** deg\(_T(v_2) \geq 4\) and \(T_{v_3} \neq DS_{3,1}\). Let us examine the following situations.

**Subcase 1.1.** \(v_3\) has at least two leaves. Let \(T'\) be the tree of order \(n'\) obtained from \(T\) by removing all leaves of \(v_2\). Note that \(n' = n - |L(v_2)|\), \(s(T') = s(T) - 1\) and \(\ell(T') = \ell(T) - |L(v_2)| + 1\). Since \(v_3\) has at least three leaves in \(T'\), we conclude that \(f'(v_3) \geq 1\). Hence the function \(f\) defined by \(f(v_2) = 2\), \(f(x) = 0\) for all \(x \in L(v_2)\) and \(f(x) = f'(x)\) for \(x \in V(T) \setminus L(v_2)\) is a PIDF of \(T\). It follows that \(\gamma_p^p(T) \leq \gamma_p^p(T') + 2\), and by the induction hypothesis we obtain

\[
\gamma_p^p(T) \leq \gamma_p^p(T') + 2 \leq \frac{4(n - |L(v_2)|) - \ell(T) + |L(v_2)| - 1 + 2s(T) - 3}{5} + 2 \\
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

**Subcase 1.2.** \(v_3\) has exactly one leaf, say \(v'\). If \(v_2\) is the unique child of \(v_3\) with depth 1, then let \(T'\) be the tree of order \(n'\) obtained from \(T\) by removing all vertices in \(T_{v_3}\) and adding two new vertices \(x_1, x_2\) attached at \(v_3\). Since \(v_3\) has at least three leaves, we have \(f'(v_3) \geq 1\), and thus the function \(f\) defined by \(f(v_2) = 2\), \(f(x) = 0\) for \(x \in L(v_2)\) and \(f(x) = f'(x)\) for \(x \in V(T) \setminus L(v_2)\) is a PIDF of \(T\). Hence \(\gamma_p^p(T) \leq \gamma_p^p(T') + 2\), and since \(T_{v_3} \neq DS_{3,1}\), we must have \(|L(v_2)| \geq 4\). It follows from the induction hypothesis that

\[
\gamma_p^p(T) \leq \gamma_p^p(T') + 2 \leq \frac{4(n + 1 - |L(v_2)|) - \ell(T) + |L(v_2)| - 2 + 2s(T) - 3}{5} + 2 \\
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Suppose that \(v_3\) has (at least) two children with depth 1, say \(a\) and \(b\) such that deg\(_T(a) \geq 4\) and deg\(_T(b) \geq 4\). Let \(T'\) be the tree formed from \(T\) by deleting all leaves of \(a\) and \(b\). Note that \(n' = n - |L(a)| - |L(b)|\), \(s(T') = s(T) - 2\) and \(\ell(T') = \ell(T) - |L(a)| - |L(b)| + 2\). Clearly, \(f'(v_3) \geq 1\) since \(v_3\) has three leaves in \(T'\). Thus the function \(f\) defined by \(f(a) = f(b) = 2\), \(f(x) = 0\) for all
$x \in L(a) \cup L(b)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus (L[a] \cup L[b])$ is a PIDF of $T$, and so $\gamma_p^p(T) \leq \gamma_p^p(T') + 4$. Using the fact $|L(a)| \geq 3$ and $|L(b)| \geq 3$ and the induction hypothesis we obtain 

$$\gamma_p^p(T) \leq \gamma_p^p(T') + 4 \leq \frac{4(n - |L(a)| - |L(b)|) - \ell(T) - |L(a)| + |L(b)| - 2 + 2s(T) - 5}{5} + 4 < \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$ 

Hence we can assume now that $v_2$ is the unique child of $v_3$ with depth one and degree at least 4. Recall that since $\deg_T(v_2) \neq 3$, we may assume that every child of $v_3$ with depth 1 that is different from $v_2$ has degree two. Note that $|C(v_3)| \geq 3$. Assume first that $|C(v_3)| \geq 4$, and let $T'$ be the tree of order $n'$ obtained from $T - T_{v_3}$ by adding three new vertices $x_1, x_2, x_3$ attached at $v_4$. Note that $n' = n - |C(v_3)| - |L(T_{v_3})| + 3$, $\ell(T') = \ell(T) - L(T_{v_3}) + 3$, $s(T') \leq s(T) - |C(v_3)| + 1$.

Now, since $v_3$ has three leaves in $T'$, we must have $f'(v_4) \geq 1$, and thus the function $f$ defined by $f(v_2) = 2$, $f(x) = 1$ for $x \in \{v', v_3\} \cup (L(T_{v_3}) \setminus L(v_2))$, $f(x) = 0$ for all $x \in (C(v_3) \setminus \{v_2, v_3\}) \cup L(v_2)$ and $f(x) = f'(x)$ for otherwise, is a PIDF of $T$. Hence $\gamma_p^p(T') \leq \gamma_p^p(T') + |C(v_3)| + 2$, and by the induction hypothesis it follows that 

$$\gamma_p^p(T) \leq \gamma_p^p(T') + |C(v_3)| + 2 \leq \frac{4(n - |C(v_3)| + 3 - |L(T_{v_3})|) - \ell(T) + |L(T_{v_3})| - 3 + 2s(T) - 2|C(v_3)| + 1}{5} + |C(v_3)| + 2 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-|C(v_3)| - 3|L(T_{v_3})| + 21}{5}.$$

Moreover, since $|L(T_{v_3})| \geq |C(v_3)| + 2$, we have $\gamma_p^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-|C(v_3)| + 15}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}$ because of $|C(v_3)| \geq 4$. Next, we can assume that $|C(v_3)| = 3$, that is $T_{v_3}$ is isomorphic to $H_1$ in Figure 1. In this case, let $T'$ be the tree formed from $T$ by removing all vertices of $T_{v_3}$ except $v_3$. Clearly $v_3$ is a leaf in $T'$. If $f'(v_3) = 0$, then $f(v_4) = 2$ and so the function $f$ defined by $f(v_3) = f(v') = f(u_1) = 1$, $f(v_2) = 2$, $f(x) = 0$ for all $x \in L(v_2) \cup \{u_2\}$ and $f(x) = f'(x)$ for otherwise is a PIDF of $T$. If $f'(v_3) = 1$, then we can extend $f'$ to be a PIDF of $T$ as above when $f'(v_3) = 0$, except that we do not assign a 1 to $v_3$. In either case, $\gamma_p^p(T) \leq \gamma_p^p(T') + 5$. It follows from the induction hypothesis that 

$$\gamma_p^p(T) \leq \gamma_p^p(T') + 5 \leq \frac{4(n - 4 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 5}{5} + 5 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$
Finally, if \( f'(v_3) = 2 \), then the function \( f \) defined by \( f(v_2) = f(u_2) = 2 \), \( f(x) = 0 \) for all \( x \in L(v_2) \cup \{u_1, v'\} \) and \( f(x) = f'(x) \) for otherwise is a PIDF of \( T \). Using the induction hypothesis we obtain

\[
\gamma_p^I(T) \leq \gamma_p^I(T') + 4 \leq \frac{4(n - 4 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 5}{5} + 4
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

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Figure 1. The trees.

\textbf{Subcase 1.3.} \( v_3 \) is not a support vertex. Suppose that \( v_3 \) has at least three children of degree at least 4, say \( a, b \) and \( c \). Let \( T' \) be the tree obtained from \( T \) by removing all leaves of \( a, b \) and \( c \). Note that \( n' = n - |L(a)| - |L(b)| - |L(c)| \), \( s(T') = s(T) - 2 \) and \( \ell(T') = \ell(T) - |L(a)| - |L(b)| - |L(c)| + 3 \). Clearly, since \( v_3 \) has three leaves in \( T' \), \( f'(v_3) \geq 1 \), and thus the function \( f \) defined by \( f(a) = f(b) = f(c) = 2 \), \( f(x) = 0 \) for all \( x \in L(a) \cup L(b) \cup L(c) \) and \( f(x) = f'(x) \) for all \( x \in V(T) \setminus (L[a] \cup L[b] \cup L[c]) \) is a PIDF of \( T \). By the induction hypothesis, it follows that

\[
\gamma_p^I(T) \leq \gamma_p^I(T') + 6
\]

\[
\leq \frac{4(n - |L(a)| - |L(b)| - |L(c)|) - \ell(T) + |L(a)| + |L(b)| + |L(c)| - 3 + 2s(T) - 5}{5} + 6
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]
Hence, \( v_3 \) has at most two children of degree at least 4, say \( v_3 \) and \( u \) (if any). Let \( T' \) be the tree of order \( n' \) obtained from \( T - T_{v_3} \) by adding three new vertices attached at \( v_4 \). Note that \( n' = n - |C(v_3)| - |L(T_{v_3})| + 2, s(T') \leq s(T) - |C(v_3)| + 1 \) and \( \ell(T') = \ell(T) - |L(T_{v_3})| + 3 \). Clearly, \( f'(v_4) \geq 1 \). Hence the function \( f \) defined by \( f(x) = 2 \) for \( x \in \{v_2, u\} \), \( f(x) = 1 \) for \( x \in (L(T_{v_3}) \cup \{v_3\}) \setminus (L(v_2) \cup L(u)) \), \( f(x) = 0 \) for \( x \in (C(v_3) \setminus \{v_2, u\}) \cup (L(v_2) \cup L(u)) \) and \( f(x) = f'(x) \) for otherwise is a PIDF of \( T \). By the induction hypothesis we obtain

\[
\gamma^p_{f}(T) \leq \gamma^p_{f}(T') + |C(v_3)| + 3
\]

\[
\leq 4(n - |C(v_3)| - |L(T_{v_3})| + 2) - \ell(T) + |L(T_{v_3})| - 3 + 2s(T) - 2|C(v_3)| + 1
\]

\[
+ \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-|C(v_3)| - 3|\ell(T_{v_3})| + 22}{5}.
\]

Since \( |L(T_{v_3})| \geq |C(v_3)| + 2 \), we have \( \gamma^p_{f}(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-4|C(v_3)| + 16}{5} \).

If \( |C(v_3)| \geq 4 \), then \( \gamma^p_{f}(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} \). Hence, \( \ell \leq |C(v_3)| \leq 3 \). If \( |C(v_3)| = 3 \) and \( v_3 \) has two children of degree at least 4, then one can easily see that \( \gamma^p_{f}(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} \) (since \( |L(T_{v_3})| \geq |C(v_3)| + 4 \)). In the sequel, we can assume that \( T_{v_3} \) is isomorphic to one of \( H_2, H_3, H_4 \) depicted in Figure 1.

In that case, let \( T'' \) be the tree formed from \( T \) by removing all vertices of \( T_{v_3} \) except \( v_3 \). Clearly \( v_3 \) is a leaf in \( T'' \). Let \( f'' \) be a \( \gamma^p_{f}(T'') \)-function. If \( f''(v_3) = 0 \), then \( f''(v_3) = 2 \) and so let \( f \) be a PIDF of \( T \) defined as follows: \( f(x) = f''(x) \) for all \( x \in V(T') \setminus \{v_3\} \) and \( f(v_3) = 1 \). Moreover, every child of \( v_3 \) of degree 2 is assigned a 0 and its unique leaf a 1; every child of \( v_3 \) of degree at least 4 is assigned a 2 and its leaves a 0. If \( f''(v_3) = 1 \), then \( f'' \) will be extended to a PIDF of \( T \) as above when \( f'(x) = 0 \), except we do not assign a 1 to \( v_3 \). Finally, if \( f''(v_3) = 2 \), then we use the following assignment for vertices of \( T_{v_3} \): assign a 2 to each child of \( v_3 \) and a 0 to each of their leaves. Now, if \( T_{v_3} = H_2 \), then in either case described above, we have \( \gamma^p_{f}(T) \leq \gamma^p_{f}(T'') + 4 \). By the induction hypothesis we obtain

\[
\gamma^p_{f}(T) \leq \gamma^p_{f}(T'') + 4 \leq \frac{4(n - 3 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 3}{5} + 4
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

If \( T_{v_3} = H_3 \), then \( \gamma^p_{f}(T) \leq \gamma^p_{f}(T'') + 5 \), and by the induction hypothesis we obtain

\[
\gamma^p_{f}(T) \leq \gamma^p_{f}(T'') + 5 \leq \frac{4(n - 2 - |L(v_2)| - |L(u)| - \ell(T) + |L(v_2)| + |L(u)| + 2s(T) - 3}{5} + 5
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]
Moreover, if $T_{v_3} = H_4$, then $\gamma^p_I(T) \leq \gamma^p_I(T'') + 6$, and by the induction hypothesis it follows that

$$\gamma^p_I(T) \leq \gamma^p_I(T'') + 6 \leq \frac{4(n - 5 - |L(v_2)|) - \ell(T) + 2 + |L(v_2)| + 2s(T) - 5}{5} + 6$$

$$< \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Before discussing Case 2, we will need the following claim.

**Claim.** Let $T$ be a wounded spider of order $n$ different from $DS_{2,1}$, with $s(T)$ support vertices and $\ell(T)$ leaves. Then we have the following.

(i) If $6s(T) - 2\ell(T) \geq 11$, then $\gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 6}{5}$.

(ii) If $6s(T) - 2\ell(T) \leq 11$, then $\gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 3}{5}$.

**Proof.** Let $v$ be the center vertex of $T$.

(i) If $6s(T) - 2\ell(T) \geq 11$, then the function $f$ defined by assigning a 1 to $v$ and every leaf of $T$, and a 0 to remaining vertices of $T$, is a PIDF of $T$ and so

$$\gamma^p_I(T) \leq \omega(f) = \ell(T) + 1 \leq \frac{4n - \ell(T) + 2s(T) - 6}{5}.$$

(ii) Let $t = |L(v)| - 1$. Clearly, $\ell(T) = s(T) + t$ and since $6s(T) - 2\ell(T) \leq 11$, then $T$ is a double star and since $T$ is not a $DS_{2,1}$, we can see that we have $4s(T) - 2t \leq 11$ and thus $t \geq 2s(T) - \frac{11}{2}$. Now if $s(T) = 2$, then $T$ is a double star and since $T$ is not a $DS_{2,1}$, we can see that $\gamma^p_I(T) \leq \frac{4n - \ell(T) + 2s(T) - 3}{5}$. Hence, let $s(T) \geq 3$. Then the function $f$ defined by assigning a 2 to the support vertices of $T$ and a 0 to remaining vertices of $T$ is a PIDF of $T$ of weight $2s(T)$. Since, $n = s(T) + \ell(T)$ and $\ell(T) = s(T) + t$, it follows that $\frac{4n - \ell(T) + 2s(T) - 3}{5} = \frac{9s(T) + 3t - 3}{5}$. Moreover, since $t \geq 2s(T) - \frac{11}{2}$ we obtain

$$\frac{9s(T) + 3t - 3}{5} \geq \frac{9s(T) + 6s(T) - \frac{33}{2} - 3}{5} = 3s(T) - \frac{39}{10}.$$

Now, if $s(T) \geq 4$, then $3s(T) - \frac{39}{10} \geq 2s(T) \geq \gamma^p_I(T)$ and so the desired result follows. Thus we assume that $s(T) = 3$. If $t \geq 2s(T) - \frac{7}{2}$, then as above we have $\frac{9s(T) + 3t - 3}{5} \geq 3s(T) - \frac{27}{10} \geq 2s(T) \geq \gamma^p_I(T)$. Hence, let $t \leq 2s(T) - \frac{7}{2} = 2.5$. Note that in this case $\ell(T) \in \{3, 4, 5\}$. Then assigning a 1 to $v$ and the leaves of $T$ and a 0 to remaining vertices of $T$ provides a PIDF of $T$ of weight $\ell(T) + 1 \leq \frac{4n - \ell(T) + 2s(T) - 3}{5}$, which completes the proof of the claim. \hfill \Box

We note from the proof of the claim that there exist PIDFs of $T$ of weight at most $\frac{4|V(T_{v_3})| - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5}$ that assign to the center vertex a 1 or 2.
Now we are ready to examine the next case.

Case 2. \( \deg T(v_2) = 2 \) or \( T_{v_3} = DS_{3,1} \). From Case 1 and since \( v_2 \) was chosen having a maximum degree, we conclude that \( T_{v_3} \) is a spider. Assume first that \( T_{v_3} \) is a healthy spider. If \( |C(v_3)| \geq 3 \), then let \( T' \) be the tree obtained by removing \( T_{v_3} \) and adding three new vertices attached at \( v_4 \). Note that \( n' = n - 2|C(v_3)| + 2 \), \( s(T') \leq s(T) - |C(v_3)| + 1 \) and \( \ell(T') = \ell(T) - |C(v_3)| + 3 \). Clearly, \( f'(v_4) \geq 1 \) (since \( v_4 \) has three leaves in \( T' \)). Thus the function \( f \) defined by \( f(x) = 1 \) for \( x \in L(T_{v_3}) \cup \{v_3\} \), \( f(x) = 0 \) for \( x \in C(v_3) \) and \( f(x) = f'(x) \) for \( x \in V(T) \setminus V(T_{v_3}) \) is a PIDF of \( T \). Hence \( \gamma_p^f(T) \leq \gamma_p^f(T') + |C(v_3)| + 1 \), and by the induction hypothesis we obtain

\[
\gamma_p^f(T) \\
\leq \gamma_p^f(T') + |C(v_3)| + 1
\]

\[
\leq \frac{4(n - 2|C(v_3)| + 2) - \ell(T) + |C(v_3)| - 3 + 2s(T) - 2|C(v_3)| + 1}{5} + |C(v_3)| + 1
\]

\[
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Now, assume that \( |C(v_3)| = 2 \), and let \( T'' = T - T_{v_3} \). If \( f'(v_4) \geq 1 \), then the function \( f \) defined by \( f(x) = 1 \) for \( x \in L(T_{v_3}) \cup \{v_3\} \), \( f(x) = 0 \) for every \( x \in C(v_3) \) and \( f(x) = f'(x) \) for all \( x \in V(T) \setminus V(T_{v_3}) \) is a PIDF of \( T \) of weight \( \gamma_p^f(T'') + 3 \). If \( f'(v_4) = 0 \), then the function \( f \) defined by \( f(x) = 1 \) for \( x \in V(T_{v_3}) \setminus \{v_3\} \), \( f(v_3) = 0 \) and \( f(x) = f'(x) \) for all \( x \in V(T) \setminus V(T_{v_3}) \) is a PIDF of \( T \) of weight \( \gamma_p^f(T'') + 4 \). In either case, \( \gamma_p^f(T) \leq \gamma_p^f(T'') + 4 \) and by the induction hypothesis we obtain

\[
\gamma_p^f(T) \leq \gamma_p^f(T'') + 4 \leq \frac{4(n - 5) - \ell(T) + 2s(T) - 3}{5} + 4
\]

\[
= \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

Suppose now that \( T_{v_3} \) is a wounded spider \( S_{k,k} \). If \( T_{v_3} = DS_{2,1} \), then let \( T'' = T - T_{v_3} \). Clearly we assume that \( n' \geq 3 \). If \( n' = 2 \), then \( \gamma_p^f(T'') \leq 5 < \frac{4n - \ell(T) + 2s(T) - 1}{5} \). Hence we assume that \( n' \geq 3 \). If \( f'(v_4) \geq 1 \), then the function \( f \) defined by \( f(v_2) = f(v_3) = 2, f(x) = 0 \) for \( x \in L(T_{v_3}) \) and \( f(x) = f'(x) \) for \( x \in V(T) \setminus V(T_{v_3}) \) is a PIDF of \( T \). If \( f'(v_4) = 0 \), then the function \( f \) defined by \( f(v_1) = f(v_2) = f(v_3) = 0 \) and \( f(x) = f'(x) \) for \( x \in V(T) \setminus V(T_{v_3}) \) is a PIDF of \( T \). In either case, \( \gamma_p^f(T) \leq \gamma_p^f(T'') + 4 \). If \( \deg T(v_4) \geq 3 \), then \( s(T') = s(T) - 2 \) and \( \ell(T') = \ell(T) - 3 \) and by the induction hypothesis we obtain

\[
\gamma_p^f(T) \leq \gamma_p^f(T') + 4 \leq \frac{4(n - 5) - \ell(T) + 3 + 2s(T) - 5}{5} + 4
\]

\[
< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]
If $\deg_T(v_4) = 2$, then $s(T') \leq s(T) - 1$ and $\ell(T') = \ell(T) - 2$ and by the induction hypothesis we obtain

$$
\gamma^p_T(T) \leq \gamma^p_T(T') + 4 \leq \frac{4(n - 5) - \ell(T) + 2 + 2s(T) - 3}{5} + 4
= \frac{4n - \ell(T) + 2s(T) - 1}{5}.
$$

From now on we may assume that $v_4$ has no child $x$ such that $T_x = DS_{2,1}$.

Let $s_1$ be the number of children of $v_4$ that are leaves and for $i \geq 2$, let $s_i$ be the number of children of $v_4$ of degree $i$ whose children are all leaves. As we assumed at the beginning of the proof, $T$ has no end support vertex with degree three, it follows that $s_3 = 0$. Let $s_{\geq 4}$ be the number of children of $v_4$ of degree at least 4 having no grandchild. Thus

$$
s_{\geq 4} = \sum_{i \geq 4} s_i.
$$

Adopting our earlier notation, for each child $v$ of $v_4$ with depth 2, let $n_v$ denote the number of children in the subtree $T_v$ of $T$. Furthermore, let $n^{+}$ denote the sum of the number of vertices in all such trees $T_v$. Also, let $s^{+}$ and $\ell^{+}$ denote the sum of the number of support vertices and leaves vertices in all such trees $T_v$, respectively. Note that every child of $v_4$ is one of the following four types: (1) a leaf; (2) a support vertex of degree 2; (3) a vertex with depth 2; (4) a support vertex of degree at least 4 whose children are all leaves. For ease of discussion, we sometimes refer to these children as Type-1, Type-2, Type-3, or Type-4, respectively. Moreover, let $m$ be the number of leaves of all Type-4 children. Consider now the following subcases.

**Subcase 2.1.** $s_1 + s_{\geq 4} \geq 3$. Let $T' = T - T_{v_3}$ be a tree of order $n'$. We claim that $f'(v_4) \geq 1$. Suppose to the contrary that $f'(v_4) = 0$. This implies that at most two children of $v_4$ in $T'$ are assigned positive values under $f'$. But since every Type-1 and Type-4 child of $v_4$ must be assigned a positive value by $f'$ when $f'(v_4) = 0$, this implies that $s_1 + s_{\geq 4} \leq 2$, a contradiction. Hence, $f'(v_4) \geq 1$. Consequently, we can extend $f'$ to a PIDF $f$ by adding to it any PIDF of $T_{v_3}$ of weight at most $\frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5}$ assigning a 1 or 2 to $v_3$ as claimed above. By the induction hypothesis we obtain

$$
\gamma^p_f(T) \leq \gamma^p_f(T') + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5}
\leq \frac{4(n - n_{v_3}) - \ell(T) + \ell(T_{v_3}) + 2s(T) - 2s(T_{v_3}) - 1}{5}
+ \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}.
$$
In the sequel, we may assume that \( s_1 + s_{\geq 4} \leq 2 \).

Subcase 2.2. \( s_1 = 2 \). Since \( s_1 + s_{\geq 4} \leq 2 \), we deduce that \( s_{\geq 4} = 0 \). Let \( F \) be the forest formed by the Type-3 children of \( v_4 \) and their descendants. We note any component of \( F \) is a wounded spider including \( T_{v_3} \) and different from \( DS_{2,1} \). Let \( T' \) be the tree obtained from \( T \) by deleting all vertices in \( V(F) \) and adding a new vertex \( a \) attached at \( v_4 \). Since \( v_4 \) has three leaf neighbors in \( T' \), we have \( f'(v_4) \geq 1 \). Let \( f \) be the PIDF of \( T \) defined as follows: \( f(x) = f'(x) \) for all \( x \in V(T') \setminus \{a\} \) and let the restriction of \( f \) to each component, say \( T_v \), in \( F \) be any PIDF of that component of weight at most \( \frac{4n - \ell(T_v) + 2s(T_v) - 3}{5} \). By our earlier observations, the total weight assigned to \( F \) is at most \( \frac{4n^* - \ell^* + 2s^* - 3}{5} \). Now, by the induction hypothesis we obtain

\[
\gamma_f^p(T) \leq \gamma_f^p(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n - n^*) + 1 - \ell(T) + \ell^* - 1 + 2s(T) - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n - n^*) + 1 - \ell(T) + \ell^* - 1 + 2s(T) - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

Hence, in the next we may assume that \( s_1 \in \{0, 1\} \).

Subcase 2.3. \( s_2 \geq 3 \). Let \( T' \) be the tree of order \( n' \) obtained from \( T - v_4 \) by adding three new vertices \( x_1, x_2, x_3 \) attached at \( v_5 \). Note that \( n' = n - n^* - s_1 - 2s_2 - s_{\geq 4} - m + 2 \), \( \ell(T') = \ell(T) - \ell_* - s_1 - s_2 - m + 3 \) and \( s(T') \leq s(T) - s^* - s_1 - s_2 - s_{\geq 4} + 1 \). Clearly, \( f'(v_5) \geq 1 \) (since \( v_5 \) has three leaves in \( T' \)). Let \( f \) be the PIDF of \( T' \) defined by \( f(x) = f'(x) \) for all \( x \in V(T') \setminus \{x_1, x_2, x_3\} \) and let \( f(v_4) = 1 \). Then assign the weights to the descendants of \( v_4 \) in \( T \) as follows: assign a 1 to each Type-1 (leaf) child of \( v_4 \) (recall that \( s_1 \in \{0, 1\} \)); assign a 0 to each Type-2 child of \( v_4 \) and a 1 to its leaf neighbor; assign a 2 to each Type-4 child of \( v_4 \) and a 0 to each of its leaves. Finally, for each Type-3 child \( v \), assign a PIDF to the subtree \( T_v \) rooted at \( v \) of weight at most \( \frac{4n_{v} - \ell(T_v) + 2s(T_v) - 3}{5} \) so that \( f(v) \geq 1 \). By our earlier observations, the total weight assigned to all Type-3 children of \( v \) and their descendants is at most \( \frac{4n^* - \ell^* + 2s^* - 3}{5} \). It follows from the induction hypothesis that

\[
\gamma_f^p(T) \leq \gamma_f^p(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n' - n^*) + 1 - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4} + 2) - \ell(T) + \ell^* + s_1 + s_2 + m - 3}{5} + \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]

\[
= \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{9 - 3m - 4s_2 + 4s_{\geq 4}}{5}.
\]
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Using the fact that $m \geq 3s \geq 4$, it follows that $\gamma^p_{I}(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{9 - 4s_2 - 5s_{\geq 4}}{5}$. Now since $s_2 \geq 3$, we deduce that $\gamma^p_{I}(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$.

By Subcase 2.3, we can assume that $s_2 \leq 2$.

Subcase 2.4. $s_2 + s_{\geq 4} \geq 1$. Let $T'$ be the tree of order $n'$ obtained by deleting all vertices of $T_{v_4}$ except $v_4$. Note that $n' = n - n^* - s_1 - 2s_2 - s_{\geq 4} - m$, $s(T') \leq s(T) - s^* - s_1 - s_2 - s_{\geq 4} + 1$ and $\ell(T') = \ell(T) - \ell^* - s_1 - s_2 - m + 1$ (since $v_4$ is a leaf vertex in $T'$). First, let $f'(v_4) = 2$ and $f$ be a PIDF of $T$ defined by $f(x) = f'(x)$ for all $x \in V(T')$: and then assign the weights to the descendants of $v_4$ in $T$ as follows: assign a 0 to each Type-1 (leaf) child of $v_4$, assign a 2 to each Type-2 child of $v_4$ and a 0 to its leaf, and assign a 2 to each Type-4 child of $v_4$ and a 0 to its leaves. Finally, for each Type-3 child $v$, assign a PIDF to the subtree $T_v$ rooted at $v$. By our earlier observations, the total weight assigned to all Type-3 children of $v$ and their descendants is at most $\frac{4n^* - \ell^* + 2s^* - 3}{5}$. By the induction hypothesis it follows that

$$\gamma^p_{I}(T) \leq \gamma^p_{I}(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s_{\geq 4}$$

$$\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s_{\geq 4}$$

$$\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4}) - \ell(T) + \ell^* + s_1 + s_2 + m - 1}{5}$$

$$+ \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s_{\geq 4}$$

$$\leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-5s_1 + s_2 - 3m + 4s_{\geq 4} - 2}{5}.$$ 

Now since $m \geq 3s_{\geq 4}$ and $s_2 \leq 2$, we get

$$\gamma^p_{I}(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-5s_1 + s_2 - 5s_{\geq 4} - 2}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$ 

Suppose now that $f'(v_4) \in \{0, 1\}$, and let $f$ be a PIDF of $T$ defined by $f(x) = f'(x)$ for all $x \in V(T')$ and let $f(v_4) = 1$. Then assign the weights to the descendants of $v_4$ in $T$ as follows: assign a 1 to each Type-1 (leaf) child of $v_4$: assign a 0 to each Type-2 child of $v_4$ and a 1 to its leaf neighbor and assign a 2 to each Type-4 child of $v_4$ and 0 to its leaves. Finally, for each Type-3 child $v$, assign a PIDF of weight at most $\frac{4n - \ell(T_v) + 2s(T_v) - 3}{5}$ to vertices of $T_v$ rooted at $v$ so that $f(v) \geq 1$. By our earlier observations, the total weight assigned to all Type-3 children of $v$ and their descendants is at most $\frac{4n^* - \ell^* + 2s^* - 3}{5}$. By the induction hypothesis we obtain
\[
\gamma_p^f(T) \leq \gamma_p^f(T') + \frac{4n^*-\ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]
\[
\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]
\[
\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4}) - \ell(T) + \ell^* + s_1 + s_2 + m - 1}{5}
\]
\[
+ \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1
\]
\[
\leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-4s_2 - 3m + 4s_{\geq 4} + 3}{5}.
\]

Now since \( m \geq 3s_{\geq 4} \), it follows that \( \gamma_p^f(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-4s_2 - 5s_{\geq 4} + 3}{5} \), and since \( s_2 + s_{\geq 4} \geq 1 \), the result follows.

**Subcase 2.5.** \( s_2 + s_{\geq 4} = 0 \). Recall that \( s_1 \in \{0, 1\} \). Let \( v' \) be the leaf neighbor of \( v_4 \) (if any). First, let \( v_4 \) has at least two children of Type-3. Let \( T' \) be the tree of order \( n' \) obtained by deleting all vertices of \( T_{v_4} \) except \( v_4 \). Note that \( n' = n - n^* - s_1, s(T') \leq s(T) - s^* - s_1 + 1 \) and \( \ell(T') = \ell(T) - \ell^* - s_1 + 1 \) (since \( v_4 \) is a leaf vertex in \( T' \)). We also notice that if \( f'(v_4) = 0 \), then since \( v_4 \) is a leaf in \( T' \), we must have \( f'(v_5) = 2 \). Now, define a PIDF \( f \) of \( T \) by \( f(x) = f'(x) \) for all \( x \in V(T') \setminus \{v_4\} \). Moreover, \( f(v') = 1 \), \( f(v_4) = 1 \) if \( f'(v_4) = 0 \) and \( f(v_4) = f'(v_4) \) if \( f'(v_4) \geq 1 \). Also, for each other child \( v \) of \( v_4 \), assign a PIDF to the subtree \( T_v \) of weight at most \( \frac{4n - \ell(T^*) + 2s(T^*) - 3}{5} \). Since there are at least two Type-3 children of \( v_4 \), the total weight assigned to such subtree \( T_v \) is \( \frac{4n^* - \ell^* + 2s^* - 5}{5} + s_1 + 1 \). Using the induction hypothesis we obtain

\[
\gamma_p^f(T) \leq \gamma_p^f(T') + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1
\]
\[
\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1
\]
\[
\leq \frac{4(n - n^* - s_1) - \ell(T) + \ell^* + s_1 - 1 + 2s(T) - 2s^* - 2s_1 + 1}{5}
\]
\[
+ \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\]

In the sequel, \( v_3 \) is the only child of \( v_4 \) of Type-3. We distinguish the following.

(i) \( T_v_3 = DS_{1, 3} \). Consider two situations depending on whether \( s_1 = 0 \) or \( s_1 = 1 \).

(a) \( s_1 = 0 \). Hence \( \deg_T(v_4) = 2 \). Let \( T' = T - T_{v_4} \). Clearly, \( n' \geq 1 \). If \( n' = 1 \), then \( T \) is a wounded spider and by the claim the result follows, and if \( n' = 2 \), then
one can easily see that $\gamma_p^p(T) = 6 < 4n - \ell(T) + 2s(T) - 1 = 7.2.$ So let $n' \geq 3.$ Note that $n' = n - 7$, $\ell(T') \geq \ell(T) - 4$ and $s(T') \leq s(T) - 1.$ Any $\gamma_p^p(T')$-function can be extended to a PIDF of $T$ by assigning a 2 to $v_2, v_3$ and a 0 to remaining vertices of $T_{v_4}$ except $v_4$ which will be assigned a 0 if $f'(v_3) = 0$ and a 1 if $f'(v_3) \geq 1.$ In either case, $\gamma_p^p(T) \leq \gamma_p^p(T') + 5.$ By the induction hypothesis we obtain

$$\gamma_p^p(T) \leq 4n' - \ell(T') + s(T') - 1 + 5 \leq 4(n - 5) - \ell(T) + 4 + 2s(T) - 3 + 5$$

(b) $s_1 = 1.$ Let $T'$ be the tree obtained from $T$ by removing all vertices $T_{v_3}$ except $v_3.$ If $f'(v_3) = 0,$ then $f'(v_4) = 2,$ and so $f'$ can be extended to a PIDF of $T$ by assigning a 2 to $v_2, v_3$ and a 0 to remaining vertices of $T_{v_3}.$ Hence $\gamma_p^p(T) \leq \gamma_p^p(T') + 4.$ If $f'(v_3) = 2,$ then $f'(v_4) = 0$ and so the other leaf neighbor of $v_4$ is assigned a 1, which is a contradiction. Hence, $f'(v_3) = 1.$ Now, if $|L(v_3)| = 1,$ then we extend $f'$ to a PIDF of $T$ by assigning a 2 to $v_2, v_3$ and a 0 to remaining vertices of $T_{v_3}.$ If $|L(v_3)| = 3,$ then we extend $f'$ to a PIDF of $T$ by assigning a 1 to $L(T_{v_3})$ and a 0 to $v_2.$ In either case, $\gamma_p^p(T) \leq \gamma_p^p(T') + 4.$ By the induction hypothesis we obtain

$$\gamma_p^p(T) \leq 4n' - \ell(T') + s(T') - 1 + 4 \leq 4(n - 5) - \ell(T) + 3 + 2s(T) - 5 + 4$$

(ii) $T_{v_3} = S_{k,t} \neq DS_{3,1}.$ We recall that $T_{v_3}$ is different from $DS_{2,1}.$ First let $6s(T_{v_3}) - 2\ell(T_{v_3}) \geq 11.$ By our Claim, $\gamma_p^p(T_{v_3}) \leq \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5}.$ Let $T'$ be the tree obtained from $T$ by removing all vertices of $T_{v_4}$ except $v_4.$ Note that $n' \geq 2.$ Moreover, if $n' = 2,$ then one can see that $\gamma_p^p(T) \leq \gamma_p^p(T_{v_3}) + 2 < 4n - \ell(T) + 2s(T) - 1.$ Hence let $n' \geq 3.$ Note that $n' = n - n_{v_3} - s_1,$ $\ell(T') = \ell(T) - \ell(T_{v_3}) - s_1 + 1$ and $s(T') \leq s(T) - s(T_{v_3}) - s_1 + 1.$ Then any $\gamma_p^p(T')$-function $f'$ can be extended to a PIDF of $T$ by adding to it a PIDF of $T_{v_3}$ of weight $\frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5}$ that assigns a 1 to $v_3.$ Moreover, the leaf neighbor of $v_4$ (if any) is assigned a 1, while $v_4$ will be assigned a 1 if $f'(v_3) = 0$ (note that in that case $f'(v_3) = 2$) or $v_4$ will keep the same assignment under $f'$ if $f'(v_3) \geq 1.$ In either case, $\gamma_p^p(T) \leq \gamma_p^p(T') + 1.$ Using the induction, we obtain

$$\gamma_p^p(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5} + s_1 + 1$$

$$\leq \frac{4(n - n_{v_3} - s_1) - \ell(T) + 2s(T) - 2s(T_{v_3}) - 2s_3 + 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 6}{5} + s_1 + 1 = \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$
Therefore, we can now assume that $6s(T_{v_3}) - 2\ell(T_{v_3}) \leq 11$. Recall that (by the proof of the Claim) there exists PIDF, say $g$, of $T_{v_3}$ of weight at most $\frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5}$ assigning a 2 to $v_3$. We now consider two situations depending on whether $s_1 = 0$ or $s_1 = 1$.

(a) $s_1 = 0$. Then $\deg_T(v_4) = 2$. Let $T' = T - T_{v_4}$. If $n' = 1$, then $T$ is a wounded spider and by the claim the result follows, and if $n' = 2$, then one can easily see that $g$ can be extended to a PIDF of $T$ by assigning a 2 to $v_6$ and a 0 to both $v_4$ and $v_5$, and thus $\gamma_f^p(T) \leq \frac{4s_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 2 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}$. So let $n' \geq 3$. In this case, any $\gamma_f^p(T')$-function can be extended to a PIDF of $T$ by adding to it the PIDF $g$ of $T_{v_3}$. Moreover, $v_4$ will be assigned a 0 if $f'(v_5) = 0$ and a 1 if $f'(v_5) \geq 1$. In either case, $\gamma_f^p(T) \leq \gamma_f^p(T') + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1$. Using the fact that $n' = n - n_{v_3} - 1$, $\ell(T') \geq \ell(T) - \ell(T_{v_3})$, $s(T') \leq s(T) - s(T_{v_3}) + 1$, it follows from the induction hypothesis that

$$\gamma_f^p(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1 \leq \frac{4(n - n_{v_3} - 1) - \ell(T) + \ell(T_{v_3}) + 2s(T) - 2s(T_{v_3}) + 1}{5}$$

$$+ \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1 = \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

(b) $s_1 = 1$. Assume first that $v_3$ has at least four leaves, and let $T' = T \setminus \{w, v_1, v_2\}$, where $w \in L(v_3)$. Since $v_3$ has at least three leaves we have $f'(v_3) \geq 1$. If $f'(v_3) = 2$, then $f'$ is extended to a PIDF of $T'$ by assigning a 2 to $v_2$ and a 0 to $w, v_1$. If $f'(v_3) = 1$, then $f'$ to a PIDF of $T$ by assigning a 1 to $v_1, v_2$ and a 0 to $v_2$. In either case, $\gamma_f^p(T) \leq \gamma_f^p(T') + 2$. By the induction hypothesis we get

$$\gamma_f^p(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + 2 \leq \frac{4(n - 3) - \ell(T) + 2 + 2s(T) - 3}{5} + 2 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Hence, we can assume that $v_3$ has at most three leaves and thus $\ell(T_{v_3}) \leq s(T_{v_3}) + 2$. Let $T'$ be the tree obtained from $T$ by removing all vertices of $T_{v_3}$ except $v_3$. Then $n' = n - n_{v_3} + 1$, $\ell(T') = \ell(T) - \ell(T_{v_3}) + 1$ and $s(T') = s(T) - s(T_{v_3})$. If $f'(v_3) = 0$, then $f'(v_4) = 2$, and $f'$ can be extended to a PIDF of $T$ by adding to it the PIDF $g$ of $T_{v_3}$, where $v_3$ is reassigned $g(v_3)$ instead of $f'(v_3)$. Applying our induction hypothesis, we obtain
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\[ \gamma^p_{\mathcal{I}}(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \]

\[ \leq \frac{4(n - n_{v_3} + 1) - \ell(T) + \ell(T_{v_3}) - 1 + 2s(T) - 2s(T_{v_3}) - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \]

If \( f'(v_3) = 2 \), then \( f'(v_4) = 0 \) and the other leaf neighbor of \( v_4 \) in \( T' \) is assigned a 1, which provides a contradiction. Hence let \( f'(v_3) = 1 \). Then we extend \( f' \) to a PIDF of \( T \) by assigning a 1 to all leaves vertices of \( T_{v_3} \) and a 0 to remaining vertices of \( T_{v_3} \) but \( v_3 \). Using the fact that \( \ell(T_{v_3}) \leq s(T_{v_3}) + 2 \), \( n_{v_3} = \ell(T_{v_3}) + s(T_{v_3}) \) and the induction hypothesis, we obtain

\[ \gamma^p_{\mathcal{I}}(T) \leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \ell(T_{v_3}) \]

\[ \leq \frac{4(n - n_{v_3} + 1) - \ell(T) + \ell(T_{v_3}) - 1 + 2s(T) - 2s(T_{v_3}) - 1}{5} + \ell(T_{v_3}) \]

\[ \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} . \]

This completes the proof. \( \blacksquare \)

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Received 4 September 2019
Revised 8 April 2020
Accepted 10 April 2020