DESCRIBING MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE 6 OR 7

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Abstract

In 1940, in attempts to solve the Four Color Problem, Henry Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $P_5$ of 3-polytopes with minimum degree 5. This description depends on 32 main parameters.

\[ (6, 6, 7, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11), \\
(5, 6, 7, 7, 7), (5, 6, 6, 7, 12), (5, 6, 6, 8, 10), (5, 6, 6, 6, 17), \\
(5, 5, 7, 7, 13), (5, 5, 7, 8, 10), (5, 5, 6, 7, 27), \\
(5, 5, 6, 6, \infty), (5, 5, 6, 8, 15), (5, 5, 6, 9, 11), \\
(5, 5, 5, 7, 41), (5, 5, 5, 8, 23), (5, 5, 5, 9, 17), \\
(5, 5, 5, 10, 14), (5, 5, 5, 11, 13). \]

Not many precise upper bounds on these parameters have been obtained as yet, even for restricted subclasses in $P_5$. In 2018, Borodin, Ivanova, Kazak proved that every forbidding vertices of degree from 7 to 11 results in a tight description $(5, 5, 6, 6, \infty)$, $(5, 6, 6, 6, 15)$, $(6, 6, 6, 6, 6)$. Recently, Borodin, Ivanova, and Kazak proved every 3-polytope in $P_5$ with no vertices of degrees 6, 7, and 8 has a 5-vertex whose neighborhood is majorized by one of the sequences $(5, 5, 5, 5, \infty)$ and $(5, 5, 10, 5, 12)$, which is tight and improves a corresponding description $(5, 5, 5, 5, \infty)$, $(5, 5, 9, 5, 17)$, $(5, 5, 10, 5, 14)$, $(5, 5, 11, 5, 13)$ that follows from the Lebesgue Theorem.

The author work was supported by the Russian Science Foundation (grant 16-11-10054-P).
The purpose of this paper is to prove that every 3-polytope with minimum degree 5 and no vertices of degree 6 or 7 has a 5-vertex whose neighborhood is majorized by one of the ordered sequences \( (5, 5, 5, 5, \infty) \), \( (5, 5, 8, 5, 14) \), or \( (5, 5, 10, 5, 12) \).

**Keywords:** planar graph, structural properties, 3-polytope, 5-star, neighborhood.

**2010 Mathematics Subject Classification:** 05C75.

1. Introduction

By a **3-polytope** \( P \) we mean a finite 3-connected plane graph. The **degree** \( d(v) \) of a vertex \( v \) (\( d(f) \) of a face \( f \)) in \( P \) is the number of edges incident with it. Let \( P_5 \) denote the class of 3-polytopes with minimum degree 5. A **k-vertex** (k-face) is a vertex (face) of degree \( k \); a **\( k^+ \)-vertex** has degree at least \( k \), etc.

By a **minor \( k \)-star** \( S_k^{(m)} \) we mean a star with \( k \) rays centered at a \( 5^- \)-vertex. The **weight** (height) of an \( S_k^{(m)} \) in \( P \) is the degree sum (maximum degree) of its boundary vertices, and \( w_k(P) \) (\( h_k(P) \)) denotes the minimum weight (height) of minor \( k \)-stars in \( P \).

In 1904, Wernicke [27] proved that every 3-polytope in \( P_5 \) has a 5-vertex adjacent to a \( 6^- \)-vertex, which was strengthened by Franklin [16] in 1922 by proving that in fact there is a 5-vertex adjacent to two \( 6^- \)-vertices. Recently, Borodin and Ivanova [2] proved that every 3-polytope in \( P_5 \) has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which description is tight.

We say that a 5-vertex \( v \) is of type \( (k_1, \ldots, k_5) \) or simply a \( (k_1, \ldots, k_5) \)-vertex if the ordered sequence of degrees of its neighbors is majorized by the vector \( (k_1, \ldots, k_5) \). If the order of certain entries in the type is irrelevant, then we put a line over them.

In 1940, the following description of the neighborhoods of 5-vertices in \( P_5 \) was given by Lebesgue [24, p. 36], which absorbs the results of Wernicke [27] and Franklin [16].

**Theorem 1** (Lebesgue [24]). Every triangulated 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

\[
(6, 6, 7, 7, 7), \ (6, 6, 6, 7, 9), \ (6, 6, 6, 6, 11), \\
(5, 6, 7, 7, 8), \ (5, 6, 6, 7, 11), \ (5, 6, 6, 8, 8), \\
(5, 6, 9, 7), \ (5, 7, 6, 6, 12), \ (5, 8, 6, 6, 10), \ (5, 6, 6, 6, 17), \\
(5, 5, 7, 7, 8), \ (5, 13, 5, 7, 7), \ (5, 10, 5, 7, 8), \\
(5, 8, 5, 7, 9), \ (5, 7, 5, 7, 10), \ (5, 7, 5, 8, 8),
\]
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(5, 5, 7, 6, 12), (5, 5, 8, 6, 10), (5, 6, 5, 7, 12),
(5, 6, 5, 8, 10), (5, 17, 5, 6, 7), (5, 11, 5, 6, 8),
(5, 11, 5, 6, 9), (5, 7, 5, 6, 13), (5, 8, 5, 6, 11), (5, 9, 5, 6, 10), (5, 6, 6, 5, ∞),
(5, 5, 7, 5, 41), (5, 5, 8, 5, 23), (5, 5, 9, 5, 17), (5, 5, 10, 5, 14), (5, 5, 11, 5, 13).

In particular, Theorem 1 implies that there is a 5-vertex with three 7^-neighbors, which means that \( h\left(S_5^{(m)}\right) \leq 7 \). Another corollary of Theorem 1 is that \( w\left(S_3^{(m)}\right) \leq 24 \), which was improved in 1996 by Jendrol’ and Madaras [21] to the sharp bound \( w\left(S_3^{(m)}\right) \leq 23 \). Furthermore, Jendrol’ and Madaras [21] gave a tight description of minor 3-stars in \( P_5 \): there is a (6,6)- or (5,6,7)-star.

Recently, Borodin and Ivanova [1], using the sharp bound \( w\left(S_4^{(m)}\right) \leq 30 \) by Borodin and Woodall [14], obtained a tight description of minor 4-stars in \( P_5 \).

Jendrol’ and Madaras [21] also showed that if a polytope \( P \) in \( P_5 \) is allowed to have a 5-vertex adjacent to four 5-vertices (such a 5-vertex is also called a minor (5,5,5,5,5,∞)-star), then \( \mu_5(P) \) (and hence \( w_5(P) \)) can be arbitrarily large. In 2014, Borodin, Ivanova, and Jensen [7] showed that the same phenomenon holds under a weaker assumption that 5-vertices are allowed to have two 5-neighbors and two 6-neighbors. Thus, the term (5,6,6,5,∞) in Theorem 1 is necessary.

Some recent sharp bounds on the height and weight of minor 5-stars in various subclasses of \( P_5 \), along with several related results, can be found in [1–9, 11–15, 17, 20, 22] and surveys [4, 23].

In particular, Borodin, Ivanova and Nikiforov [13] obtained a sharp bound \( h\left(S_5^{(m)}\right) \leq 17 \) under the absence of 6-vertices, which improves the upper bound 41 that follows from Theorem 1.

In 2013, Ivanova and Nikiforov [18] corrected two misprints in the statement of Theorem 1: 11 in (5,11,5,6,8) should be replaced by 15, and in (5,17,5,6,7) there should be 27 instead of 17. Later on, they improved [19, 26] thus corrected version of Theorem 1 by replacing 41 and 23 in the types (5,5,7,5,41) and (5,5,8,5,23) to 31 and 22, respectively.

**Theorem 2** (Ivanova, Nikiforov [18, 19, 26]). Every 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

(6, 6, 7, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11),
(5, 8, 6, 7, 7), (5, 7, 6, 8, 7), (5, 6, 6, 7, 11), (5, 6, 6, 6, 8),
(5, 7, 6, 6, 12), (5, 8, 6, 6, 10), (5, 6, 6, 6, 17),
(5, 5, 7, 7, 8), (5, 13, 5, 7, 7), (5, 10, 5, 7, 8), (5, 8, 5, 7, 9),
(5, 7, 5, 7, 10), (5, 7, 5, 8, 8), (5, 5, 7, 6, 12), (5, 5, 8, 6, 10),
(5, 6, 5, 7, 12), (5, 6, 5, 8, 10), (5, 27, 5, 6, 7), (5, 15, 5, 6, 8),
(5, 11, 5, 6, 9), (5, 7, 5, 6, 13), (5, 8, 5, 6, 11), (5, 9, 5, 6, 10),
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(5, 6, 6, 5, ∞),
(5, 5, 7, 5, 31), (5, 5, 8, 5, 22), (5, 5, 9, 5, 17), (5, 5, 10, 5, 14), (5, 5, 11, 5, 13).

Recently, Li, Rao, and Wang [25] obtained two descriptions of minor 5-stars in plane graphs with minimum degree 5, in which some parameters are better and some are worse than in Theorems 1 and 2.

Recently, Borodin, Ivanova, and Kazak proved in [8] that forbidding vertices of degree from 7 to 11 in \( P_5 \) results in a tight description \((5, 5, 6, 5, 15), (6, 6, 6, 6), \) which improves a description \((5, 5, 6, 5, 6, ∞), (6, 6, 6, 6) \) that follows from Theorem 1.

If vertices of degrees 6, 7, and 8 are forbidden, then Theorem 1 implies a 5-vertex of one of the following types: (5, 5, 5, 5, 5, ∞), (5, 5, 5, 10, 5, 14), (5, 5, 11, 5, 13). Recently, Borodin, Ivanova, and Kazak [10] proved a precise description of 5-stars in this subclass of \( P_5 \): (5, 5, 5, 5, ∞) and (5, 5, 10, 5, 12), where all parameters are best possible.

The purpose of this paper is to extend and strengthen the description in [10] as follows.

**Theorem 3.** Every 3-polytope with minimum degree 5 and without vertices of degrees of 6 or 7 has a 5-vertex of one of the following types: (5, 5, 5, 5, ∞), (5, 5, 8, 5, 14), or (5, 5, 10, 5, 12).

2. Proof of Theorem 3

2.1. The tightness

To confirm the tightness of the term (5, 5, 10, 5, 12), we start with the (5, 6, 6)-Archimedean solid, which is a cubic 3-polytope whose each vertex is incident with a 5-face and two 6-faces, replace all its vertices by small 3-faces, and cap each 10+-face obtained.

The resulting 3-polytope has only 5-vertices, 10-vertices, and 12-vertices, and all 5-vertices are of type (5, 5, 10, 5, 12) or (5, 5, 12, 5, 12), as desired.

The construction confirming the tightness of (5, 5, 5, 5, ∞) is due to Jendrol’ and Madaras [21].

To confirm the tightness of the term (5, 5, 8, 5, 14) we start with the (3, 4, 4, 4) Archimedean solid \( A(3, 4, 4, 4) \), which is a 4-regular 3-polytope whose each vertex is incident with a 3-face and three 4-faces. Now cap each 4-face of \( A(3, 4, 4, 4) \) to obtain a triangulation \( T \) whose each face is incident with a 4-vertex and two \( 7^+ \)-vertices. The dual \( D \) of \( T \) is a cubic 3-polytope, and we replace all its vertices by small 3-faces.

The resulting 3-polytope \( R \) is cubic and such that each vertex is incident with a 3-face, 8-face, and \( 14^+ \)-face. Capping all \( 8^+ \)-faces of \( R \) yields a desired
3-polytope in which every 5-vertex has a $14^+$-neighbor and another $8^+$-neighbor, where these two major neighbors are non-consecutive.

### 2.2. Discharging

Suppose that a 3-polytope $P'_5$ is a counterexample to the main statement of Theorem 3. In particular, each 5-vertex in $P'_5$ has at most three 5-neighbors and is adjacent either to at most two 5-vertices, or otherwise to two consecutive $8^+$-vertices, or a 8-vertex non-consecutive with a $15^+$-vertex, or a vertex of degree 9 or 10 non-consecutive with a $13^+$-vertex, or two non-consecutive $11^+$-vertices.

Let $P_5$ be a counterexample with the most edges on the same vertices as $P'_5$.

**Remark 4.** $P_5$ has no $4^+$-face with two non-consecutive $8^+$-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with a greater number of edges.

Let $V$, $E$, and $F$ be the sets of vertices, edges, and faces of $P_5$. Euler’s formula $|V| - |E| + |F| = 2$ implies

$$
\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12.
$$

(1)

We assign an **initial charge** $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of $P_5$ as a counterexample to Theorem 3, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to $-12$.

The **final charge** $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R8 below (see Figure 1).

For a vertex $v$, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to $v$ in a cyclic order. A vertex is **simplicial** if it is completely surrounded by 3-faces. A 5-vertex $v$ is **strong** if $d(v_1) = d(v_2) = 5$, $d(v_3) \geq 8$, $d(v_4) \geq 8$, and there is a 3-face $vv_1v_2$. Note that $v$ also is incident to 3-faces $v_3v_4v_1$ and $v_4v_3v_5$ due to Remark 4.

A simplicial 5-vertex $v$ such that $d(v_1) = d(v_2) = d(v_4) = 5$, $8 \leq d(v_3) \leq 10$, and hence $d(v_5) \geq 13$ is **poor**, and $v_1$ is **paired** with $v$.

We note that the poor and paired neighbors in the neighborhood of each $13^+$-vertex $w$ are in one-to-one correspondence with each other. Indeed, if $w_2$ were paired with two poor vertices $w_1$ and $w_3$, then $w_2$ would have four 5-neighbors, a contradiction. On the other hand, if $w_1$, $w_2$, $w_3$ are poor neighbors of $w$, where $w_1$ and $w_2$ have a common neighbor of degree from 8 to 10, then $w_2$ is paired with $w_3$, but not with $w_1$ due to a unique 3-face incident with three 5-vertices at a poor vertex. We also see that a paired vertex $v_1$ is poor itself if and only if $v_2$ is strong.
A simplicial 5-vertex \( v \) such that \( d(v_1) = d(v_2) = d(v_3) = 5, d(v_4) = 8 \), and hence \( d(v_5) \geq 8 \) is bad, and \( v_3 \) is conjugate with \( v \). By symmetry, \( v_1 \) is also conjugate with \( v \) if \( d(v_5) = 8 \).

**R1.** A \( 4^+ \)-face \( f = v_1 \cdots v_{d(f)} \) gives each incident 5-vertex \( v_2 \):

(a) \( \frac{1}{2} \) if \( d(v_1) = d(v_3) = 5 \), or

(b) \( \frac{3}{4} \) if \( d(v_1) \geq 8 \) and \( d(v_3) = 5 \).

**R2.** A 5-vertex \( v \) with \( d(v_1) \geq 8 \) receives the following charge from its \( 8^+ \)-neighbor \( v_2 \):

(a) if \( d(v_3) = 5 \), then \( \frac{3}{8}, \frac{1}{2}, \frac{7}{12}, \) or \( \frac{3}{4} \) in the cases \( d(v_2) = 8, 9 \leq d(v_2) \leq 12, 13 \leq d(v_2) \leq 14, \) or \( d(v_2) \geq 15 \), respectively, and

(b) \( \frac{1}{2} \) if \( d(v_3) \geq 8 \).

**R3.** A non-simplicial 5-vertex \( v \) with \( d(v_1) = d(v_3) = d(v_4) = 5 \) receives \( \frac{1}{4} \) from each of its \( 8^+ \)-neighbors \( v_2 \) and \( v_5 \).

**R4.** A strong 5-vertex \( v \) with \( d(v_1) = d(v_2) = 5 \) gives \( \frac{1}{8} \) or \( \frac{1}{6} \) to \( v_1 \) if \( d(v_5) = 8 \) or \( d(v_5) \geq 9 \), respectively, and the same is valid for \( v_2 \) depending on \( d(v_3) \) by symmetry.

**R5.** A simplicial 5-vertex \( v \) with \( d(v_1) = d(v_2) = d(v_4) = 5 \) receives from \( v_5 \):

(a) \( \frac{1}{4} \) if \( d(v_5) = 8 \),

(b) \( \frac{1}{3} \) if \( 9 \leq d(v_5) \leq 10 \), and

(c) \( \frac{1}{2} \) if \( 11 \leq d(v_5) \leq 12 \).

**R6.** If a simplicial vertex \( v \) satisfies \( d(v_1) = d(v_2) = d(v_4) = 5, d(v_3) \geq 8, \) and \( d(v_5) \geq 13 \), then \( v_5 \) gives \( \frac{1}{2} \) or \( \frac{3}{4} \) to \( v \) if \( 13 \leq d(v_5) \leq 14 \) or \( d(v_5) \geq 15 \), respectively, with the following two exceptions:

(ex1) if \( 13 \leq d(v_5) \leq 14 \), \( 9 \leq d(v_3) \leq 10 \) and \( v_2 \) is not strong (hence \( v_2 \) has three 5-neighbors and a \( 13^+ \)-neighbor), then \( v_5 \) gives \( \frac{7}{12} \) to \( v \):

(ex2) if \( 13 \leq d(v_5) \leq 14 \), \( v_1 \) is a poor vertex paired with \( v \), and \( v_2 \) is not strong (so \( v_2 \) has three 5-neighbors), then \( v_5 \) also gives \( \frac{7}{12} \) to \( v \).

**R7.** Every poor 5-vertex \( v \) with a non-strong neighbor \( v_2 \) receives from its paired vertex \( v_1 \):

(a) \( \frac{1}{8} \) if \( v \) has an 8-neighbor \( v_3 \), or

(b) \( \frac{1}{12} \) if \( 9 \leq d(v_3) \leq 10 \).

**R8.** If vertex \( v \) satisfies \( d(v_1) = d(v_3) = 5, d(v_2) \geq 8, d(v_4) \geq 8 \) and \( d(v_5) = 8 \), then \( v \) receives \( \frac{1}{3} \) from \( v_2 \).

**R9.** A bad 5-vertex \( v \) receives \( \frac{1}{8} \) from each conjugate vertex that is neither strong nor simplicial.
R10. If a bad 5-vertex $v$ has a conjugate neighbor $v_3$ that is simplicial and non-strong (so $v_3$ is poor with a $15^+$-neighbor), then $v$ receives $\frac{1}{8}$ from the 5-vertex $v_2$ across the face $v_2v_3$. By symmetry, the same holds for $v_1$ and $v_1v_2$ if $d(v_3) = 8$.

2.3. Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$

If $f$ is a $4^+$-face, then the donation of $\frac{3}{4}$ by R1b may be interpreted as giving $\frac{1}{2}$ to the 5-vertex and $\frac{1}{4}$ to the neighbor $8^+$-vertex along the boundary $\partial(f)$ of $f$. As a result, each vertex in $\partial(f)$ receives at most $2 \times \frac{1}{4}$ from $f$ after this averaging, so we have $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3d(f)-4}{2} \geq 0$.

Now suppose $v \in V$.

Case 1. $d(v) = 5$. If $v$ is adjacent to at least four $8^+$-vertices, then $\mu'(v) \geq 5 - 6 + 4 \times \frac{3}{8} > 0$ by R2, since $v$ does not give charge away by R4, R7, R9 or R10.

Suppose $v$ has precisely three $8^+$-neighbors. If they are consecutive round $v$, say $v_1, v_2, v_3$, then $v$ receives at least $\frac{1}{2} + 2 \times \frac{3}{8} > 1$ from them by R2 in view of
Remark 4. Also, \( v \) can give \( \frac{1}{8} \) or \( \frac{1}{6} \) to each of the two 5-neighbors \( v_4 \) and \( v_5 \) by R4, and \( \frac{1}{2} \) or \( \frac{1}{12} \) to one of \( v_4 \) and \( v_5 \) by R7, if \( v \) is strong.

More specifically, if \( d(v_3) = 8 \) then \( v_4 \) receives \( \frac{1}{8} \) from \( v \) while \( v \) receives \( \frac{3}{8} \) from \( v_3 \) by R2a, so \( v_3 \) brings \( v \) the total of \( \frac{1}{4} + \frac{3}{8} - \frac{5}{8} = 0 \). If \( d(v_3) \geq 9 \), then \( v_4 \) receives \( \frac{1}{6} \) from \( v \) by R4 while \( v \) receives at least \( \frac{1}{2} \) from \( v_3 \) by R2a, so \( v_3 \) actually brings at least \( \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \) to \( v \).

Thus each of \( v_1 \) and \( v_3 \) thus brings \( v \) the total of at least \( \frac{1}{4} \) by R2 combined with R4, while \( v_2 \) brings \( \frac{1}{2} \) to \( v \) by R2b, so \( \mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0 \) if \( v \) does not give charge by R7.

If \( v \) gives \( \frac{1}{8} \) by R7a, then \( v \) receives \( \frac{3}{4} \) from each of \( v_1, v_3 \) by R2a, so \( \mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{3}{8} - \frac{5}{8} = 0 \) in view of R2 and R4. If \( v \) gives \( \frac{1}{12} \) by R7b, then \( v \) receives \( \frac{7}{12} \) from each of \( v_1, v_3 \) by R2a, so \( \mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{7}{12} - \frac{1}{12} - \frac{5}{8} \geq 0 \) in view of R2 and R4.

Now suppose \( d(v_1) = d(v_3) = 5 \). Here, \( v \) does not give charge to \( v_1 \) and \( v_3 \) by R4 or R7, so it suffices for \( v \) to collect the total of at least 1 from its three \( 8^+ \)-neighbors. If \( d(v_4) \geq 9 \) and \( d(v_5) \geq 9 \), then \( \mu'(v) \geq -1 + 2 \times \frac{1}{6} = 0 \) by R2a in view of Remark 4; otherwise, we have \( d(v_4) = 8 \) and \( d(v_5) \geq 8 \) by symmetry, which yields \( \mu'(v) \geq -1 + 2 \times \frac{3}{8} + \frac{1}{8} = 0 \) by R2a combined with R8, as desired.

It remains to assume that \( v \) has precisely two \( 8^+ \)-neighbors due to the absence of \( (5, 5, 5, 5, \infty) \)-vertex. First suppose \( d(v_4) \geq 8 \) and \( d(v_5) \geq 8 \). If \( v \) is not simplicial, then \( \mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{3}{8} - 2 \times \frac{1}{8} = 0 \) by R1, R2a, R4, R7 and R10. So suppose \( v \) is simplicial.

We next show that the total balance of \( v \) caused by donations from \( v_4 \) according to R2a, from \( v_3 \) due to R9, and from \( v_2 \) across the face \( v_2v_3v_5 \) by R10, in view of possible giving charge from \( v \) to a poor vertex \( v_3 \) by R7 and, when \( d(v_4) \geq 15 \), to a bad vertex \( v_2 \) by R10. By symmetry between \( v_4 \) and \( v_5 \) this will result in \( \mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0 \).

First suppose \( d(v_4) = 8 \). Now \( v \) receives \( \frac{3}{8} \) from \( v_4 \) by R2a and does not loose charge by R7, but can gives \( \frac{1}{8} \) by R10. If \( v \) gives \( \frac{1}{8} \) by R10, then \( d(v_5) \geq 15 \) and \( v \) receives \( \frac{2}{3} \) from \( v_5 \) by R2a, so \( \mu'(v) \geq -1 + \frac{2}{3} + \frac{3}{4} - \frac{1}{8} = 0 \). If \( v \) does not give \( \frac{1}{8} \) by R10, then the required \( \frac{1}{8} \) comes from \( v_3 \) either by R4 if \( v_3 \) is strong, or by R9 if \( v_3 \) is not simplicial, or by R10 (the same is true for \( v_1 \)), hence \( \mu'(v) \geq -1 + 2 \times \frac{3}{8} + 2 \times \frac{1}{8} = 0 \).

If \( 9 \leq d(v_4) \leq 12 \), then it suffices to observe that \( v \) receives \( \frac{1}{4} \) by R2a and does not give charge away by R7. If \( v \) gives \( \frac{1}{8} \) by R10, then \( v \) receives \( \frac{3}{4} \) from \( 15^+ \)-neighbor by R2a, and we have \( \mu'(v) \geq -1 + \frac{3}{4} + \frac{3}{8} - \frac{1}{8} > 0 \).

When \( 13 \leq d(v_4) \leq 14 \), our \( v \) receives \( \frac{7}{12} \) by R2a and can give away \( \frac{1}{12} \) by R7b if \( v \) is paired with a poor vertex \( v_3 \) or \( \frac{1}{8} \) to \( v_2 \) by R10.

Finally, if \( d(v_4) \geq 15 \) then \( v \) receives \( \frac{3}{4} \) by R2a and can give away \( \frac{1}{8} \) to a poor vertex \( v_3 \) by R7b and also \( \frac{1}{8} \) to a bad vertex \( v_2 \) by R10. So again the balance of \( v_3 \) is at least \( \frac{1}{2} - \frac{3}{8} = \frac{3}{8} - 2 \times \frac{1}{8} = 0 \), as desired.
From now on suppose \( d(v_1) \geq 8 \) and \( d(v_3) \geq 8 \). If \( v \) is not simplicial, then \( v \) receives \( 2 \times \frac{1}{3} \) from \( v_1 \) and \( v_3 \) by R3 and at least \( \frac{1}{2} \) from an incident 4-face by R1. Thus we are done unless \( v \) gives \( \frac{1}{12} \) or \( \frac{1}{8} \) to at least one of \( v_4 \) and \( v_5 \) by R7 or R9, which can happen only if the face \( f = \cdots v_1 v_5 v_3 \) is a triangle. However, then \( v \) actually receives \( \frac{3}{8} \) by R1b at least once, and we have \( \mu'(v) \geq -1 + \frac{2}{3} + 2 \times \frac{1}{12} - 2 \times \frac{1}{8} = 0 \).

Finally, suppose \( v \) is simplicial. Now \( v \) does not give charge by R9. If \( v \) gives \( \frac{1}{5} \) or \( \frac{1}{12} \) to \( v_5 \) by R7, so that \( v \) is paired with a poor vertex \( v_5 \), then \( d(v_1) \geq 15 \) or \( d(v_1) \geq 13 \), respectively, due to the absence \((5,5,5,8,14)\) and \((5,5,5,10,12)\)-vertex by assumption. (Hereafter, we consider two possibilities in parallel, depending on whether \( v_5 \) has an 8-neighbor or a neighbor of degree 9 or 10.) Furthermore, \( v_4 \) is not strong, which implies that \( v_4 \) has a 5-neighbor different from \( v \) and \( v_5 \). In turn, this means that \( d(v_3) \geq 15 \) or \( d(v_3) \geq 13 \), respectively, since otherwise we would have a \((5,5,5,8,14)\)-vertex or \((5,5,10,5,12)\)-vertex, a contradiction.

Thus \( v \) receives from \( v_1 \) either \( \frac{5}{8} \) by R6 or \( \frac{7}{12} \) by R6ex2, respectively, and hence \( v_1 \) brings the total of \( \frac{1}{2} = \frac{5}{8} - \frac{3}{8} = \frac{7}{12} - \frac{1}{12} \) to \( v \). By symmetry, the same is true for \( v_3 \): no matter whether it is paired with \( v_4 \) or not, it brings \( \frac{1}{2} \) either by R6 or by R6ex2 combined with R7.

Thus we have \( \mu'(v) = -1 + 2 \times \frac{1}{2} = 0 \) when \( v \) gives away \( \frac{1}{8} \) or \( \frac{1}{12} \) at least once to a poor neighbor according to R7, so from now we can assume that \( v \) is not a donator of charge by R7.

We know that each 11-neighbor gives \( v \) at least \( \frac{1}{2} \) by R5c and R6, so it remains to assume that \( d(v_1) \leq 10 \), which means that \( v \) is poor.

First suppose \( d(v_1) = 8 \); then \( d(v_3) \geq 15 \) since we have no \((5,5,5,8,14)\)-vertex by assumption. No matter whether \( v_5 \) is strong or otherwise, our \( v \) receives \( \frac{1}{8} \) either from \( v_5 \) by R4 or from its paired vertex \( v_4 \) by R7a, respectively. Also, \( v \) receives \( \frac{1}{4} \) from \( v_1 \) by R5a and \( \frac{5}{8} \) from \( v_3 \) by R6a, so we have \( \mu'(v) = 0 \) in both options.

Now, if \( 9 \leq d(v_1) \leq 10 \) then \( d(v_3) \geq 13 \) due to the absence \((5,5,10,5,12)\)-vertex. Now if \( v_5 \) is strong, then \( v \) receives \( \frac{1}{6} \) from \( v_5 \) by R4, \( \frac{1}{3} \) from \( v_1 \) by R5b, and \( \frac{1}{2} \) from \( v_3 \) by R6a, so we have \( \mu'(v) = 0 \). Otherwise, \( v \) receives \( \frac{1}{12} \) from \( v_4 \) by R7b and \( \frac{1}{7} \) from \( v_1 \). Also, \( v \) receives from \( v_3 \) either \( \frac{7}{12} \) by R6ex1 if \( d(v_3) \leq 14 \) or \( \frac{5}{8} \) (which is greater than \( \frac{7}{12} \)) by R6 if \( d(v_3) \geq 15 \). This again makes \( \mu'(v) \geq 0 \), as desired.

Finally, if \( 11 \leq d(v_1) \leq 12 \) and \( 11 \leq d(v_3) \leq 12 \), then \( \mu'(v) = 0 \) by R5c.

Case 2. \( d(v) = 8 \). We can average donations of \( v \) to its 5-neighbors according to R2, R3, R5a, and R8 as follows. If \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) \geq 8 \), which is the situation of R2a, then \( v \) instead gives \( \frac{1}{4} \) to \( v_2 \) and \( \frac{1}{3} \) to \( v_3 \). Similarly, instead of giving \( \frac{1}{2} \) to a 5-neighbor \( v_2 \) by R2b, our \( v \) now gives \( \frac{1}{4} \) to \( v_2 \) and \( \frac{1}{3} \) to each of the 8-face \( v_1 \) and \( v_3 \). As a result, each neighbor receives at most
\[ \frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{3}{8} - \frac{1}{8} \text{ from } v \text{ after averaging, so } \mu'(v) \geq d(v) - 6 - \frac{d(v)}{4} = \frac{3(d(v)-8)}{4} \geq 0, \text{ as desired.} \]

**Case 3.** \( 9 \leq d(v) \leq 10. \) We now average donations of \( v \) to its 5-neighbors according to R2, R3, R5b, and R8 in the same fashion. Instead of giving \( \frac{1}{2} \) to a 5-neighbor \( v_2 \) by R2b, our \( v \) gives \( \frac{1}{2} \) to each of the vertices \( v_1, v_2, \) and \( v_3. \) If \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) \geq 9, \) which happens in R2a, then \( v \) rather gives \( \frac{1}{2} \) to \( v_2 \) and \( \frac{1}{6} \) to \( v_3. \) As a result, each neighbor receives at most \( \frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{2} - \frac{1}{6} \) from \( v, \) so \( \mu'(v) \geq d(v) - 6 - \frac{d(v)}{3} = \frac{2(d(v)-9)}{3} \geq 0, \) and we are done.

**Case 4.** \( 11 \leq d(v) \leq 12. \) We note that \( v \) gives each neighbor at most \( \frac{1}{2} \) by R2, R3, R5c, and R8, so \( \mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2}, \) which settles the case \( d(v) = 12. \)

So suppose \( d(v) = 11. \) If \( v \) has an \( 8^+ \)-neighbor, then \( \mu'(v) \geq 11 - 6 - 10 \times \frac{1}{2} = 0. \) Thus we can assume that \( v \) is completely surrounded by 5-vertices. If \( v \) is incident with a \( 4^+ \)-face \( \cdots v_1 v_2 v_3 \cdots, \) then each of \( v_1 \) and \( v_2 \) is non-simplicial and hence can only receive \( \frac{1}{2} \) from \( v \) by R3 or R8. Indeed, if the neighbors of \( v_1 \) in a cyclic order are \( \ldots, x_1, v, y_1, \ldots, \) then \( d(x_1) = d(y_1) = 5 \) due to Remark 1, and the same argument works for \( v_2. \) This implies \( \mu'(v) \geq 5 - 2 \times \frac{1}{2} = 11 - 2 \times \frac{1}{2} = 0. \)

Therefore, it remains to assume in addition that \( v \) is simplicial. Now if there is a \( 4^+ \)-face \( \cdots v'_1 v'_1 v'_2 v'_2, \) then each of \( v_1 \) and \( v_2 \) receives at most \( \frac{1}{2} \) from \( v, \) either by R3, which happens when \( v_1 \) has three 5-neighbors, or possibly by R8, otherwise. So again \( \mu'(v) \geq 0. \)

Thus we are done unless there are vertices \( w_1, \ldots, w_{11} \) lying in 3-faces \( w_k v_k v_{k+1} \) whenever \( 1 \leq k \leq 11 \) (addition mod 11 throughout proving Case 4). If so, then we cannot have \( d(w_k) \leq 8 \geq d(w_{k+1}) \) for any \( k, \) for otherwise \( w(S_5(v_{k+1})) \leq 3 \times 5 + 2 \times 8 + 11 = 42, \) which is impossible. By the oddness of 11, this implies that, say, \( d(w_1) \geq 9 \) and \( d(w_2) \geq 9. \) It follows from Remark 1 that there is a 3-face \( w_1 v_2 w_2, \) and it suffices to observe that \( v \) gives no charge to \( v_2 \) by R8 or any other our rule. Hence we have \( \mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0. \)

**Case 5.** \( 13 \leq d(v) \leq 14. \) We know that \( v \) gives at most \( \frac{7}{12} \) to each adjacent 5-vertex by R1–R8. Since \( \mu(v) = d(v) - 6 - \frac{7d(v)}{12} = 5d(v) - \frac{72}{12}, \) it follows that \( \mu'(v) \geq \frac{7}{12} \) for \( d(v) = 14, \) and \( \mu'(v) \geq \frac{7}{12} \) for \( d(v) = 13. \) Therefore, we use some additional reasons to improve these rough estimations in order to prove \( \mu'(v) \geq 0. \)

First of all, we can assume that \( v \) is completely surrounded by 5-vertices, for otherwise \( \mu'(v) \geq d(v) - 6 - \frac{7(d(v)-1)}{12} = \frac{5(d(v)-13)}{12} \geq 0, \) as desired.

Secondly, if \( v \) is not simplicial then \( v \) gives at most \( \frac{1}{2} \) to each of at least two vertices incident with a common \( 4^+ \)-face with \( v \) due to the argument used in Case 4, which means that in fact \( \mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v)-2)}{12} \geq \frac{5(d(v)-13)}{12} + \frac{1}{12} > 0. \)
Thus we are done unless \( v \) is simplicial and completely surrounded by 5-vertices. Furthermore, if there is a 4\(^{+}\)-face \( \cdots v'_{1}v_{1}v_{2}v'_{2}, \) then we similarly have \( \mu'(v) \geq \frac{1}{12}. \)

So again there is a cyclic sequence \( W_{d(v)} = w_{1}, \ldots, w_{d(v)} \) such that there are 3-faces \( w_{k}v_{k}v_{k+1} \) whenever \( 1 \leq k \leq d(v) \) (addition mod \( d(v) \)). As before, there are no two consecutive 5-vertices in \( W_{d(v)} \) since each \( v_{k} \) must have an 8\(^{+}\)-neighbor other than \( v \).

If there is an 8-vertex in \( W_{d(v)} \), say \( w_{2} \), then \( d(w_{1}) \geq 8 \) and \( d(w_{3}) \geq 8 \), since \( 43 - 3 \times 5 - 13 - 8 = 7 \). Thus, in fact each of \( v_{2} \) and \( v_{3} \) receives at most \( \frac{1}{4} \) from \( v \) by R3, R8 rather than \( \frac{7}{12} \), and we again have \( \mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v) - 2)}{12} > 0 \), as above. In what follows, we can assume that \( d(w_{i}) \geq 9 \) or \( d(w_{i}) = 5 \) whenever \( 1 \leq k \leq d(v) \).

If there are two consecutive 9\(^{+}\)-vertices in \( W_{d(v)} \), say \( w_{1} \) and \( w_{2} \), then \( v_{2} \) receives no charge from \( 1 \leq w_{8} \times w_{14} \) already have \( \mu \) as above. In what follows, we can assume that \( d(v) \geq 13 \) due to the oddness of 13.

So suppose \( d(v) = 14 \), all neighbors of \( v \) are simplicial, and \( d(w_{1}) = d(w_{3}) = \cdots = d(w_{14}) = 5 \), for otherwise \( v \) gives at most \( \frac{1}{4} \) to one of its neighbors, and we already have \( \mu'(v) \geq -\frac{2}{12} + \frac{7}{12} - \frac{1}{4} > 0 \).

Now if at least one of 5-vertices in \( W_{14} \), say \( w_{1} \), is strong, that is \( w_{1} \) has an 8\(^{+}\)-neighbor outside \( W_{14} \), then each of \( v_{1} \) and \( v_{2} \) receives \( \frac{1}{2} \) by R6a rather than \( \frac{7}{12} \) by R6ex1 or R6ex2, which yields \( \mu'(v) \geq 8 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0. \)

Thus we can assume that all \( w_{1}, w_{3}, \ldots, w_{13} \) are non-strong, that is each of them has a 5-neighbor outside \( W_{14} \). Among the seven 9\(^{+}\)-vertices \( w_{2}, w_{4}, \ldots, w_{14} \), there are no two consecutive (cyclically) 10\(^{-}\)-vertices, for otherwise we would have a minor 5-star with weight at most 40, which is impossible.

By parity reasons and symmetry, we can assume that \( d(w_{14}) \geq 11 \) and \( d(w_{2}) \geq 11 \). So each of \( v_{1} \) and \( v_{2} \) obeys the general rule R6 rather than its exceptions R6ex1 or R6ex2. This means that again \( \mu'(v) \geq 14 - 6 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0, \) as desired.

**Case 6.** \( d(v) \geq 15 \). We know that \( v \) gives at most \( \frac{5}{8} \) to each adjacent 5-vertex by R1–R8, except for giving \( \frac{3}{4} \) in R2a.

We now average these donations so that each 8\(^{+}\)-neighbor will receive at most \( 2 \times \frac{1}{4} \) from \( v \), while each 5-neighbor will receive at most \( \frac{5}{8} \). To this end, it suffices to switch \( \frac{1}{8} \) from the donation of \( \frac{3}{4} \) to a 5-vertex \( v_{2} \) by R2a to the neighbor 8\(^{+}\)-vertex \( v_{1}. \)

Since \( \mu(v) = d(v) - 5d(v) = \frac{3d(v) - 16}{8} \), it follows that our averaging results in \( \mu'(v) \geq 0 \) for \( d(v) \geq 16. \)

Finally, suppose \( d(v) = 15 \). If \( v \) has an 8\(^{+}\)-neighbor or a non-simplicial 5-neighbor, then \( \mu'(v) \geq 15 - 6 - \frac{1}{4} - 14 \times \frac{5}{8} = 0 \) by R1–R8.
Thus we can assume that \( v \) is completely surrounded by simplicial 5-vertices, which means that the sequence \( W_{15} \) introduced in Case 5 is actually a 15-cycle. Again, \( W_{15} \) has no two consecutive 5-vertices, which implies by parity reasons and symmetry that \( d(w_1) \geq 8 \) and \( d(w_2) \geq 8 \). Since \( v_2 \) receives \( \frac{1}{4} \) from \( v \) by R8 and nothing by any other our rule, we are done.

Thus we have proved \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \), which contradicts (1) and completes the proof of Theorem 3.

References


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Received 3 October 2019
Revised 13 December 2019
Accepted 13 December 2019