DESCRIBING MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE 6 OR 7

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Abstract

In 1940, in attempts to solve the Four Color Problem, Henry Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $P_5$ of 3-polytopes with minimum degree 5. This description depends on 32 main parameters.

$$(6, 6, 6, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11),$$
$$(5, 5, 6, 7, 8), (5, 6, 6, 7, 12), (5, 6, 6, 8, 10), (5, 6, 6, 6, 17),$$
$$(5, 5, 7, 7, 13), (5, 5, 7, 8, 10), (5, 5, 6, 7, 27),$$
$$(5, 5, 6, 6, \infty), (5, 5, 6, 8, 15), (5, 5, 6, 9, 11),$$
$$(5, 5, 7, 41), (5, 5, 5, 8, 23), (5, 5, 5, 9, 17),$$
$$(5, 5, 5, 10, 14), (5, 5, 5, 11, 13).$$

Not many precise upper bounds on these parameters have been obtained as yet, even for restricted subclasses in $P_5$. In 2018, Borodin, Ivanova, Kazak proved that every forbidding vertices of degree from 7 to 11 results in a tight description $(5, 5, 6, 6, \infty), (5, 6, 6, 6, 15), (6, 6, 6, 6, 6)$. Recently, Borodin, Ivanova, and Kazak proved every 3-polytope in $P_5$ with no vertices of degrees 6, 7, and 8 has a 5-vertex whose neighborhood is majorized by one of the sequences $(5, 5, 5, 5, \infty)$ and $(5, 5, 10, 5, 12)$, which is tight and improves a corresponding description $(5, 5, 5, 5, \infty), (5, 5, 9, 5, 17), (5, 5, 10, 5, 14), (5, 5, 11, 5, 13)$ that follows from the Lebesgue Theorem.

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The purpose of this paper is to prove that every 3-polytope with minimum degree 5 and no vertices of degree 6 or 7 has a 5-vertex whose neighborhood is majorized by one of the ordered sequences \((5, 5, 5, 5, \infty)\), \((5, 5, 8, 5, 14)\), or \((5, 5, 10, 5, 12)\).

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1. Introduction

By a 3-polytope \(P\) we mean a finite 3-connected plane graph. The degree \(d(v)\) of a vertex \(v\) (\(d(f)\) of a face \(f\)) in \(P\) is the number of edges incident with it. Let \(P_5\) denote the class of 3-polytopes with minimum degree 5. A \(k\)-vertex (\(k\)-face) is a vertex (face) of degree \(k\); a \(k^+\)-vertex has degree at least \(k\), etc.

By a minor \(k\)-star \(S_k^{(m)}\) we mean a star with \(k\) rays centered at a \(5^-\)-vertex. The weight (height) of an \(S_k^{(m)}\) in \(P\) is the degree sum (maximum degree) of its boundary vertices, and \(w_k(P)\) (\(h_k(P)\)) denotes the minimum weight (height) of minor \(k\)-stars in \(P\).

In 1904, Wernicke [27] proved that every 3-polytope in \(P_5\) has a 5-vertex adjacent to a 6\(^-\)-vertex, which was strengthened by Franklin [16] in 1922 by proving that in fact there is a 5-vertex adjacent to two 6\(^-\)-vertices. Recently, Borodin and Ivanova [2] proved that every 3-polytope in \(P_5\) has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which description is tight.

We say that a 5-vertex \(v\) is of type \((k_1, \ldots, k_5)\) or simply a \((k_1, \ldots, k_5)\)-vertex if the ordered sequence of degrees of its neighbors is majorized by the vector \((k_1, \ldots, k_5)\). If the order of certain entries in the type is irrelevant, then we put a line over them.

In 1940, the following description of the neighborhoods of 5-vertices in \(P_5\) was given by Lebesgue [24, p. 36], which absorbs the results of Wernicke [27] and Franklin [16].

**Theorem 1** (Lebesgue [24]). Every triangulated 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

\[
(6, 6, 7, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11), \\
(5, 6, 7, 7, 8), (5, 6, 6, 7, 11), (5, 6, 6, 8, 8), \\
(5, 6, 6, 9, 7), (5, 7, 6, 6, 12), (5, 8, 6, 6, 10), (5, 6, 6, 6, 17), \\
(5, 5, 7, 7, 8), (5, 13, 5, 7, 7), (5, 10, 5, 7, 8), \\
(5, 8, 5, 7, 9), (5, 7, 5, 7, 10), (5, 7, 5, 8, 8),
\]

By a 3-polytope \(P\) we mean a finite 3-connected plane graph. The degree \(d(v)\) of a vertex \(v\) (\(d(f)\) of a face \(f\)) in \(P\) is the number of edges incident with it. Let \(P_5\) denote the class of 3-polytopes with minimum degree 5. A \(k\)-vertex (\(k\)-face) is a vertex (face) of degree \(k\); a \(k^+\)-vertex has degree at least \(k\), etc.
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(5, 5, 7, 6, 12), (5, 5, 8, 6, 10), (5, 6, 5, 7, 12),
(5, 6, 5, 8, 10), (5, 17, 5, 6, 7), (5, 11, 5, 6, 8),
(5, 11, 5, 6, 9), (5, 5, 7, 5, 13), (5, 8, 5, 6, 11),
(5, 5, 7, 5, 41), (5, 8, 5, 23), (5, 9, 5, 17),
(5, 5, 10, 5, 14), (5, 5, 11, 5, 13).

In particular, Theorem 1 implies that there is a 5-vertex with three 7-neighbors, which means that \( h\left(S_{m}^{(5)}\right) \leq 7 \). Another corollary of Theorem 1 is that \( w\left(S_{m}^{(5)}\right) \leq 24 \), which was improved in 1996 by Jendrol’ and Madaras [21] to the sharp bound \( w\left(S_{m}^{(5)}\right) \leq 23 \). Furthermore, Jendrol’ and Madaras [21] gave a tight description of minor 3-stars in \( P_5 \): there is a (6,6,6)- or (5,6,7)-star. Recently, Borodin and Ivanova [1], using the sharp bound \( w\left(S_{4}^{(m)}\right) \leq 30 \) by Borodin and Woodall [14], obtained a tight description of minor 4-stars in \( P_5 \).

Jendrol’ and Madaras [21] also showed that if a polytope \( P \) in \( P_5 \) is allowed to have a 5-vertex adjacent to four 5-vertices (such a 5-vertex is also called a minor (5,5,5,5,∞)-star), then \( h_5(P) \) (and hence \( w_5(P) \)) can be arbitrarily large. In 2014, Borodin, Ivanova, and Jensen [7] showed that the same phenomenon holds under a weaker assumption that 5-vertices are allowed to have two 5-neighbors and two 6-neighbors. Thus, the term (5,6,6,5,∞) in Theorem 1 is necessary.

Some recent sharp bounds on the height and weight of minor 5-stars in various subclasses of \( P_5 \), along with several related results, can be found in [1–9,11–15, 17,20,22] and surveys [4,23].

In particular, Borodin, Ivanova and Nikiforov [13] obtained a sharp bound \( h\left(S_{m}^{(5)}\right) \leq 17 \) under the absence of 6-vertices, which improves the upper bound 41 that follows from Theorem 1.

In 2013, Ivanova and Nikiforov [18] corrected two misprints in the statement of Theorem 1: 11 in (5,11,5,6,8) should be replaced by 15, and in (5,17,5,6,7) there should be 27 instead of 17. Later on, they improved [19, 26] thus corrected version of Theorem 1 by replacing 41 and 23 in the types (5,5,7,5,41) and (5,5,8,5,23) to 31 and 22, respectively.

**Theorem 2** (Ivanova, Nikiforov [18, 19, 26]). Every 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

(6,6,7,7,7), (6,6,6,7,9), (6,6,6,6,11),
(5,8,6,7,7), (5,7,6,8,7), (5,6,6,7,11), (5,6,6,5,8),
(5,7,6,6,12), (5,8,6,6,10), (5,6,6,6,17),
(5,5,7,7,8), (5,13,5,7,7), (5,10,5,7,8), (5,8,5,7,9),
(5,7,5,7,10), (5,7,5,8,8), (5,5,7,6,12), (5,5,8,6,10),
(5,6,5,7,12), (5,6,5,8,10), (5,27,5,6,7), (5,15,5,6,8),
(5,11,5,6,9), (5,7,5,6,13), (5,8,5,6,11), (5,9,5,6,10),
Recently, Li, Rao, and Wang [25] obtained two descriptions of minor 5-stars in plane graphs with minimum degree 5, in which some parameters are better and some are worse than in Theorems 1 and 2.

Recently, Borodin, Ivanova, and Kazak proved in [8] that forbidding vertices of degree from 7 to 11 in $P_5$ results in a tight description $(5, 5, 5, 5, 31), (5, 5, 5, 6, 6), (5, 5, 9, 5, 17), (5, 5, 10, 5, 14), (5, 5, 11, 5, 13)$.

If vertices of degrees 6, 7, and 8 are forbidden, then Theorem 1 implies a 5-vertex of one of the following types: $(5, 5, 5, 5, 5, 5, ∞), (5, 5, 5, 5, 10, 5, 12)$. Recently, Borodin, Ivanova, and Kazak [10] proved a precise description of 5-stars in this subclass of $P_5$: $(5, 5, 5, 5, ∞), (5, 5, 5, 10, 5, 12)$, where all parameters are best possible.

The purpose of this paper is to extend and strengthen the description in [10] as follows.

**Theorem 3.** Every 3-polytope with minimum degree 5 and without vertices of degrees of 6 or 7 has a 5-vertex of one of the following types: $(5, 5, 5, 5, 5, ∞), (5, 5, 8, 5, 14), (5, 5, 10, 5, 12)$.

2. **Proof of Theorem 3**

2.1. The tightness

To confirm the tightness of the term $(5, 5, 10, 5, 12)$, we start with the $(5, 6, 6)$-Archimedean solid, which is a cubic 3-polytope whose each vertex is incident with a 5-face and two 6-faces, replace all its vertices by small 3-faces, and cap each $10^+$-face obtained.

The resulting 3-polytope has only 5-vertices, 10-vertices, and 12-vertices, and all 5-vertices are of type $(5, 5, 10, 5, 12)$ or $(5, 5, 12, 5, 12)$, as desired.

The construction confirming the tightness of $(5, 5, 5, 5, ∞)$ is due to Jendrol’ and Madaras [21].

To confirm the tightness of the term $(5, 5, 8, 5, 14)$ we start with the $(3, 4, 4, 4)$ Archimedean solid $A(3, 4, 4, 4)$, which is a 4-regular 3-polytope whose each vertex is incident with a 3-face and three 4-faces. Now cap each 4-face of $A(3, 4, 4, 4)$ to obtain a triangulation $T$ whose each face is incident with a 4-vertex and two $7^+$-vertices. The dual $D$ of $T$ is a cubic 3-polytope, and we replace all its vertices by small 3-faces.

The resulting 3-polytope $R$ is cubic and such that each vertex is incident with a 3-face, 8-face, and $14^+$-face. Capping all $8^+$-faces of $R$ yields a desired
3-polytope in which every 5-vertex has a $14^+$-neighbor and another $8^+$-neighbor, where these two major neighbors are non-consecutive.

2.2. Discharging

Suppose that a 3-polytope $P'_5$ is a counterexample to the main statement of Theorem 3. In particular, each 5-vertex in $P'_5$ has at most three 5-neighbors and is adjacent either to at most two 5-vertices, or otherwise to two consecutive $8^+$-vertices, or a 8-vertex non-consecutive with a $15^+$-vertex, or a vertex of degree 9 or 10 non-consecutive with a $13^+$-vertex, or two non-consecutive $11^+$-vertices.

Let $P_5$ be a counterexample with the most edges on the same vertices as $P'_5$.

Remark 4. $P_5$ has no $4^+$-face with two non-consecutive $8^+$-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with a greater number of edges.

Let $V$, $E$, and $F$ be the sets of vertices, edges, and faces of $P_5$. Euler’s formula $|V| - |E| + |F| = 2$ implies

\[
\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12.
\]

We assign an initial charge $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of $P_5$ as a counterexample to Theorem 3, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to $-12$.

The final charge $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R8 below (see Figure 1).

For a vertex $v$, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to $v$ in a cyclic order. A vertex is simplicial if it is completely surrounded by 3-faces. A 5-vertex $v$ is strong if $d(v_1) = d(v_2) = 5$, $d(v_3) \geq 8$, $d(v_4) \geq 8$, and there is a 3-face $vv_1v_2$. Note that $v$ also is incident to 3-faces $v_3vv_4$ and $v_4vv_5$ due to Remark 4.

A simplicial 5-vertex $v$ such that $d(v_1) = d(v_2) = d(v_4) = 5$, $8 \leq d(v_3) \leq 10$, and hence $d(v_5) \geq 13$ is poor, and $v_1$ is paired with $v$.

We note that the poor and paired neighbors in the neighborhood of each $13^+$-vertex $w$ are in one-to-one correspondence with each other. Indeed, if $w_2$ were paired with two poor vertices $w_1$ and $w_3$, then $w_2$ would have four 5-neighbors, a contradiction. On the other hand, if $w_1, w_2, w_3$ are poor neighbors of $w$, where $w_1$ and $w_2$ have a common neighbor of degree from 8 to 10, then $w_2$ is paired with $w_3$, but not with $w_1$ due to a unique 3-face incident with three 5-vertices at a poor vertex. We also see that a paired vertex $v_1$ is poor itself if and only if $v_2$ is strong.
A simplicial 5-vertex \( v \) such that \( d(v_1) = d(v_2) = d(v_3) = 5 \), \( d(v_4) = 8 \), and hence \( d(v_5) \geq 8 \) is bad, and \( v_3 \) is conjugate with \( v \). By symmetry, \( v_1 \) is also conjugate with \( v \) if \( d(v_5) = 8 \).

**R1.** A \( 4^+\)-face \( f = v_1 \cdots v_{d(f)} \) gives each incident 5-vertex \( v_2 \):

(a) \( \frac{1}{2} \) if \( d(v_1) = d(v_3) = 5 \), or
(b) \( \frac{3}{4} \) if \( d(v_1) \geq 8 \) and \( d(v_3) = 5 \).

**R2.** A 5-vertex \( v \) with \( d(v_1) \geq 8 \) receives the following charge from its \( 8^+\)-neighbor \( v_2 \):

(a) if \( d(v_2) = 5 \), then \( \frac{3}{8}, \frac{1}{2}, \frac{7}{12}, \) or \( \frac{3}{4} \) in the cases \( d(v_2) = 8 \), \( 9 \leq d(v_2) \leq 12 \), \( 13 \leq d(v_2) \leq 14 \), or \( d(v_2) \geq 15 \), respectively, and
(b) \( \frac{1}{2} \) if \( d(v_3) \geq 8 \).

**R3.** A non-simplicial 5-vertex \( v \) with \( d(v_1) = d(v_3) = d(v_4) = 5 \) receives \( \frac{1}{4} \) from each of its \( 8^+\)-neighbors \( v_2 \) and \( v_5 \).

**R4.** A strong 5-vertex \( v \) with \( d(v_1) = d(v_2) = 5 \) gives \( \frac{1}{8} \) or \( \frac{1}{6} \) to \( v_1 \) if \( d(v_3) = 8 \) or \( d(v_3) \geq 9 \), respectively, and the same is valid for \( v_2 \) depending on \( d(v_3) \) by symmetry.

**R5.** A simplicial 5-vertex \( v \) with \( d(v_1) = d(v_2) = d(v_4) = 5 \) receives from \( v_5 \):

(a) \( \frac{1}{4} \) if \( d(v_5) = 8 \),
(b) \( \frac{1}{3} \) if \( 9 \leq d(v_5) \leq 10 \), and
(c) \( \frac{1}{2} \) if \( 11 \leq d(v_5) \leq 12 \).

**R6.** If a simplicial vertex \( v \) satisfies \( d(v_1) = d(v_2) = d(v_4) = 5 \), \( d(v_3) \geq 8 \), and \( d(v_5) \geq 13 \), then \( v_5 \) gives \( \frac{1}{4} \) or \( \frac{3}{8} \) to \( v \) if \( 13 \leq d(v_5) \leq 14 \) or \( d(v_5) \geq 15 \), respectively, with the following two exceptions:

(ex1) if \( 13 \leq d(v_5) \leq 14 \), \( 9 \leq d(v_3) \leq 10 \) and \( v_2 \) is not strong (hence \( v_2 \) has three \( 5\)-neighbors and a \( 13^+\)-neighbor), then \( v_5 \) gives \( \frac{7}{12} \) to \( v \);

(ex2) if \( 13 \leq d(v_3) \leq 14 \), \( v_1 \) is a poor vertex paired with \( v \), and \( v_2 \) is not strong (so \( v_2 \) has three \( 5\)-neighbors), then \( v_5 \) also gives \( \frac{7}{12} \) to \( v \).

**R7.** Every poor 5-vertex \( v \) with a non-strong neighbor \( v_2 \) receives from its paired vertex \( v_1 \):

(a) \( \frac{1}{8} \) if \( v \) has an \( 8\)-neighbor \( v_3 \), or
(b) \( \frac{1}{12} \) if \( 9 \leq d(v_3) \leq 10 \).

**R8.** If vertex \( v \) satisfies \( d(v_1) = d(v_3) = 5 \), \( d(v_2) \geq 8 \), \( d(v_4) \geq 8 \) and \( d(v_5) = 8 \), then \( v \) receives \( \frac{1}{4} \) from \( v_2 \).

**R9.** A bad 5-vertex \( v \) receives \( \frac{1}{3} \) from each conjugate vertex that is neither strong nor simplicial.
R10. If a bad 5-vertex \( v \) has a conjugate neighbor \( v_3 \) that is simplicial and non-strong (so \( v_3 \) is poor with a \( 15^+ \)-neighbor), then \( v \) receives \( \frac{1}{8} \) from the 5-vertex \( v_2 \) across the face \( v_2v_3 \). By symmetry, the same holds for \( v_1 \) and \( v_1v_2 \) if \( d(v_5) = 8 \).

![Figure 1. Rules of discharging.](image)

### 2.3. Checking \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \)

If \( f \) is a \( 4^+ \)-face, then the donation of \( \frac{3}{4} \) by R1b may be interpreted as giving \( \frac{1}{2} \) to the 5-vertex and \( \frac{1}{4} \) to the neighbor \( 8^+ \)-vertex along the boundary \( \partial(f) \) of \( f \). As a result, each vertex in \( \partial(f) \) receives at most \( 2 \times \frac{3}{4} \) from \( f \) after this averaging, so we have \( \mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \geq 0 \).

Now suppose \( v \in V \).

**Case 1.** \( d(v) = 5 \). If \( v \) is adjacent to at least four \( 8^+ \)-vertices, then \( \mu'(v) \geq 5 - 6 + 4 \times \frac{3}{8} > 0 \) by R2, since \( v \) does not give charge away by R4, R7, R9 or R10.

Suppose \( v \) has precisely three \( 8^+ \)-neighbors. If they are consecutive round \( v \), say \( v_1, v_2, v_3 \), then \( v \) receives at least \( \frac{1}{2} + 2 \times \frac{3}{8} > 1 \) from them by R2 in view of
Remark 4. Also, \(v\) can give \(\frac{1}{8}\) or \(\frac{1}{6}\) to each of the two 5-neighbors \(v_4\) and \(v_5\) by R4, and \(\frac{1}{2}\) or \(\frac{1}{12}\) to one of \(v_4\) and \(v_5\) by R7, if \(v\) is strong.

More specifically, if \(d(v_3) = 8\) then \(v_4\) receives \(\frac{1}{8}\) from \(v\) while \(v\) receives \(\frac{3}{8}\) from \(v_3\) by R2a, so \(v_3\) brings \(v\) the total of \(\frac{1}{4} = \frac{3}{8} - \frac{1}{8}\). If \(d(v_3) \geq 9\), then \(v_4\) receives \(\frac{1}{4}\) from \(v\) by R4 while \(v\) receives at least \(\frac{1}{2}\) from \(v_3\) by R2a, so \(v_3\) actually brings at least \(\frac{1}{2} = \frac{1}{2} - \frac{1}{8}\) to \(v\).

Thus each of \(v_1\) and \(v_3\) thus brings \(v\) the total of at least \(\frac{1}{2}\) by R2 combined with R4, while \(v_2\) brings \(\frac{1}{2}\) to \(v\) by R2b, so \(\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0\) if \(v\) does not give charge by R7.

If \(v\) gives \(\frac{1}{8}\) by R7a, then \(v\) receives \(\frac{3}{8}\) from each of \(v_1\), \(v_3\) by R2a, so \(\mu'(v) \geq -1 + \frac{3}{8} + \frac{1}{8} + 2 \times \frac{1}{6} > 0\) in view of R2 and R4. If \(v\) gives \(\frac{1}{12}\) by R7b, then \(v\) receives \(\frac{7}{12}\) from each of \(v_1\), \(v_3\) by R2a, so \(\mu'(v) \geq -1 + \frac{7}{12} + 2 \times \frac{1}{12} - \frac{1}{12} - 2 \times \frac{1}{6} > 0\) in view of R2 and R4.

Now suppose \(d(v_1) = d(v_3) = 5\). Here, \(v\) does not give charge to \(v_1\) and \(v_3\) by R4 or R7, so it suffices for \(v\) to collect the total of at least 1 from its three \(8^+\)-neighbors. If \(d(v_1) \geq 9\) and \(d(v_3) \geq 9\), then \(\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0\) by R2a in view of Remark 4; otherwise, we have \(d(v_4) = 8\) and \(d(v_5) \geq 8\) by symmetry, which yields \(\mu'(v) \geq -1 + 2 \times \frac{3}{8} + \frac{1}{8} = 0\) by R2a combined with R8, as desired.

It remains to assume that \(v\) has precisely two \(8^+\)-neighbors due to the absence of \((5,5,5,5,\infty)\)-vertex. First suppose \(d(v_4) \geq 8\) and \(d(v_5) \geq 8\). If \(v\) is not simplicial, then \(\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{3}{8} - 2 \times \frac{1}{8} = 0\) by R1, R2a, R4, R7 and R10. So suppose \(v\) is simplicial.

We next show that the total balance of \(v\) caused by donations from \(v_4\) according to R2a, from \(v_3\) due to R9, and from \(v_2\) across the face \(v_2v_3\) by R10, in view of possible giving charge from \(v\) to a poor vertex \(v_3\) by R7 and, when \(d(v_4) \geq 15\), to a bad vertex \(v_2\) by R10. By symmetry between \(v_4\) and \(v_5\) this will result in \(\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0\).

First suppose \(d(v_4) = 8\). Now \(v\) receives \(\frac{3}{8}\) from \(v_4\) by R2a and does not loose charge by R7, but can gives \(\frac{1}{8}\) by R10. If \(v\) gives \(\frac{1}{8}\) by R10, then \(d(v_5) \geq 15\) and \(v\) receives \(\frac{3}{8}\) from \(v_5\) by R2a, so \(\mu'(v) \geq -1 + \frac{3}{8} + \frac{3}{8} - \frac{1}{8} = 0\). If \(v\) does not give \(\frac{1}{8}\) by R10, then the required \(\frac{1}{8}\) comes from \(v_3\) either by R4 if \(v_3\) is strong, or by R9 if \(v_3\) is not simplicial, or by R10 (the same is true for \(v_1\)), hence \(\mu'(v) \geq -1 + 2 \times \frac{3}{8} + 2 \times \frac{1}{8} = 0\).

If \(9 \leq d(v_4) \leq 12\), then it suffices to observe that \(v\) receives \(\frac{1}{2}\) by R2a and does not give charge away by R7. If \(v\) gives \(\frac{1}{8}\) by R10, then \(v\) receives \(\frac{1}{2}\) from \(15^+\)-neighbor by R2a, and we have \(\mu'(v) \geq -1 + \frac{1}{8} + \frac{3}{8} - \frac{1}{8} > 0\).

When \(13 \leq d(v_4) \leq 14\), our \(v\) receives \(\frac{7}{12}\) by R2a and can give away \(\frac{1}{12}\) by R7b if \(v\) is paired with a poor vertex \(v_3\) or \(\frac{1}{8}\) to \(v_2\) by R10.

Finally, if \(d(v_4) \geq 15\) then \(v\) receives \(\frac{3}{4}\) by R2a and can give away \(\frac{1}{8}\) to a poor vertex \(v_3\) by R7b and also \(\frac{1}{8}\) to a bad vertex \(v_2\) by R10. So again the balance of \(v_3\) is at least \(\frac{1}{2} = \frac{3}{4} - 2 \times \frac{1}{8}\), as desired.
From now on suppose \( d(v_1) \geq 8 \) and \( d(v_3) \geq 8 \). If \( v \) is not simplicial, then \( v \) receives \( 2 \times \frac{1}{3} \) from \( v_1 \) and \( v_3 \) by R3 and at least \( \frac{1}{2} \) from an incident 4\(^{+}\)-face by R1. Thus we are done unless \( v \) gives \( \frac{1}{12} \) or \( \frac{1}{8} \) to at least one of \( v_4 \) and \( v_5 \) by R7 or R9, which can happen only if the face \( f = \cdots v_4 v_5 \) is a triangle. However, then \( v \) actually receives \( \frac{3}{4} \) by R1b at least once, and we have \( \mu'(v) \geq -1 + \frac{2}{7} + 2 \times \frac{1}{12} - 2 \times \frac{1}{8} = 0 \).

Finally, suppose \( v \) is simplicial. Now \( v \) does not give charge by R9. If \( v \) gives \( \frac{1}{8} \) or \( \frac{1}{12} \) to \( v_5 \), then \( d(v_1) \geq 15 \) or \( d(v_1) \geq 13 \), respectively, due to the absence \((5,5,5,8,14)\) and \((5,5,5,10,12)\)-vertex by assumption. (Hereafter, we consider two possibilities in parallel, depending on whether \( v_5 \) has an 8-neighbor or a neighbor of degree 9 or 10.) Furthermore, \( v_4 \) is not strong, which implies that \( v_4 \) has a 5-neighbor different from \( v \) and \( v_5 \). In turn, this means that \( d(v_3) \geq 15 \) or \( d(v_3) \geq 13 \), respectively, since otherwise we would have a \((5,5,5,8,14)\)-vertex or \((5,5,10,5,12)\)-vertex, a contradiction.

Thus \( v \) receives from \( v_1 \) either \( \frac{5}{8} \) by R6 or \( \frac{7}{12} \) by R6ex2, respectively, and hence \( v_1 \) brings the total of \( \frac{1}{2} = \frac{5}{8} - \frac{7}{8} = \frac{7}{12} - \frac{1}{12} \) to \( v \). By symmetry, the same is true for \( v_3 \): no matter whether it is paired with \( v_4 \) or not, it brings \( \frac{1}{2} \) either by R6 or by R6ex2 combined with R7.

Thus we have \( \mu'(v) = -1 + 2 \times \frac{1}{2} = 0 \) when \( v \) gives away \( \frac{1}{8} \) or \( \frac{1}{12} \) at least once to a poor neighbor according to R7, so from now we can assume that \( v \) is not a donator of charge by R7.

We know that each 11\(^{+}\)-neighbor gives \( v \) at least \( \frac{1}{2} \) by R5c and R6, so it remains to assume that \( d(v_1) \leq 10 \), which means that \( v \) is poor.

First suppose \( d(v_1) = 8 \); then \( d(v_3) \geq 15 \) since we have no \((5,5,5,8,14)\)-vertex by assumption. No matter whether \( v_5 \) is strong or otherwise, our \( v \) receives \( \frac{1}{8} \) either from \( v_5 \) by R4 or from its paired vertex \( v_1 \) by R7a, respectively. Also, \( v \) receives \( \frac{2}{5} \) from \( v_1 \) by R5a and \( \frac{3}{5} \) from \( v_3 \) by R6a, so we have \( \mu'(v) = 0 \) in both options.

Now, if \( 9 \leq d(v_1) \leq 10 \) then \( d(v_3) \geq 13 \) due to the absence \((5,5,10,5,12)\)-vertex. Now if \( v_5 \) is strong, then \( v \) receives \( \frac{1}{6} \) from \( v_5 \) by R4, \( \frac{1}{5} \) from \( v_1 \) by R5b, and \( \frac{1}{2} \) from \( v_3 \) by R6a, so we have \( \mu'(v) = 0 \). Otherwise, \( v \) receives \( \frac{1}{12} \) from \( v_1 \) by R7b and \( \frac{1}{7} \) from \( v_1 \). Also, \( v \) receives from \( v_3 \) either \( \frac{7}{12} \) by R6ex1 if \( d(v_3) \leq 14 \) or \( \frac{5}{8} \) (which is greater than \( \frac{7}{12} \)) by R6 if \( d(v_3) \geq 15 \). This again makes \( \mu'(v) \geq 0 \), as desired.

Finally, if \( 11 \leq d(v_1) \leq 12 \) and \( 11 \leq d(v_3) \leq 12 \), then \( \mu'(v) = 0 \) by R5c.

Case 2. \( d(v) = 8 \). We can average donations of \( v \) to its 5-neighbors according to R2, R3, R5a, and R8 as follows. If \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) \geq 8 \), which is the situation of R2a, then \( v \) instead gives \( \frac{1}{2} \) to \( v_2 \) and \( \frac{1}{6} \) to \( v_3 \). Similarly, instead of giving \( \frac{1}{2} \) to a 5-neighbor \( v_2 \) by R2b, our \( v \) now gives \( \frac{1}{4} \) to \( v_2 \) and \( \frac{1}{8} \) to each of the 8\(^{+}\)-vertices \( v_1 \) and \( v_3 \). As a result, each neighbor receives at most
\[
\frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{3}{8} - \frac{1}{8} \text{ from } v \text{ after averaging, so } \mu'(v) \geq d(v) - 6 - \frac{d(v)}{4} = \frac{3d(v) - 8}{4} \geq 0, 
\]
as desired.

**Case 3.** \(9 \leq d(v) \leq 10\). We now average donations of \(v\) to its 5-neighbors according to R2, R3, R5b, and R8 in the same fashion. Instead of giving \(\frac{1}{2}\) to a 5-neighbor \(v_2\) by R2b, our \(v\) gives \(\frac{1}{6}\) to each of the vertices \(v_1, v_2, v_3\). If \(d(v_1) = d(v_2) = 5\) and \(d(v_3) \geq 9\), which happens in R2a, then \(v\) rather gives \(\frac{1}{3}\) to \(v_2\) and \(\frac{1}{6}\) to \(v_3\). As a result, each neighbor receives at most \(\frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{2} - \frac{1}{6}\) from \(v\), so \(\mu'(v) \geq d(v) - 6 - \frac{d(v)}{3} = \frac{2(d(v) - 9)}{3} \geq 0\), and we are done.

**Case 4.** \(11 \leq d(v) \leq 12\). We note that \(v\) gives each neighbor at most \(\frac{1}{2}\) by R2, R3, R5c, and R8, so \(\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v) - 12}{2}\), which settles the case \(d(v) = 12\).

So suppose \(d(v) = 11\). If \(v\) has an \(8^+\)-neighbor, then \(\mu'(v) \geq 11 - 6 - 10 \times \frac{1}{2} = 0\). Thus we can assume that \(v\) is completely surrounded by 5-vertices. If \(v\) is incident with a \(4^+\)-face \(\cdots v_1v_2v_3\), then each of \(v_1\) and \(v_2\) is non-simplicial and hence can only receive \(\frac{1}{4}\) from \(v\) by R3 or R8. Indeed, if the neighbors of \(v_1\) in a cyclic order are \(\ldots x_1, v, y_1, \ldots\), then \(d(x_1) = d(y_1) = 5\) due to Remark 1, and the same argument works for \(v_2\). This implies \(\mu'(v) \geq 5 - 2 \times \frac{1}{2} - (11 - 2) \times \frac{1}{2} = 0\).

Therefore, it remains to assume in addition that \(v\) is simplicial. Now if there is a \(4^+\)-face \(\cdots v_1'v_2v_3\), then each of \(v_1\) and \(v_2\) receives at most \(\frac{1}{6}\) from \(v\): either by R3, which happens when \(v_1\) has three 5-neighbors, or possibly by R8, otherwise. So again \(\mu'(v) \geq 0\).

Thus we are done unless there are vertices \(w_1, \ldots, w_{11}\) lying in 3-faces \(w_kv_kv_{k+1}\) whenever \(1 \leq k \leq 11\) (addition mod 11 throughout proving Case 4). If so, then we cannot have \(d(w_k) \leq 8 \geq d(w_{k+1})\) for any \(k\), for otherwise \(w(S_3(v_{k+1})) \leq 3 \times 5 + 2 \times 8 + 11 = 42\), which is impossible. By the oddness of 11, this implies that, say, \(d(w_1) \geq 9\) and \(d(w_2) \geq 9\). It follows from Remark 1 that there is a 3-face \(w_1v_2w_3\), and it suffices to observe that \(v\) gives no charge to \(v_2\) by R8 or any other our rule. Hence we have \(\mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0\).

**Case 5.** \(13 \leq d(v) \leq 14\). We know that \(v\) gives at most \(\frac{8}{77}\) to each adjacent 5-vertex by R1–R8. Since \(\mu(v) = d(v) - 6 - \frac{7d(v)}{12} = \frac{5d(v) - 72}{12}\), it follows that \(\mu'(v) \geq -\frac{8}{12}\) for \(d(v) = 14\), and \(\mu'(v) \geq -\frac{7}{12}\) for \(d(v) = 13\). Therefore, we use some additional reasons to improve these rough estimations in order to prove \(\mu'(v) \geq 0\).

First of all, we can assume that \(v\) is completely surrounded by 5-vertices, for otherwise \(\mu'(v) \geq d(v) - 6 - \frac{7(d(v) - 1)}{12} = \frac{5(d(v) - 13)}{12} \geq 0\), as desired.

Secondly, if \(v\) is not simplicial then \(v\) gives at most \(\frac{1}{4}\) to each of at least two vertices incident with a common \(4^+\)-face with \(v\) due to the argument used in Case 4, which means that in fact \(\mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v) - 2)}{12} \geq \frac{5(d(v) - 13)}{12} + \frac{1}{12} > 0\).
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Thus we are done unless \( v \) is simplicial and completely surrounded by 5-vertices. Furthermore, if there is a \( 4^+ \)-face \( \cdots v'_1v_1v_2v'_2, \) then we similarly have \( \mu'(v) \geq \frac{1}{12}. \)

So again there is a cyclic sequence \( W_{d(v)} = w_1, \ldots, w_{d(v)} \) such that there are 3-faces \( w_kv_kv_{k+1} \) whenever \( 1 \leq k \leq d(v) \) (addition mod \( d(v) \)). As before, there are no two consecutive 5-vertices in \( W_{d(v)} \) since each \( v_k \) must have an \( 8^+ \)-neighbor other than \( v. \)

If there is an \( 8^- \)-vertex in \( W_{d(v)}, \) say \( w_2, \) then \( d(w_1) \geq 8 \) and \( d(w_3) \geq 8, \) since \( 43 - 3 \times 5 - 13 - 8 = 7. \) Thus, in fact each of \( v_2 \) and \( v_3 \) receives at most \( \frac{1}{7} \) from \( v \) by R3, R8 rather than \( \frac{7}{12}, \) and we again have \( \mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{7} - \frac{7(d(v)-2)}{12} > 0, \) as above. In what follows, we can assume that \( d(w_i) \geq 9 \) or \( d(w_i) = 5 \) whenever \( 1 \leq k \leq d(v). \)

If there are two consecutive \( 9^- \)-vertices in \( W_{d(v)}, \) say \( w_1 \) and \( w_2, \) then \( v_2 \) receives no charge from \( v \) by R1–R8, so we can improve our rough estimation \( \mu'(v) \geq -\frac{2}{7} \) to \( \mu'(v) \geq -\frac{2}{7} + \frac{7}{12} \geq 0, \) as desired. This completes the proof for \( d(v) = 13 \) due to the oddness of 13.

So suppose \( d(v) = 14, \) all neighbors of \( v \) are simplicial, and \( d(w_1) = d(w_3) = \cdots = d(w_{13}) = 5, \) for otherwise \( v \) gives at most \( \frac{1}{4} \) to one of its neighbors, and we already have \( \mu'(v) \geq -\frac{2}{7} + \frac{7}{12} - \frac{1}{4} > 0. \)

Now if at least one of 5-vertices in \( W_{14}, \) say \( w_1, \) is strong, that is \( w_1 \) has an \( 8^- \)-neighbor outside \( W_{14}, \) then each of \( v_1 \) and \( v_2 \) receives \( \frac{1}{7} \) by R6a rather than \( \frac{7}{12} \) by R6ex1 or R6ex2, which yields \( \mu'(v) \geq 8 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0. \)

Thus we can assume that all \( w_1, w_3, \ldots, w_{13} \) are non-strong, that is each of them has a 5-neighbor outside \( W_{14}. \) Among the seven \( 9^- \)-vertices \( w_2, w_4, \ldots, w_{14}, \) there are no two consecutive (cyclically) \( 10^- \)-vertices, for otherwise we would have a minor 5-star with weight at most 40, which is impossible.

By parity reasons and symmetry, we can assume that \( d(w_{14}) \geq 11 \) and \( d(w_2) \geq 11. \) So each of \( v_1 \) and \( v_2 \) obeys the general rule R6 rather than its exceptions R6ex1 or R6ex2. This means that again \( \mu'(v) \geq 14 - 6 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0, \) as desired.

Case 6. \( d(v) \geq 15. \) We know that \( v \) gives at most \( \frac{5}{8} \) to each adjacent 5-vertex by R1–R8, except for giving \( \frac{3}{4} \) in R2a.

We now average these donations so that each \( 8^- \)-neighbor will receive at most \( 2 \times \frac{1}{4} \) from \( v, \) while each 5-neighbor will receive at most \( \frac{5}{8}. \) To this end, it suffices to switch \( \frac{1}{8} \) from the donation of \( \frac{3}{4} \) to a 5-vertex \( v_2 \) by R2a to the neighbor \( 8^- \)-vertex \( v_1. \)

Since \( \mu(v) = d(v) - 5d(v) = \frac{3(d(v)-16)}{8}, \) it follows that our averaging results in \( \mu'(v) \geq 0 \) for \( d(v) \geq 16. \)

Finally, suppose \( d(v) = 15. \) If \( v \) has an \( 8^- \)-neighbor or a non-simplicial 5-neighbor, then \( \mu'(v) \geq 15 - 6 - \frac{1}{4} - 14 \times \frac{5}{8} = 0 \) by R1–R8.
Thus we can assume that $v$ is completely surrounded by simplicial 5-vertices, which means that the sequence $W_{15}$ introduced in Case 5 is actually a 15-cycle. Again, $W_{15}$ has no two consecutive 5-vertices, which implies by parity reasons and symmetry that $d(w_1) \geq 8$ and $d(w_2) \geq 8$. Since $v_2$ receives $\frac{1}{4}$ from $v$ by R8 and nothing by any other our rule, we are done.

Thus we have proved $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 3.

References


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