DESCRIBING MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE 6 OR 7

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Abstract

In 1940, in attempts to solve the Four Color Problem, Henry Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $P_5$ of 3-polytopes with minimum degree 5. This description depends on 32 main parameters.

$$(6,6,7,7), (6,6,6,7,9), (6,6,6,6,11),$$
$$(5,6,7,7,8), (5,6,6,7,12), (5,6,6,8,10), (5,6,6,6,17),$$
$$(5,5,7,7,13), (5,5,7,8,10), (5,5,6,7,27),$$
$$(5,5,6,6,∞), (5,5,6,8,15), (5,5,6,9,11),$$
$$(5,5,5,7,41), (5,5,5,8,23), (5,5,5,9,17),$$
$$(5,5,5,10,14), (5,5,5,11,13).$$

Not many precise upper bounds on these parameters have been obtained as yet, even for restricted subclasses in $P_5$. In 2018, Borodin, Ivanova, Kazak proved that every forbidding vertices of degree from 7 to 11 results in a tight description $(5,5,6,6,∞)$, $(5,6,6,6,15)$. Recently, Borodin, Ivanova, and Kazak proved every 3-polytope in $P_5$ with no vertices of degrees 6, 7, and 8 has a 5-vertex whose neighborhood is majorized by one of the sequences $(5,5,5,5,∞)$ and $(5,5,10,5,12)$, which is tight and improves a corresponding description $(5,5,5,5,∞)$, $(5,5,9,5,17)$, $(5,5,10,5,14)$, $(5,5,11,5,13)$ that follows from the Lebesgue Theorem.

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The purpose of this paper is to prove that every 3-polytope with minimum degree 5 and no vertices of degree 6 or 7 has a 5-vertex whose neighborhood is majorized by one of the ordered sequences (5, 5, 5, 5, ∞), (5, 5, 8, 5, 14), or (5, 5, 10, 5, 12).

**Keywords:** planar graph, structural properties, 3-polytope, 5-star, neighborhood.

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1. Introduction

By a 3-polytope $P$ we mean a finite 3-connected plane graph. The degree $d(v)$ of a vertex $v$ (of a face $f$) in $P$ is the number of edges incident with it. Let $P_5$ denote the class of 3-polytopes with minimum degree 5. A $k$-vertex ($k$-face) is a vertex (face) of degree $k$; a $k^+$-vertex has degree at least $k$, etc.

By a minor $k$-star $S_k^{(m)}$ we mean a star with $k$ rays centered at a $5^-$-vertex. The weight (height) of an $S_k^{(m)}$ in $P$ is the degree sum (maximum degree) of its boundary vertices, and $w_k(P)$ ($h_k(P)$) denotes the minimum weight (height) of minor $k$-stars in $P$.

In 1904, Wernicke [27] proved that every 3-polytope in $P_5$ has a 5-vertex adjacent to a 6$^-$-vertex, which was strengthened by Franklin [16] in 1922 by proving that in fact there is a 5-vertex adjacent to two 6$^-$-vertices. Recently, Borodin and Ivanova [2] proved that every 3-polytope in $P_5$ has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which description is tight.

We say that a 5-vertex $v$ is of type $(k_1, \ldots, k_5)$ or simply a $(k_1, \ldots, k_5)$-vertex if the ordered sequence of degrees of its neighbors is majorized by the vector $(k_1, \ldots, k_5)$. If the order of certain entries in the type is irrelevant, then we put a line over them.

In 1940, the following description of the neighborhoods of 5-vertices in $P_5$ was given by Lebesgue [24, p. 36], which absorbs the results of Wernicke [27] and Franklin [16].

**Theorem 1** (Lebesgue [24]). *Every triangulated 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:*

\[
\begin{align*}
& (6,6,7,7,7), (6,6,6,7,9), (6,6,6,6,11), \\
& (5,6,7,7,8), (5,6,6,7,11), (5,6,6,8,8), \\
& (5,6,6,9,7), (5,7,6,6,12), (5,8,6,6,10), (5,6,6,6,17), \\
& (5,5,7,7,8), (5,13,5,7,7), (5,10,5,7,8), \\
& (5,8,5,7,9), (5,7,5,7,10), (5,7,5,8,8),
\end{align*}
\]
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(5, 5, 7, 6, 12), (5, 5, 8, 6, 10), (5, 6, 5, 7, 12),
(5, 6, 5, 8, 10), (5, 17, 5, 6, 7), (5, 11, 5, 6, 8),
(5, 11, 5, 6, 9), (5, 7, 5, 6, 13), (5, 8, 5, 6, 11), (5, 9, 5, 6, 10), (5, 6, 6, 5, ∞),
(5, 5, 7, 5, 41), (5, 5, 8, 5, 23), (5, 5, 9, 5, 17), (5, 5, 10, 5, 14), (5, 5, 11, 5, 13).

In particular, Theorem 1 implies that there is a 5-vertex with three 7^-neighbors, which means that \( h \left( S_3^{(m)} \right) \leq 7 \). Another corollary of Theorem 1 is that \( w \left( S_3^{(m)} \right) \leq 24 \), which was improved in 1996 by Jendrol’ and Madaras [21] to the sharp bound \( w \left( S_3^{(m)} \right) \leq 23 \). Furthermore, Jendrol’ and Madaras [21] gave a tight description of minor 3-stars in \( P_5 \): there is a (6,6,6)- or (5,6,7)-star.

Recently, Borodin and Ivanova [1], using the sharp bound \( w \left( S_4^{(m)} \right) \leq 30 \) by Borodin and Woodall [14], obtained a tight description of minor 4-stars in \( P_5 \).

Jendrol’ and Madaras [21] also showed that if a polytope \( P \) in \( P_5 \) is allowed to have a 5-vertex adjacent to four 5-vertices (such a 5-vertex is also called a minor (5,5,5,5,∞)-star), then \( h_5(P) \) (and hence \( w_5(P) \)) can be arbitrarily large. In 2014, Borodin, Ivanova, and Jensen [7] showed that the same phenomenon holds under a weaker assumption that 5-vertices are allowed to have two 5-neighbors and two 6-neighbors. Thus, the term (5,6,6,5,∞) in Theorem 1 is necessary.

Some recent sharp bounds on the height and weight of minor 5-stars in various subclasses of \( P_5 \), along with several related results, can be found in [1–9,11–15,17,20,22] and surveys [4,23].

In particular, Borodin, Ivanova and Nikiforov [13] obtained a sharp bound \( h \left( S_5^{(m)} \right) \leq 17 \) under the absence of 6-vertices, which improves the upper bound 41 that follows from Theorem 1.

In 2013, Ivanova and Nikiforov [18] corrected two misprints in the statement of Theorem 1: 11 in (5,11,5,6,8) should be replaced by 15, and in (5,17,5,6,7) there should be 27 instead of 17. Later on, they improved [19, 26] thus corrected version of Theorem 1 by replacing 41 and 23 in the types (5,5,7,5,41) and (5,5,8,5,23) to 31 and 22, respectively.

**Theorem 2** (Ivanova, Nikiforov [18, 19, 26]). Every 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:

\[
(6, 6, 7, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11),
(5, 8, 6, 7, 7), (5, 7, 6, 8, 7), (5, 6, 6, 7, 11), (5, 6, 6, 5, 8),
(5, 7, 6, 6, 12), (5, 8, 6, 6, 10), (5, 6, 6, 6, 17),
(5, 5, 7, 7, 8), (5, 13, 5, 7, 7), (5, 10, 5, 7, 8), (5, 8, 5, 7, 9),
(5, 7, 5, 7, 10), (5, 7, 5, 8, 8), (5, 5, 7, 6, 12), (5, 5, 8, 6, 10),
(5, 6, 5, 7, 12), (5, 6, 5, 8, 10), (5, 27, 5, 6, 7), (5, 15, 5, 6, 8),
(5, 11, 5, 6, 9), (5, 7, 5, 6, 13), (5, 8, 5, 6, 11), (5, 9, 5, 6, 10),
\]
Recently, Li, Rao, and Wang [25] obtained two descriptions of minor 5-stars in plane graphs with minimum degree 5, in which some parameters are better and some are worse than in Theorems 1 and 2.

Recently, Borodin, Ivanova, and Kazak proved in [8] that forbidding vertices of degree from 7 to 11 in \( P_5 \) results in a tight description \((5,5,6,6,\infty),(5,5,7,5,31),(5,5,8,5,22),(5,5,9,5,17),(5,5,10,5,14),(5,5,11,5,13)\), which improves a description \((5,5,5,\infty),(5,5,6,6,6),(5,6,6,6,15),(6,6,6,6,6)\) that follows from Theorem 1.

If vertices of degrees 6, 7, and 8 are forbidden, then Theorem 1 implies a 5-vertex of one of the following types: \((5,5,5,5,\infty),(5,5,5,10,5,14),(5,5,5,11,5,13)\). Recently, Borodin, Ivanova, and Kazak [10] proved a precise description of 5-stars in this subclass of \( P_5 \): \((5,5,5,5,\infty),(5,5,5,10,5,12)\), where all parameters are best possible.

The purpose of this paper is to extend and strengthen the description in [10] as follows.

**Theorem 3.** Every 3-polytope with minimum degree 5 and without vertices of degrees of 6 or 7 has a 5-vertex of one of the following types: \((5,5,5,\infty),(5,5,5,5,5,14),(5,5,8,5,14),(5,5,10,5,12)\).

### 2. Proof of Theorem 3

#### 2.1. The tightness

To confirm the tightness of the term \((5,5,10,5,12)\), we start with the \((5,6,6)\)-Archimedean solid, which is a cubic 3-polytope whose each vertex is incident with a 5-face and two 6-faces, replace all its vertices by small 3-faces, and cap each \(10^+\)-face obtained.

The resulting 3-polytope has only 5-vertices, 10-vertices, and 12-vertices, and all 5-vertices are of type \((5,5,10,5,12)\) or \((5,5,12,5,12)\), as desired.

The construction confirming the tightness of \((5,5,5,\infty)\) is due to Jendrol’ and Madaras [21].

To confirm the tightness of the term \((5,5,8,5,14)\) we start with the \((3,4,4,4)\)-Archimedean solid \( A(3,4,4,4) \), which is a 4-regular 3-polytope whose each vertex is incident with a 3-face and three 4-faces. Now cap each 4-face of \( A(3,4,4,4) \) to obtain a triangulation \( T \) whose each face is incident with a 4-vertex and two \(7^+\)-vertices. The dual \( D \) of \( T \) is a cubic 3-polytope, and we replace all its vertices by small 3-faces.

The resulting 3-polytope \( R \) is cubic and such that each vertex is incident with a 3-face, 8-face, and \(14^+\)-face. Capping all \(8^+\)-faces of \( R \) yields a desired
3-polytope in which every 5-vertex has a 14+ -neighbor and another 8+ -neighbor, where these two major neighbors are non-consecutive.

2.2. Discharging

Suppose that a 3-polytope $P'_5$ is a counterexample to the main statement of Theorem 3. In particular, each 5-vertex in $P'_5$ has at most three 5-neighbors and is adjacent either to at most two 5-vertices, or otherwise to two consecutive 8+-vertices, or a 8-vertex non-consecutive with a 15+-vertex, or a vertex of degree 9 or 10 non-consecutive with a 13+-vertex, or two non-consecutive 11+-vertices.

Let $P_5$ be a counterexample with the most edges on the same vertices as $P'_5$.

**Remark 4.** $P_5$ has no 4+-face with two non-consecutive 8+-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with a greater number of edges.

Let $V, E, F$ be the sets of vertices, edges, and faces of $P_5$. Euler’s formula $|V| - |E| + |F| = 2$ implies

$$(1) \quad \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12.$$  

We assign an initial charge $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of $P_5$ as a counterexample to Theorem 3, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to $-12$.

The final charge $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R8 below (see Figure 1).

For a vertex $v$, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to $v$ in a cyclic order. A vertex is **simplicial** if it is completely surrounded by 3-faces. A 5-vertex $v$ is **strong** if $d(v_1) = d(v_2) = 5$, $d(v_3) \geq 8$, $d(v_4) \geq 8$, and there is a 3-face $vv_1v_2$. Note that $v$ also is incident to 3-faces $v_3v_4$ and $v_4v_5$ due to Remark 4.

A simplicial 5-vertex $v$ such that $d(v_1) = d(v_2) = d(v_4) = 5$, $8 \leq d(v_3) \leq 10$, and hence $d(v_5) \geq 13$ is **poor**, and $v_1$ is **paired** with $v$.

We note that the poor and paired neighbors in the neighborhood of each 13+-vertex $w$ are in one-to-one correspondence with each other. Indeed, if $w_2$ were paired with two poor vertices $w_1$ and $w_3$, then $w_2$ would have four 5-neighbors, a contradiction. On the other hand, if $w_1, w_2, w_3$ are poor neighbors of $w$, where $w_1$ and $w_2$ have a common neighbor of degree from 8 to 10, then $w_2$ is paired with $w_3$, but not with $w_1$ due to a unique 3-face incident with three 5-vertices at a poor vertex. We also see that a paired vertex $v_1$ is poor itself if and only if $v_2$ is strong.
A simplicial 5-vertex \( v \) such that \( d(v_1) = d(v_2) = d(v_3) = 5 \), \( d(v_4) = 8 \), and hence \( d(v_5) \geq 8 \) is bad, and \( v_3 \) is conjugate with \( v \). By symmetry, \( v_1 \) is also conjugate with \( v \) if \( d(v_5) = 8 \).

**R1.** A \( 4^+ \)-face \( f = v_1 \cdots v_{d(f)} \) gives each incident 5-vertex \( v_2 \):

(a) \( \frac{1}{2} \) if \( d(v_1) = d(v_3) = 5 \), or
(b) \( \frac{2}{3} \) if \( d(v_1) \geq 8 \) and \( d(v_3) = 5 \).

**R2.** A 5-vertex \( v \) with \( d(v_1) \geq 8 \) receives the following charge from its \( 8^+ \)-neighbor \( v_2 \):

(a) if \( d(v_3) = 5 \), then \( \frac{3}{8}, \frac{1}{2}, \frac{7}{12} \), or \( \frac{5}{8} \) in the cases \( d(v_2) = 8 \), \( 9 \leq d(v_2) \leq 12 \), \( 13 \leq d(v_2) \leq 14 \), or \( d(v_2) \geq 15 \), respectively, and
(b) \( \frac{1}{2} \) if \( d(v_3) \geq 8 \).

**R3.** A non-simplicial 5-vertex \( v \) with \( d(v_1) = d(v_3) = d(v_4) = 5 \) receives \( \frac{1}{4} \) from each of its \( 8^+ \)-neighbors \( v_2 \) and \( v_5 \).

**R4.** A strong 5-vertex \( v \) with \( d(v_1) = d(v_2) = 5 \) gives \( \frac{1}{5} \) or \( \frac{1}{6} \) to \( v_1 \) if \( d(v_3) = 8 \) or \( d(v_5) \geq 9 \), respectively, and the same is valid for \( v_2 \) depending on \( d(v_3) \) by symmetry.

**R5.** A simplicial 5-vertex \( v \) with \( d(v_1) = d(v_2) = d(v_4) = 5 \) receives \( \frac{1}{4} \) from \( v_5 \):

(a) \( \frac{1}{4} \) if \( d(v_5) = 8 \),
(b) \( \frac{1}{3} \) if \( 9 \leq d(v_5) \leq 10 \), and
(c) \( \frac{1}{2} \) if \( 11 \leq d(v_5) \leq 12 \).

**R6.** If a simplicial vertex \( v \) satisfies \( d(v_1) = d(v_2) = d(v_4) = 5 \), \( d(v_3) \geq 8 \), and \( d(v_5) \geq 13 \), then \( v_5 \) gives \( \frac{1}{5} \) or \( \frac{1}{3} \) to \( v \) if \( 13 \leq d(v_5) \leq 14 \) or \( d(v_5) \geq 15 \), respectively, with the following two exceptions:

(ex1) if \( 13 \leq d(v_5) \leq 14 \), \( 9 \leq d(v_3) \leq 10 \) and \( v_2 \) is not strong (hence \( v_2 \) has three \( 5 \)-neighbors and a \( 13^+ \)-neighbor), then \( v_5 \) gives \( \frac{7}{12} \) to \( v \);

(ex2) if \( 13 \leq d(v_5) \leq 14 \), \( v_1 \) is a poor vertex paired with \( v \), and \( v_2 \) is not strong (so \( v_2 \) has three \( 5 \)-neighbors), then \( v_5 \) also gives \( \frac{7}{12} \) to \( v \).

**R7.** Every poor 5-vertex \( v \) with a non-strong neighbor \( v_2 \) receives from its paired vertex \( v_1 \):

(a) \( \frac{1}{8} \) if \( v \) has an \( 8 \)-neighbor \( v_3 \), or
(b) \( \frac{1}{12} \) if \( 9 \leq d(v_3) \leq 10 \).

**R8.** If vertex \( v \) satisfies \( d(v_1) = d(v_3) = 5 \), \( d(v_2) \geq 8 \), \( d(v_4) \geq 8 \) and \( d(v_5) = 8 \), then \( v \) receives \( \frac{1}{4} \) from \( v_2 \).

**R9.** A bad 5-vertex \( v \) receives \( \frac{1}{2} \) from each conjugate vertex that is neither strong nor simplicial.
R10. If a bad 5-vertex $v$ has a conjugate neighbor $v_3$ that is simplicial and non-strong (so $v_3$ is poor with a $15^+$-neighbor), then $v$ receives $\frac{1}{5}$ from the 5-vertex $v_2$ across the face $v_2v_3$. By symmetry, the same holds for $v_1$ and $v_1v_2$ if $d(v_3) = 8$.

![Diagram of rules](image)

Figure 1. Rules of discharging.

2.3. Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$

If $f$ is a $4^+$-face, then the donation of $\frac{3}{4}$ by R1b may be interpreted as giving $\frac{1}{2}$ to the 5-vertex and $\frac{1}{4}$ to the neighbor $8^+$-vertex along the boundary $\partial(f)$ of $f$. As a result, each vertex in $\partial(f)$ receives at most $2 \times \frac{1}{4}$ from $f$ after this averaging, so we have $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \geq 0$.

Now suppose $v \in V$.

Case 1. $d(v) = 5$. If $v$ is adjacent to at least four $8^+$-vertices, then $\mu'(v) \geq 5 - 6 + 4 \times \frac{3}{8} > 0$ by R2, since $v$ does not give charge away by R4, R7, R9 or R10.

Suppose $v$ has precisely three $8^+$-neighbors. If they are consecutive round $v$, say $v_1, v_2, v_3$, then $v$ receives at least $\frac{1}{2} + 2 \times \frac{3}{8} > 1$ from them by R2 in view of
Remark 4. Also, $v$ can give $\frac{1}{8}$ or $\frac{1}{6}$ to each of the two 5-neighbors $v_4$ and $v_5$ by $R_4$, and $\frac{1}{8}$ or $\frac{1}{12}$ to one of $v_4$ and $v_5$ by $R_7$, if $v$ is strong.

More specifically, if $d(v_3) = 8$ then $v_4$ receives $\frac{1}{8}$ from $v$ while $v$ receives $\frac{3}{8}$ from $v_3$ by $R_2a$, so $v_3$ brings $v$ the total of $\frac{1}{4} = \frac{3}{8} - \frac{1}{8}$. If $d(v_3) \geq 9$, then $v_4$ receives $\frac{1}{8}$ from $v$ by $R_4$ while $v$ receives at least $\frac{1}{2}$ from $v_3$ by $R_2a$, so $v_3$ actually brings at least $\frac{1}{2} = \frac{3}{8} - \frac{1}{8}$ to $v$.

Thus each of $v_1$ and $v_3$ thus brings $v$ the total of at least $\frac{1}{4}$ by $R_2$ combined with $R_4$, while $v_2$ brings $\frac{1}{2}$ to $v$ by $R_2b$, so $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ if $v$ does not give charge by $R_7$.

If $v$ gives $\frac{1}{8}$ by $R_7a$, then $v$ receives $\frac{3}{4}$ from each of $v_1$, $v_3$ by $R_2a$, so $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} - \frac{1}{8} - 2 \times \frac{1}{8} > 0$ in view of $R_2$ and $R_4$. If $v$ gives $\frac{1}{12}$ by $R_7b$, then $v$ receives $\frac{7}{12}$ from each of $v_1$, $v_3$ by $R_2a$, so $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} - \frac{1}{12} - 2 \times \frac{1}{8} > 0$ in view of $R_2$ and $R_4$.

Now suppose $d(v_1) = d(v_3) = 5$. Here, $v$ does not give charge to $v_1$ and $v_3$ by $R_4$ or $R_7$, so it suffices for $v$ to collect the total of at least 1 from its three $8^+$-neighbors. If $d(v_4) \geq 9$ and $d(v_5) \geq 9$, then $\mu'(v) \geq -1 + 2 \times \frac{1}{4} = 0$ by $R_2a$ in view of Remark 4; otherwise, we have $d(v_4) = 8$ and $d(v_5) \geq 8$ by symmetry, which yields $\mu'(v) \geq -1 + 2 \times \frac{3}{8} + \frac{1}{4} = 0$ by $R_2a$ combined with $R_8$, as desired.

It remains to assume that $v$ has precisely two $8^+$-neighbors due to the absence of $(5, 5, 5, 5, \infty)$-vertex. First suppose $d(v_4) \geq 8$ and $d(v_5) \geq 8$. If $v$ is not simplicial, then $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{3}{8} - 2 \times \frac{1}{8} = 0$ by $R_1$, $R_2a$, $R_4$, $R_7$ and $R_{10}$. So suppose $v$ is simplicial.

We next show that the total balance of $v$ caused by donations from $v_4$ according to $R_2a$, from $v_3$ due to $R_9$, and from $v_2$ across the face $v_2v_3v_5$ by $R_{10}$, in view of possible giving charge from $v$ to a poor vertex $v_3$ by $R_7$ and, when $d(v_4) \geq 15$, to a bad vertex $v_2$ by $R_{10}$. By symmetry between $v_4$ and $v_5$ this will result in $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$.

First suppose $d(v_4) = 8$. Now $v$ receives $\frac{3}{8}$ from $v_4$ by $R_2a$ and does not loose charge by $R_7$, but can gives $\frac{1}{8}$ by $R_{10}$. If $v$ gives $\frac{1}{8}$ by $R_10$, then $d(v_5) \geq 15$ and $v$ receives $\frac{3}{8}$ from $v_5$ by $R_2a$, so $\mu'(v) \geq -1 + \frac{3}{8} + \frac{3}{8} - \frac{1}{8} = 0$. If $v$ does not give $\frac{1}{8}$ by $R_{10}$, then the required $\frac{1}{8}$ comes from $v_3$ either by $R_4$ if $v_3$ is strong, or by $R_9$ if $v_3$ is not simplicial, or by $R_{10}$ (the same is true for $v_1$), hence $\mu'(v) \geq -1 + 2 \times \frac{3}{8} + 2 \times \frac{1}{8} = 0$.

If $9 \leq d(v_4) \leq 12$, then it suffices to observe that $v$ receives $\frac{1}{2}$ by $R_2a$ and does not give charge away by $R_7$.

If $9 \leq d(v_4) \leq 12$, then it suffices to observe that $v$ receives $\frac{1}{2}$ by $R_2a$ and does not give charge away by $R_7$.

If $9 \leq d(v_4) \leq 12$, then it suffices to observe that $v$ receives $\frac{1}{2}$ by $R_2a$ and does not give charge away by $R_7$.

Finally, if $d(v_4) \geq 15$ then $v$ receives $\frac{3}{8}$ by $R_2a$ and can give away $\frac{1}{8}$ to a poor vertex $v_3$ by $R_7b$ and also $\frac{1}{2}$ to a bad vertex $v_2$ by $R_{10}$. So again the balance of $v_3$ is at least $\frac{1}{2} = \frac{3}{8} - 2 \times \frac{1}{8}$, as desired.
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From now on suppose \( d(v_1) \geq 8 \) and \( d(v_3) \geq 8 \). If \( v \) is not simplicial, then \( v \) receives \( 2 \times \frac{1}{4} \) from \( v_1 \) and \( v_3 \) by R3 and at least \( \frac{1}{2} \) from an incident 4\(^+\)-face by R1. Thus we are done unless \( v \) gives \( \frac{1}{12} \) or \( \frac{1}{8} \) to at least one of \( v_4 \) and \( v_5 \) by R7 or R9, which can happen only if the face \( f = \cdots v_1 v_5 v_3 \) is a triangle. However, then \( v \) actually receives \( \frac{3}{4} \) by R1b at least once, and we have \( \mu'(v) \geq -1 + \frac{2}{3} + 2 \times \frac{1}{4} - 2 \times \frac{1}{2} = 0 \).

Finally, suppose \( v \) is simplicial. Now \( v \) does not give charge by R9. If \( v \) gives \( \frac{1}{8} \) or \( \frac{1}{12} \) to \( v_5 \) by R7, so that \( v \) is paired with a poor vertex \( v_5 \), then \( d(v_1) \geq 15 \) or \( d(v_1) \geq 13 \), respectively, due to the absence \((5, 5, 5, 8, 14)-\) and \((5, 5, 10, 12)-\)vertex by assumption. (Hereafter, we consider two possibilities in parallel, depending on whether \( v_5 \) has an 8-neighbor or a neighbor of degree 9 or 10.) Furthermore, \( v_4 \) is not strong, which implies that \( v_4 \) has a 5-neighbor different from \( v \) and \( v_5 \). In turn, this means that \( d(v_3) \geq 15 \) or \( d(v_3) \geq 13 \), respectively, since otherwise we would have a \((5, 5, 5, 8, 14)-\)vertex or \((5, 5, 10, 5, 12)-\)vertex, a contradiction.

Thus \( v \) receives from \( v_1 \) either \( \frac{5}{8} \) by R6 or \( \frac{7}{12} \) by R6ex2, respectively, and hence \( v_1 \) brings the total of \( \frac{1}{2} = \frac{5}{8} - \frac{1}{8} = \frac{7}{12} - \frac{1}{12} \) to \( v \). By symmetry, the same is true for \( v_3 \); no matter whether it is paired with \( v_4 \) or not, it brings \( \frac{1}{2} \) either by R6 or by R6ex2 combined with R7.

Thus we have \( \mu'(v) = -1 + 2 \times \frac{1}{2} = 0 \) when \( v \) gives away \( \frac{1}{8} \) or \( \frac{1}{12} \) at least once to a poor neighbor according to R7, so from now we can assume that \( v \) is not a donator of charge by R7.

We know that each 11\(^+\)-neighbor gives \( v \) at least \( \frac{1}{2} \) by R5c and R6, so it remains to assume that \( d(v_1) \leq 10 \), which means that \( v \) is poor.

First suppose \( d(v_1) = 8 \); then \( d(v_3) \geq 15 \) since we have no \((5, 5, 5, 8, 14)-\)vertex by assumption. No matter whether \( v_5 \) is strong or otherwise, our \( v \) receives \( \frac{5}{8} \) either from \( v_5 \) by R4 or from its paired vertex \( v_4 \) by R7a, respectively. Also, \( v \) receives \( \frac{1}{4} \) from \( v_1 \) by R5a and \( \frac{5}{8} \) from \( v_3 \) by R6a, so we have \( \mu'(v) = 0 \) in both options.

Now, if \( 9 \leq d(v_1) \leq 10 \) then \( d(v_3) \geq 13 \) due to the absence \((5, 5, 10, 5, 12)-\)vertex. Now if \( v_5 \) is strong, then \( v \) receives \( \frac{1}{6} \) from \( v_5 \) by R4, \( \frac{1}{3} \) from \( v_1 \) by R5b, and \( \frac{1}{2} \) from \( v_3 \) by R6a, so we have \( \mu'(v) = 0 \). Otherwise, \( v \) receives \( \frac{1}{12} \) from \( v_1 \) by R7b and \( \frac{1}{3} \) from \( v_1 \). Also, \( v \) receives from \( v_3 \) either \( \frac{7}{12} \) by R6ex1 if \( d(v_3) \leq 14 \) or \( \frac{5}{8} \) (which is greater than \( \frac{7}{12} \)) by R6 if \( d(v_3) \geq 15 \). This again makes \( \mu'(v) \geq 0 \), as desired.

Finally, if \( 11 \leq d(v_1) \leq 12 \) and \( 11 \leq d(v_3) \leq 12 \), then \( \mu'(v) = 0 \) by R5c.

**Case 2.** \( d(v) = 8 \). We can average donations of \( v \) to its 5-neighbors according to R2, R3, R5a, and R8 as follows. If \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) \geq 8 \), which is the situation of R2a, then \( v \) instead gives \( \frac{1}{2} \) to \( v_2 \) and \( \frac{1}{5} \) to \( v_3 \). Similarly, instead of giving \( \frac{1}{2} \) to a 5-neighbor \( v_2 \) by R2b, our \( v \) now gives \( \frac{1}{2} \) to \( v_2 \) and \( \frac{1}{5} \) to each of the 8\(^+\)-vertices \( v_1 \) and \( v_3 \). As a result, each neighbor receives at most
\[ \frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{3}{8} - \frac{1}{8} \] from \( v \) after averaging, so \( \mu'(v) \geq d(v) - 6 - \frac{d(v)}{4} = \frac{3(d(v)-8)}{4} \geq 0, \) as desired.

**Case 3.** \( 9 \leq d(v) \leq 10. \) We now average donations of \( v \) to its 5-neighbors according to R2, R3, R5b, and R8 in the same fashion. Instead of giving \( \frac{1}{2} \) to a 5-neighbor \( v_2 \) by R2b, our \( v \) gives \( \frac{1}{6} \) to each of the vertices \( v_1, v_2, \) and \( v_3. \) If \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) \geq 9, \) which happens in R2a, then \( v \) rather gives \( \frac{1}{3} \) to \( v_2 \) and \( \frac{1}{6} \) to \( v_3. \) As a result, each neighbor receives at most \( \frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{2} - \frac{1}{6} \) from \( v, \) so \( \mu'(v) \geq d(v) - 6 - \frac{d(v)}{6} = \frac{2(d(v)-9)}{3} \geq 0, \) and we are done.

**Case 4.** \( 11 \leq d(v) \leq 12. \) We note that \( v \) gives each neighbor at most \( \frac{1}{6} \) by R2, R3, R5c, and R8, so \( \mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2}, \) which settles the case \( d(v) = 12. \)

So suppose \( d(v) = 11. \) If \( v \) has an \( 8^+ \)-neighbor, then \( \mu'(v) \geq 11 - 6 - 10 \times \frac{1}{2} = 0. \) Thus we can assume that \( v \) is completely surrounded by 5-vertices. If \( v \) is incident with a \( 4^+ \)-face \( \cdots v_1v_2v_3, \) then each of \( v_1 \) and \( v_2 \) is non-simplicial and hence can only receive \( \frac{1}{4} \) from \( v \) by R3 or R8. Indeed, if the neighbors of \( v_1 \) in a cyclic order are \( \ldots, x_1, v, y_1, \) then \( d(x_1) = d(y_1) = 5 \) due to Remark 1, and the same argument works for \( v_2. \) This implies \( \mu'(v) \geq 5 - 2 \times \frac{1}{2} - (11-2) \times \frac{1}{2} = 0. \)

Therefore, it remains to assume in addition that \( v \) is simplicial. Now if there is a \( 4^+ \)-face \( \cdots v_1'v_2'v_2', \) then each of \( v_1 \) and \( v_2 \) receives at most \( \frac{1}{4} \) from \( v \); either by R3, which happens when \( v_1 \) has three 5-neighbors, or possibly by R8, otherwise. So again \( \mu'(v) \geq 0. \)

Thus we are done unless there are vertices \( w_1, \ldots, w_{11} \) lying in 3-faces \( w_kv_kv_{k+1} \) whenever \( 1 \leq k \leq 11 \) (addition mod 11 throughout proving Case 4). If so, then we cannot have \( d(w_k) \leq 8 \geq d(w_{k+1}) \) for any \( k, \) for otherwise \( w(S_5(v_{k+1})) \leq 3 \times 5 + 2 \times 8 + 11 = 42, \) which is impossible. By the oddness of 11, this implies that, say, \( d(w_1) \geq 9 \) and \( d(w_2) \geq 9. \) It follows from Remark 1 that there is a 3-face \( w_1w_2w_2, \) and it suffices to observe that \( v \) gives no charge to \( v_2 \) by R8 or any other our rule. Hence we have \( \mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0. \)

**Case 5.** \( 13 \leq d(v) \leq 14. \) We know that \( v \) gives at most \( \frac{7}{12} \) to each adjacent 5-vertex by R1–R8. Since \( \mu(v) = d(v) - 6 - \frac{7(d(v)}{12} = \frac{5d(v)-72}{12}, \) it follows that \( \mu'(v) \geq -\frac{7}{12} \text{ for } d(v) = 14, \) and \( \mu'(v) \geq -\frac{7}{12} \text{ for } d(v) = 13. \) Therefore, we use some additional reasons to improve these rough estimations in order to prove \( \mu'(v) \geq 0. \)

First of all, we can assume that \( v \) is completely surrounded by 5-vertices, for otherwise \( \mu'(v) \geq d(v) - 6 - \frac{7(d(v)-1)}{12} = \frac{5(d(v)-13)}{12} \geq 0, \) as desired.

Secondly, if \( v \) is not simplicial then \( v \) gives at most \( \frac{1}{4} \) to each of at least two vertices incident with a common \( 4^+ \)-face with \( v \) due to the argument used in Case 4, which means that in fact \( \mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v)-2)}{12} \geq \frac{5(d(v)-13)}{12} + \frac{1}{12} > 0. \)
Thus we are done unless \( v \) is simplicial and completely surrounded by 5-vertices. Furthermore, if there is a 4\(^{+}\)-face \( \cdots v'_{1}v_{1}v_{2}v'_{2} \), then we similarly have \( \mu'(v) \geq \frac{1}{12} \).

So again there is a cyclic sequence \( W_{d(v)} = w_{1}, \ldots, w_{d(v)} \) such that there are 3-faces \( w_{k}v_{k}v_{k+1} \) whenever \( 1 \leq k \leq d(v) \) (addition mod \( d(v) \)). As before, there are no two consecutive 5-vertices in \( W_{d(v)} \) since each \( v_{k} \) must have an 8\(^{+}\)-neighbor other than \( v \).

If there is an 8-vertex in \( W_{d(v)} \), say \( w_{2} \), then \( d(w_{1}) \geq 8 \) and \( d(w_{3}) \geq 8 \), since \( 43 - 3 \times 5 - 13 - 8 = 7 \). Thus, in fact each of \( v_{2} \) and \( v_{3} \) receives at most \( \frac{1}{4} \) from \( v \) by R3, R8 rather than \( \frac{7}{12} \), and we again have \( \mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v) - 2)}{12} > 0 \), as above. In what follows, we can assume that \( d(w_{i}) \geq 9 \) or \( d(w_{i}) = 5 \) whenever \( 1 \leq k \leq d(v) \).

If there are two consecutive 9\(^{+}\)-vertices in \( W_{d(v)} \), say \( w_{1} \) and \( w_{2} \), then \( v_{2} \) receives no charge from \( v \) by R1–R8, so we can improve our rough estimation \( \mu'(v) \geq -\frac{2}{12} + \frac{7}{12} \geq 0 \), as desired. This completes the proof for \( d(v) = 13 \) due to the oddness of 13.

So suppose \( d(v) = 14 \), all neighbors of \( v \) are simplicial, and \( d(w_{1}) = d(w_{3}) = \cdots = d(w_{13}) = 5 \), for otherwise \( v \) gives at most \( \frac{1}{4} \) to one of its neighbors, and we already have \( \mu'(v) \geq -\frac{2}{12} + \frac{7}{12} - \frac{1}{4} > 0 \).

Now if at least one of 5-vertices in \( W_{14} \), say \( w_{1} \), is strong, that is \( w_{1} \) has an 8\(^{+}\)-neighbor outside \( W_{14} \), then each of \( v_{1} \) and \( v_{2} \) receives \( \frac{1}{2} \) by R6a rather than \( \frac{7}{12} \) by R6ex1 or R6ex2, which yields \( \mu'(v) \geq 8 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0 \).

Thus we can assume that all \( w_{1}, w_{3}, \ldots, w_{13} \) are non-strong, that is each of them has a 5-neighbor outside \( W_{14} \). Among the seven 9\(^{+}\)-vertices \( w_{2}, w_{4}, \ldots, w_{14} \), there are no two consecutive (cyclically) 10\(^{-}\)-vertices, for otherwise we would have a minor 5-star with weight at most 40, which is impossible.

By parity reasons and symmetry, we can assume that \( d(w_{14}) \geq 11 \) and \( d(w_{2}) \geq 11 \). So each of \( v_{1} \) and \( v_{2} \) obeys the general rule R6 rather than its exceptions R6ex1 or R6ex2. This means that again \( \mu'(v) \geq 14 - 6 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0 \), as desired.

**Case 6.** \( d(v) \geq 15 \). We know that \( v \) gives at most \( \frac{5}{8} \) to each adjacent 5-vertex by R1–R8, except for giving \( \frac{3}{4} \) in R2a.

We now average these donations so that each 8\(^{+}\)-neighbor will receive at most \( 2 \times \frac{1}{4} \) from \( v \), while each 5-neighbor will receive at most \( \frac{5}{8} \). To this end, it suffices to switch \( \frac{1}{8} \) from the donation of \( \frac{3}{4} \) to a 5-vertex \( v_{2} \) by R2a to the neighbor 8\(^{+}\)-vertex \( v_{1} \).

Since \( \mu(v) = d(v) - \frac{5d(v)}{8} = \frac{3(d(v) - 16)}{8} \), it follows that our averaging results in \( \mu'(v) \geq 0 \) for all \( d(v) \geq 16 \).

Finally, suppose \( d(v) = 15 \). If \( v \) has an 8\(^{+}\)-neighbor or a non-simplicial 5-neighbor, then \( \mu'(v) \geq 15 - 6 - \frac{1}{4} - 14 \times \frac{5}{8} = 0 \) by R1–R8.
Thus we can assume that $v$ is completely surrounded by simplicial 5-vertices, which means that the sequence $W_{15}$ introduced in Case 5 is actually a 15-cycle. Again, $W_{15}$ has no two consecutive 5-vertices, which implies by parity reasons and symmetry that $d(w_1) \geq 8$ and $d(w_2) \geq 8$. Since $v_2$ receives $\frac{1}{4}$ from $v$ by R8 and nothing by any other our rule, we are done.

Thus we have proved $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 3.

References


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