TREES WHOSE EVEN-DEGREE VERTICES INDUCE A PATH ARE ANTIMAGIC

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Abstract
An antimagic labeling of a connected graph $G$ is a bijection from the set of edges $E(G)$ to $\{1, 2, \ldots, |E(G)|\}$ such that all vertex sums are pairwise distinct, where the vertex sum at vertex $v$ is the sum of the labels assigned to edges incident to $v$. A graph is called antimagic if it has an antimagic labeling. In 1990, Hartsfield and Ringel conjectured that every simple connected graph other than $K_2$ is antimagic; however the conjecture remains open, even for trees. In this note we prove that trees whose vertices of even degree induce a path are antimagic, extending a result given by Liang, Wong, and Zhu [Anti-magic labeling of trees, Discrete Math. 331 (2014) 9–14].

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1. Introduction

All graphs considered in this work are finite, undirected and simple. Given a graph \( G = (V(G), E(G)) \) and a vertex \( v \in V(G) \), we denote by \( E_G(v) \) the set of edges incident to \( v \) and by \( d_G(v) = |E_G(v)| \) the degree of \( v \) in \( G \). We will just write \( E(v) \) and \( d(v) \) when \( G \) is clear from the context. A tree is a connected and acyclic graph, and a forest is a disjoint union of trees. Undefined terms in this work can be found in [2].

An (edge) labeling of a graph \( G \) is a mapping from \( E(G) \) to the set of non-negative integers. A labeling \( \phi \) of a connected graph \( G \) is called antimagic if it is a bijection \( \phi : E(G) \to \{1, 2, \ldots, |E(G)|\} \) such that all vertex sums are pairwise distinct, where the vertex sum \( s(v) \) at vertex \( v \in V(G) \) is defined as \( s(v) = \sum_{e \in E(v)} \phi(e) \). A graph is called antimagic if it has an antimagic labeling.

In 1990, Hartsfield and Ringel [5] conjectured that every simple connected graph other than \( K_2 \) is antimagic. The conjecture has received much attention (see [4]); but it is widely open in general, even for trees. Nevertheless, several classes of trees are known to be antimagic (see [1,3,5–10]).

Given a tree \( T \), \( V_{\text{even}}(T) \) (respectively, \( V_{\text{odd}}(T) \)) denotes the set of even (respectively, odd) degree vertices of \( T \). Regarding trees such that \( V_{\text{even}} \) induces a path, Liang, Wong, and Zhu [7] proved the following two theorems.

**Theorem 1** [7]. If \( T \) is a tree such that \( V_{\text{even}}(T) \) induces a path and \( |V_{\text{even}}(T)| \) is odd, then \( T \) is antimagic.

**Theorem 2** [7]. Let \( T \) be a tree such that \( V_{\text{even}}(T) \) induces a path of order \( 2p \), \((v_1, \ldots, v_{2p}) \). Let \( v_0 \) (respectively, \( v_{2p+1} \)) be a neighbor of \( v_1 \) (respectively, \( v_{2p} \)) different from \( v_2 \) (respectively, \( v_{2p-1} \)). If \( d(v_p) \neq d(v_{2p+1}) + 1 \) or \( d(v_{p+1}) \neq d(v_0) + 1 \), then \( T \) is antimagic.

The aim of this note is to extend Theorem 2 to all cases, that is, to prove the antimagicness of trees such that \( V_{\text{even}}(T) \) induces a path whenever \( |V_{\text{even}}(T)| \) is even, obtaining as a consequence that trees whose even-degree vertices induce a path are antimagic.

2. Constructing an Antimagic Labeling

In the proof of the next theorem we follow and extend the main idea developed by Liang, Wong, and Zhu in [7]. We denote by \([a, b]\) the set of consecutive integers \( \{a, a+1, \ldots, b\} \), where \( a \leq b \).

**Theorem 3.** If \( T \) is a tree such that \( V_{\text{even}}(T) \) induces a path and \( |V_{\text{even}}(T)| \) is even, then \( T \) is antimagic.
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**Proof.** Let $|V_{\text{even}}(T)| = 2p$, it is known that trees without vertices of degree 2 are antimagic [6,7], hence we may assume $p \geq 1$. Let $P = (v_0, v_1, v_2, \ldots, v_{2p}, v_{2p+1})$ be a path induced by $V_{\text{even}}(T)$ extended with endpoints in $V_{\text{odd}}(T)$, that is, $V_{\text{even}}(T) = \{v_1, \ldots, v_{2p}\}$ and $\{v_0, v_{2p+1}\} \subseteq V_{\text{odd}}(T)$. For every $1 \leq i \leq 2p + 1$, we denote by $e_i$ the edge $v_{i-1}v_i$.

We prove the theorem by constructing an antimagic labeling $\phi$ of $T$ in two steps. The first step produces a labeling of a subtree of $T$ containing the path $P$ and satisfying a particular additional condition. This labeling will be extended to an antimagic labeling of $T$ at the second step.

Let $m = |E(T)|$. We will use the residues modulo $m + 2$ to compare vertex sums: since vertex sums are distinct if they are distinct modulo $m + 2$, it is enough to compare vertex sums whenever they are equal modulo $m + 2$ in order to check that they all are pairwise distinct.

For each (not necessarily connected) subgraph $T'$ of $T$, we set $L_\phi(T') = \{\phi(e) : e \in E(T')\}$ and $s_{T'}(v) = \sum_{e \in E_{T'}(v)} \phi(e)$ for every $v \in V(T')$ such that $d_{T'}(v) \geq 1$. Obviously, if $T' = T$, then $s_{T'} = s$. The set of all vertex sums modulo $m + 2$ in $T'$ will be denoted by $R_{m+2}(T')$, that is,

$$R_{m+2}(T') = \{s_{T'}(v) \mod m+2 : v \in V(T') \text{ and } d_{T'}(v) \geq 1\} \subseteq \{0, 1, \ldots, m+1\}.$$

**Step I.** The labeling of the tree $T_1$ constructed at this step will satisfy the following condition: all vertex sums in $T_1$ will be pairwise distinct modulo $m + 2$ with at most one exception; moreover, if the vertex sums are equal modulo $m + 2$ for a pair of vertices, then exactly one of them will be a leaf in $T$, and the vertex sums in $T_1$ for both vertices in the pair will be different.

As a starting point, let $T_1 := P$ and define

$$\phi(e_{2i+1}) := i + 1, \quad \text{for } 0 \leq i \leq p;$$

$$\phi(e_{2i}) := m - p + i, \quad \text{for } 1 \leq i \leq p.$$ 

Hence,

$$L_\phi(T_1) = [1, p + 1] \cup [m - p + 1, m].$$

Moreover, $s_{T_1}(v_0) = 1$, $s_{T_1}(v_{2p+1}) = p + 1$, and for $1 \leq i \leq 2p$, $s_{T_1}(v_i) = m - p + i + 1$. Next, we calculate the set $R_{m+2}(T_1)$ according to the values of $p$ and to the degrees of $v_0$ and $v_{2p+1}$.

**Case 1.** $p = 1$. In this case, $|V(T_1)| = 4$ and

$$R_{m+2}(T_1) = \{0, 1, 2, m + 1\}.$$

Hence, vertex sums at the vertices of $T_1$ are distinct.
**Case 2.** \( p > 1 \). In such a case,

\[
R_{m+2}(T_1) = \left( [0, p + 1] \setminus \{ p \} \right) \cup [m - p + 2, m + 1],
\]

and only the residues of vertex sums at vertices \( v_0 \) and \( v_{p+2} \) are equal. In fact, we have that \( s_{T_1}(v_0) \equiv 1 \equiv s_{T_1}(v_{p+2}) \pmod{m + 2} \) (see an example in Figures 1(a) and 2(a)).

![Figure 1. Labeling of \( T_1 \) for \( p = 5 \) and \( m = 21 \); (a) before the swaps, and (b) after the swaps. The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label.](image1.png)

![Figure 2. Labeling of \( T_1 \) for \( p = 4 \) and \( m = 21 \); (a) before the swap, and (b) after the swap. The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label.](image2.png)

Now we distinguish two subcases.

**Subcase 2.1.** At least one of the vertices \( v_0 \) or \( v_{2p+1} \) is a leaf in \( T \). Notice that by properly relabeling the vertices of \( T_1 \), we may assume \( d_T(v_0) = 1 \). Then, \( s_{T_1}(v_0) \equiv 1 \equiv s_{T_1}(v_{p+2}) \pmod{m + 2} \), but \( s_{T_1}(v_0) = 1 < m + 3 = s_{T_1}(v_{p+2}) \).

**Subcase 2.2.** Neither \( v_0 \) nor \( v_{2p+1} \) are leaves in \( T \). In this case, for \( 1 \leq i \leq \left\lfloor \frac{p-1}{2} \right\rfloor \), we swap the labels of the edges \( e_{2i-1} \) and \( e_{2i} \), that is,

\[
\phi(e_{2i-1}) := m - p + i, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{p-1}{2} \right\rfloor;
\]

\[
\phi(e_{2i}) := i, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{p-1}{2} \right\rfloor.
\]

Notice that the endpoints of the subpath of \( T_1 \) involved in the swaps are \( v_0 \) and \( v_p \), where \( k = p - 2 \), if \( p \) is even; and \( k = p - 1 \), if \( p \) is odd. After the swaps, \( s_{T_1}(v_0) = m - p + 1 \); also it can be easily checked that \( s_{T_1}(v_k) = p - 1 \), if \( p \) is even; and \( s_{T_1}(v_k) = p \), if \( p \) is odd; and the vertex sum at any other vertex in \( T_1 \) remains unchanged. We distinguish cases depending on the parity of \( p \).

**Subcase 2.2.a.** \( p \) odd. In this case we have that \( s_{T_1}(v_k) = p \), implying that the residues modulo \( m + 2 \) at the vertices of \( T_1 \) are pairwise distinct. Concretely,

\[
R_{m+2}(T_1) = [0, p + 1] \cup \left( [m - p + 1, m + 1] \setminus \{ m \} \right).
\]
(b) \( p \) even. In this case, even if \( p = 2 \) (in which case no swaps take place), we have that \( s_{T_1}(v_k) = p - 1 \equiv s_{T_1}(v_{2p}) \) (mod \( m + 2 \)), and thus

\[
R_{m+2}(T_1) = \left( [0, p+1] \setminus \{p\} \right) \cup \left( [m-p+1, m+1] \setminus \{m-1\} \right).
\]

Notice that only \( s_{T_1}(v_k) \) and \( s_{T_1}(v_{2p}) \) have the same residue in \( T_1 \) (see an example in Figure 2(b)). Now, let \( x_0 = v_{2p+1} \) and let \( P' = (x_0, x_1, \ldots, x_{\ell}) \) be a maximal subpath of \( T \) starting at \( x_0 \) and with vertices in \( V_{\text{odd}}(T) \). Observe that, in such a case, \( x_i \) is a leaf in \( T \) and there exist vertices \( y_0, \ldots, y_{\ell-1} \in V_{\text{odd}}(T) \) such that \( x_i y_i \in E(T) \) (see an example in Figure 3). We update \( T_1 \) as the tree induced by the set of vertices of the paths \( P \) and \( P' \), and \( \{y_0, \ldots, y_{\ell-1}\} \):

\[
T_1 := T[\{v_0, \ldots, v_{2p}\} \cup \{x_0, \ldots, x_{\ell}\} \cup \{y_0, \ldots, y_{\ell-1}\}].
\]

We define the labels of the new edges and update the label of the edge \( e_{2p+1} \) as follows:

\[
\phi(e_{2p+1}) := p + \ell + 1;
\]

\[
\phi(x_i x_{i+1}) := p + i + 1, \quad \text{for } 0 \leq i \leq \ell - 1;
\]

\[
\phi(x_i y_i) := m - p - i, \quad \text{for } 0 \leq i \leq \ell - 1.
\]

Thus, we have

\[
L_{\phi}(T_1) = [1, p + \ell + 1] \cup [m - p - \ell + 1, m],
\]

and

\[
s_{T_1}(x_i) = m + p + i + 1 \equiv p + i - 1 \pmod{m + 2}, \text{ for } 1 \leq i \leq \ell - 1,
\]

\[
s_{T_1}(v_{2p}) = m + p + \ell + 1 \equiv p + \ell - 1 \pmod{m + 2},
\]

\[
s_{T_1}(x_0) = m + p + \ell + 2 \equiv p + \ell \pmod{m + 2},
\]

\[
s_{T_1}(x_i) = p + \ell \equiv p + \ell \pmod{m + 2},
\]

\[
s_{T_1}(y_i) = m - p - i \equiv m - p - i \pmod{m + 2}, \text{ for } 0 \leq i \leq \ell - 1.
\]

Therefore, taking into account equality (5):

\[
R_{m+2}(T_1) = [0, p + \ell] \cup \left( [m - p - \ell + 1, m+1] \setminus \{m-1\} \right),
\]
where \( m \geq 2p + 2\ell + 1 \). So, only \( s_{T_1}(x_0) \) and \( s_{T_1}(x_\ell) \) have the same residue, concretely \( p + \ell \). However, \( s_{T_1}(x_0) = m + p + \ell + 2 > p + \ell = s_{T_2}(x_\ell) \), and hence all vertex sums in \( T_1 \) are different (see an example in Figure 3).

Notice that, in each of the above cases, \( |L_\phi(T_1)| = |E(T_1)| \). Hence \( \phi \), restricted to \( E(T_1) \), is a bijection from \( E(T_1) \) to \( L_\phi(T_1) \).

**Step II.** Now, let \( T_2 \) be the forest obtained by removing all the edges of \( T_1 \). Each component of \( T_2 \) has exactly one vertex in \( T_1 \). Therefore, if \( T_2(v) \) denotes the component of \( T_2 \) containing \( v \),

\[
T_2 := T - E(T_1) = \bigcup_{v \in V(T_1)} T_2(v).
\]

Clearly, \( T_2(v) \) can be viewed as a directed rooted tree with root at \( v \), where every edge is directed away from the root. Moreover, since every vertex of \( T_2(v) \) different from \( v \) has odd degree in \( T \), each vertex in \( T_2(v) \) has an even number of children in this rooted tree and, therefore, \( |E(T_2(v))| \) is even for every \( v \in V(T_1) \).

Hence, \( |E(T_2)| \) is even. If we set \( \ell = 0 \) whenever \( T_1 = P \), then, by equalities (1) and (6), the available labels for the edges of \( T_2 \) are

\[
L_\phi(T_2) := [p + \ell + 2, m - p - \ell].
\]

As \( L_\phi(T_2) = [a, b] \), where \( a + b = m + 2 \), and each \( w \in V(T_2) \) has an even number of children, we can label the edges of \( T_2 \) with integers in \( L_\phi(T_2) \) fulfilling the following additional condition: if a vertex \( w \) has an outgoing edge with label \( t \) in the corresponding rooted tree of \( T_2 \), then \( w \) has another outgoing edge with label \( m + 2 - t \) (see an example in Figure 4).

![Figure 4. An antimagic labeling of a tree with \( m = 21 \). Thicker edges correspond to the forest \( T_2 \) and are labeled in Step II (in this example, the forest \( T_2 \) has two nontrivial components). The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label, but different vertex sums.](image)

Clearly, by the previous discussion, the labeling \( \phi \) already constructed is a bijection from \( E(T) \) to \( [1, m] \). Finally, we just need to show that the vertex sums defined by \( \phi \) in \( T \) are pairwise distinct.

Observe that in \( T_3 \), the sum of the labels of the outgoing edges of \( v \) is a multiple of \( m + 2 \). Thus, the following two conditions hold.
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1. For every $v \in V(T_1)$, $s(v) \equiv s_{T_1}(v) \pmod{m+2}$.

2. For every $v \in V(T) \setminus V(T_1)$, $s(v) \equiv \phi(f) \pmod{m+2}$, where $f$ is the incoming edge of $v$ in $T_2$.

Let $u, v \in V(T)$. We consider the following cases.

(a) $u, v \in V(T) \setminus V(T_1)$. Since $\phi : E(T) \to [1, m]$ is a bijection, condition 2 implies that $s(u) \neq s(v)$.

(b) $u, v \in V(T_1)$. If $s_{T_1}(u) \neq s_{T_1}(v) \pmod{m+2}$, then by condition 1 we have that $s(u) \neq s(v)$. Otherwise, as we have seen in Cases 2.1 and 2.2(b) of Step I, just one of these vertices, say $u$, is a leaf in $T$ and $s_{T_1}(u) < s_{T_1}(v)$. Therefore, we have that $s(u) = s_{T_1}(u) < s_{T_1}(v) \leq s(v)$, as we wanted to prove.

(c) One of the vertices belongs to $V(T_1)$ and the other to $V(T) \setminus V(T_1)$. We can assume without loss of generality that $u \in V(T_1)$ and $v \in V(T) \setminus V(T_1)$. By condition 1, $s(u) \pmod{m+2} \in R_{m+2}(T_1)$. Moreover, by condition 2, $s(v) \pmod{m+2} \in L_{\phi}(T_2)$. By equalities (2), (3), (4), (7), and (8), we have $R_{m+2}(T_1) \cap L_{\phi}(T_2) = \emptyset$. Hence, $s(u) \neq s(v)$.

Thus, the theorem holds.

The next result follows from Theorems 1 and 3.

**Corollary 4.** If $T$ is a tree such that $V_{\text{even}}(T)$ induces a path, then $T$ is antimagic.

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