ORIENTED CHROMATIC NUMBER OF CARTESIAN PRODUCTS $P_M \square P_N$ AND $C_M \square P_N$

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Abstract

We consider oriented chromatic number of Cartesian products of two paths $P_m \square P_n$ and of Cartesian products of paths and cycles, $C_m \square P_n$. We say that the oriented graph $\overrightarrow{G}$ is colored by an oriented graph $\overrightarrow{H}$ if there is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. In this paper we show that there exists an oriented tournament $\overrightarrow{H}_{10}$ with ten vertices which colors every orientation of $P_8 \square P_n$ and every orientation of $C_m \square P_n$, for $m = 3, 4, 5, 6, 7$ and $n \geq 1$.

We also show that there exists an oriented graph $\overrightarrow{T}_{16}$ with sixteen vertices which colors every orientation of $C_m \square P_n$.

Keywords: graphs, oriented coloring, oriented chromatic number.

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1. Introduction

An oriented graph is a digraph $\overrightarrow{G}$ obtained from an undirected graph $G$ by assigning to each edge one of two possible directions. We say that $\overrightarrow{G}$ is an orientation of $G$ and $G$ is the underlying graph of $\overrightarrow{G}$. A tournament $\overrightarrow{T}$ is an orientation of a complete graph. If there is a homomorphism $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{T})$, then we say that $\overrightarrow{G}$ is colored by $\overrightarrow{T}$ or that $\overrightarrow{T}$ colors $\overrightarrow{G}$. We also say that $\overrightarrow{T}$ is a coloring graph (tournament). The oriented chromatic number of the oriented graph $\overrightarrow{G}$, denoted by $\chi(\overrightarrow{G})$, is the smallest integer $k$ such that $\overrightarrow{G}$ is colored by a tournament with $k$ colors (vertices). The oriented chromatic number $\chi(G)$ of an undirected graph $G$ is the maximal chromatic number over all possible orientations of $G$. The oriented chromatic number of a family of
graphs is the maximal oriented chromatic number over all possible graphs of the family. The upper oriented chromatic number $\chi^+(G)$ of an undirected graph $G$ is the minimum order of an oriented graph $\overrightarrow{H}$ such that every orientation $\overrightarrow{G}$ of $G$ admits a homomorphism to $\overrightarrow{H}$.

It is easy to see that for every undirected graph $G$, $\chi(G) \leq \chi^+(G) \leq \chi^+(G)$, see [19]. The Cartesian product $G \square H$ of two undirected graphs $G$ and $H$ is the graph with the vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. We use $P_k$ to denote the path on $k$ vertices. Sopena [19] considered upper oriented chromatic number of strong, Cartesian and direct products of graphs.

**Theorem 1** [19]. If $G$ and $H$ are two undirected graphs, then $\chi^+(G \square H) \leq \chi^+(G) \cdot \chi^+(H) \cdot \min\{\chi(G), \chi(H)\}$.

Oriented coloring has been studied in recent years [1, 2, 6, 8–10, 12, 14, 16–20, 22], see [15] for a survey of the main results. Several authors established or bounded chromatic numbers for some families of graphs, such as oriented planar graphs [12,14], outerplanar graphs [12,17,18], graphs with bounded degree three [10,17,20], $k$-trees [17], Halin graphs [5,9], graphs with given excess [8] or grids [3,4,6,13,22].

In this paper we focus on the oriented chromatic number of Cartesian products of two paths, called 2-dimensional grids $G_{m,n} = P_m \square P_n$, and Cartesian products of cycles and paths, called stacked prism graphs $Y_{m,n} = C_m \square P_n$.

**Theorem 2** [16,21]. Let $G$ be an undirected graph. Then:

(a) If $G$ is a forest with at least three vertices, then $\chi^+(G) = 3$.

(b) $\chi^+(C_5) = 5$. Moreover, every orientation of $C_5$ can be colored by $\overrightarrow{H}_2$ (see Figure 1(b)).

(c) For each $k \leq 3$, $k \neq 5$, we have $\chi^+(C_k) = 4$. Moreover, every orientation of a cycle $C_k$ with $k \leq 3$ and $k \neq 5$ can be colored by $\overrightarrow{H}_1$ (see Figure 1(a)).

Theorems 1 and 2 imply that $\chi^+(P_m \square P_n) \leq 3 \cdot 3 \cdot 2 = 18$. Furthermore, we know that

- $\chi^+(P_m \square P_n) \leq 11$, for every $m,n \geq 2$ [6],
- there exists an orientation of $P_4 \square P_5$ which requires 7 colors for oriented coloring [6],
- there exists an orientation of $P_1 \square P_{212}$ which requires 8 colors for oriented coloring [3],
- $\chi^+(P_2 \square P_2) = 4$, $\chi^+(P_2 \square P_3) = 5$ and $\chi^+(P_2 \square P_n) = 6$, for $n \geq 6$ [6],
- $\chi^+(P_3 \square P_n) = 6$, for every $3 \leq n \leq 6$, and $\chi^+(P_3 \square P_n) = 7$, for every $n \geq 7$ [6,22],
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Figure 1. Coloring graphs $\vec{H}_1$ and $\vec{H}_2$.

- $\chi'(P_4 \square P_4) = 6$ and $\chi'(P_4 \square P_n) = 7$, for every $n \geq 5$ [6, 22],
- $\chi'(P_3 \square P_n) \leq 9$, for every $n \geq 5$ [4].

Since $\chi'(C_5) = 5$ and $\chi'(C_k) \leq 4$, for $k \neq 5$, by Theorem 1, we have
- $\chi'(C_5 \square P_n) \leq 2 \cdot 3 \cdot 5 = 30$, for $n \geq 3$,
- $\chi'(C_m \square P_n) \leq 2 \cdot 3 \cdot 4 = 24$, for $m \neq 5$, $n \geq 3$.

In this paper we show that there exists an oriented tournament $\vec{H}_{10}$, see Figure 2, which colors every orientation of every grid $P_8 \square P_n$ and every orientation of $C_m \square P_n$, with $m = 3, 4, 5, 6, 7$ and $n \geq 1$. We also show that there exists an oriented graph $\vec{T}_{16}$ which colors every orientation of $C_m \square P_n$, for $m \geq 8$ and $n \geq 1$. These imply that
- $\chi'(P_8 \square P_n) \leq \chi'(P_8 \square P_n) \leq 10$, for every $n$,
- $\chi'(C_m \square P_n) \leq \chi'(C_m \square P_n) \leq 10$, for $m = 3, 4, 5, 6, 7$ and $n \geq 1$,
- $\chi'(C_m \square P_n) \leq \chi'(C_m \square P_n) \leq 16$, for $m \geq 8$ and $n \geq 1$.

2. Coloring Graphs

2.1. Paley tournament

Let $p$ be a prime number such that $p \equiv 3 \mod 4$, and let $Z_p = \{0, \ldots, p - 1\}$ be the ring of integers modulo $p$. We denote by $QR_p = \{r : r \neq 0, r = s^2, \text{for some } s \in Z_p\}$ — the set of nonzero quadratic residues of $Z_p$. All arithmetic operation in this section are made in the ring $Z_p$.

Definition 3. The directed graph $\vec{T}_p$ with the set of vertices $V(\vec{T}_p) = Z_p$ and the set of arcs $A(\vec{T}_p) = \{(x, y) : x, y \in V(\vec{T}_p) \text{ and } y - x \in QR_p\}$ is called the Paley tournament of order $p$. Observe that $\vec{T}_p$ is a tournament.
Lemma 4. If \( a \in \mathbb{Q}R_p \) and \( b \in \mathbb{Z}_p \), then the mapping \( f : \overrightarrow{T_p} \rightarrow \overrightarrow{T_p} \) defined by \( f(x) = a \cdot x + b \) is an automorphism.

Lemma 5 [7]. The Paley tournament \( \overrightarrow{T_p} \) is arc-transitive; i.e., for any two pairs of arcs \((u, v), (x, y) \in A(\overrightarrow{T_p})\), there exists an automorphism \( h \) such that \( h(u) = x \) and \( h(v) = y \).

Lemma 6. The Paley tournament \( \overrightarrow{T_p} \) is self-converse; i.e., \( \overrightarrow{T_p} \) and its converse \( \overrightarrow{T_p^R} \) are isomorphic.

Proof. Consider the function \( f : \overrightarrow{T_p^R} \rightarrow \overrightarrow{T_p} \) defined by \( f(x) = -x \). Then \((x, y) \in A(\overrightarrow{T_p^R})\) if and only if \((-x, -y) \in A(\overrightarrow{T_p})\).

2.2. Coloring graph \( \overrightarrow{H_{10}} \)

Consider the coloring graph \( \overrightarrow{H_{10}} \) obtained from the Paley tournament \( \overrightarrow{T_{11}} \) by removing the vertex 0, i.e., \( V(\overrightarrow{H_{10}}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) and \((u, v) \in A(\overrightarrow{H_{10}})\) if \((v - u) \in \{1, 3, 4, 5, 9\}\), see Figure 2.

![Figure 2. Coloring graph \( \overrightarrow{H_{10}} \).](image-url)
Lemma 7. (a) For every \( a \in \{1, 3, 4, 5, 9\} \), the function \( h_a(x) = ax \) (mod 11) is an automorphism of \( \overrightarrow{H}_{10} \).
(b) For every \( x \in \{1, 3, 4, 5, 9\} \) there is an automorphism \( h_a \) such that \( h_a(x) = 1 \).
(c) For every \( x \in \{2, 6, 7, 8, 10\} \) there is an automorphism \( h_a \) such that \( h_a(x) = 10 \).

Lemma 8. Let \( \overrightarrow{G} \) be an orientation of a grid and let \( v \) be one of its vertices. Then the following two statements are equivalent.
(a) There exists an oriented coloring (homomorphism) \( c : \overrightarrow{G} \to \overrightarrow{H}_{10} \).
(b) There exists an oriented coloring (homomorphism) \( c' : \overrightarrow{G} \to \overrightarrow{H}_{10} \) such that \( c'(v) \in \{1, 10\} \).

2.3. Tromp graph

Definition 9. Let \( \overrightarrow{G} \) be an oriented graph. We build the Tromp graph \( \overrightarrow{T}(\overrightarrow{G}) \) in the following way.
- Let \( \overrightarrow{G}' \) be an isomorphic copy of \( \overrightarrow{G} \).
- \( \infty, \infty' \) be two additional vertices.
- Let \( t : V(\overrightarrow{G}) \cup \{\infty\} \to V(\overrightarrow{G}') \cup \{\infty'\} \) be an isomorphism with \( t(\infty) = \infty' \). For every \( u \in V(\overrightarrow{G}) \cup \{\infty\} \) by \( u' \) we denote \( t(u) \) and for every \( u \in V(\overrightarrow{G}') \cup \{\infty'\} \) by \( u' \) we denote \( t^{-1}(u) \). The pair \( (u, u') \) is called a pair of twin vertices.
- The set of vertices \( V(\overrightarrow{T}(\overrightarrow{G})) = V(\overrightarrow{G}) \cup V(\overrightarrow{G}') \cup \{\infty, \infty'\} \).
- The set of arcs is defined by
  \[ \forall u \in V(\overrightarrow{G}), (u, \infty), (\infty, u'), (u', \infty'), (\infty', u) \in A(\overrightarrow{T}(\overrightarrow{G})), \]
  \[ \forall u, v \in V(\overrightarrow{G}), (u, v), (u', v'), (v, u'), (v', u) \in A(\overrightarrow{T}(\overrightarrow{G})). \]

Let \( \overrightarrow{T}_{16} = \overrightarrow{T}(\overrightarrow{T}_{7}) \) be the Tromp graph on sixteen vertices obtained from the Paley tournament \( \overrightarrow{T}_{7} \), see Figure 3.

Suppose that \( i \) and \( j \) are integers such that \( i \geq 1 \) and \( j \geq 1 \). Consider the star \( K_{1,i} \) with the set of vertices \( V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_i\} \) and edges of the form \( \{x, v_k\} \) for \( 1 \leq k \leq i \); and a Tromp graph \( \overrightarrow{T}(\overrightarrow{G}) \). Let \( \overrightarrow{K} \) be an orientation of the star \( K_{1,i} \) and \( c : \overrightarrow{K} \to \overrightarrow{T}(\overrightarrow{G}) \) be a homomorphism. We say that the sequence of colors \( (c(v_1), c(v_2), \ldots, c(v_i)) \) chosen for leaves of the star is compatible with orientation \( \overrightarrow{K} \) if for every pair of vertices \( v_k, v_l \) it holds:
- \( c(v_k) \neq c(v_l) \) if \( (v_k, x) \) and \( (x, v_l) \in \overrightarrow{K} \) or if \( (v_l, x) \) and \( (x, v_k) \in \overrightarrow{K} \), and
- \( c(v_k) \neq c(v_l)' \) if \( (v_k, x) \) and \( (x, v_l) \in \overrightarrow{K} \) or if \( (x, v_l) \) and \( (x, v_k) \in \overrightarrow{K} \).
Definition 10. We say that the Tromp graph $\overrightarrow{T}$ has the property $P_{c}(i, j)$ if $|V(\overrightarrow{T})| \geq i$ and for every orientation $\overrightarrow{K}$ of the star $K_{1,i}$ and every sequence of colors $(c(v_1), c(v_2), \ldots, c(v_k))$ chosen for leaves compatible with $\overrightarrow{K}$ we can choose $j$ different ways to color $x$, the central vertex of the star.

Lemma 11 [11]. The Tromp graph $\overrightarrow{T_{16}}$ has the properties $P_{c}(1, 7)$, $P_{c}(2, 3)$ and $P_{c}(3, 1)$.

3. Grids $G_{8,n} = P_8 \Box P_n$

Definition 12. The comb $R_8$ is an undirected graph with the set of vertices $V(R_8) = \{(1,1), \ldots, (8,1), (1,2), \ldots, (8,2)\}$ and edges of the form $\{(i,1), (i,2)\}$ for $1 \leq i \leq 8$, or $\{(i,2), (i+1,2)\}$ for $1 \leq i < 8$; see Figure 4. The vertices $(1,1), \ldots, (8,1)$ form the first column of the comb $R_8$, while $(1,2), \ldots, (8,2)$ form the second column.

Definition 13. A set $S \subseteq (V(\overrightarrow{H_{10}}))^8$ is closed under extension if
(a) for every orientation $\overrightarrow{P}$ of the path $P_8 = (v_1, \ldots, v_8)$, there exists a coloring $c : \overrightarrow{P} \rightarrow \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) \in S$.

(b) for every orientation $\overrightarrow{R}$ of the comb $R_8$ and for every sequence $(c_1, \ldots, c_8) \in S$, there exists a coloring $c : \overrightarrow{R} \rightarrow \overrightarrow{H}_{10}$ and an automorphism $h_a$ of $\overrightarrow{H}_{10}$ such that

1. $(c(1,1), \ldots, c(8,1)) = (c_1, \ldots, c_8)$, and
2. $h_a(c(1,2), \ldots, c(8,2)) \in S$.

Lemma 14. There exists a set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ which is closed under extension.

Proof. In order to proof the lemma we use a computer. We have designed an algorithm that finds a proper set $S$. Let

$$S_{\text{max}}(P_8) = \{(c_1, \ldots, c_8) : c_1 \in \{1, 10\}, \text{ and } \forall 2 \leq i \leq 8 \ c_i \in V(\overrightarrow{H}_{10}), \text{ and } c_{i-1} \neq c_i\}.$$ 

Note, that for every sequence $t = (t_1, \ldots, t_8) \in S_{\text{max}}(P_8)$, there exists an orientation $\overrightarrow{P}$ of the path $P_8 = (v_1, \ldots, v_8)$ and a coloring $c : \overrightarrow{P} \rightarrow \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) = t$. For a set $T$, a sequence $t = (t_1, \ldots, t_8) \in T$, and an orientation $\overrightarrow{R}$ of the comb $R_8$, we say that $t$ can be extended in $T$ on $\overrightarrow{R}$ if there exists a coloring $c : \overrightarrow{R} \rightarrow \overrightarrow{H}_{10}$ and a homomorphism $h_a$ such that

- $(c(1,1), \ldots, c(8,1)) = t$, and
- $h_a(c(1,2), \ldots, c(8,2)) \in S$.

The algorithm starts with $T = S_{\text{max}}(P_8)$. In the while loop, for each sequence $t \in T$ and for each orientation $\overrightarrow{R}$ of the comb $R_8$, the algorithm checks if $t$ can be extended in $T$ on $\overrightarrow{R}$. If the sequence $t$ can not be extended, then $t$ is removed from $T$. After the while loop, the set $T$ satisfies the condition (b) of Definition 13. It is easy to see that if $T$ is not empty, then it also satisfies the condition (a). In this case $S = T$ is returned. If $T$ is empty, then the algorithm returns NO.
Algorithm ComputeSet S
OUTPUT: a set $S \subset (V(\overrightarrow{H}_{10}))^8$ closed under extension or NO if such a set does not exist.

1. compute the set $S_{\text{max}}(P_8)$
2. $T := S_{\text{max}}(P_8)$
3. SetIsReady := false
4. while not SetIsReady
5. SetIsReady := true
6. for every sequence $t = (t_1, \ldots, t_8) \in T$
7. color the first column of the comb $R_8$
8. by setting $c(i, 1) = t_i$, for $1 \leq i \leq 8$
9. SeqCanBeExtended := true
10. for every orientation $\overrightarrow{R}$ of the comb $R_8$
11. if $t$ cannot be extended on $\overrightarrow{R}$
12. SeqCanBeExtended := false
13. if not SeqCanBeExtended
14. $T := T - t$
15. SetIsReady := false
16. if $T = \emptyset$
17. return NO
18. else
19. $S := T$
20. return the set $S$

Using Algorithm ComputeSet S we have found a nonempty set $S$ closed under extension. The set $S$ is posted on the website https://inf.ug.edu.pl/grids/.

Theorem 15. Every orientation of every grid with eight rows can be colored by the coloring graph $\overrightarrow{H}_{10}$.

Proof. For a given orientation $\overrightarrow{G}$ of $G(8, n)$ and $i \leq n$, by $\overrightarrow{G}(i)$ we denote the induced subgraph of $\overrightarrow{G}$ formed by the first $i$ columns of $\overrightarrow{G}$. It is easy to show by induction that, for every $i$, there is a coloring $c : \overrightarrow{G}(i) \to \overrightarrow{H}_{10}$ such that $c(\text{i-th column}) \in S$. ■

4. Stacked Prism Graphs $Y_{m,n} = C_m \square P_n$

Theorem 16. Every orientation of $C_m \square P_n$ with $m \geq 3$ and $n \geq 1$ can be colored by the Tromp graph $\overrightarrow{T}_{16}$.
Proof. Let \( \overrightarrow{Y} \) be any orientation of stacked prism graph \( Y_{m,n} = C_m \square P_n \). We identify each vertex \( u \in \overrightarrow{Y} \) with the pair of its coordinates \((i,j)\), \(1 \leq i \leq m\), \(1 \leq j \leq n\). We shall show that \( \overrightarrow{Y} \) can be colored by \( \overrightarrow{T}_{16} \). We color the vertices of \( \overrightarrow{Y} \) row by row. For the first row, clearly, it is always possible to color any oriented cycle by homomorphism to \( \overrightarrow{T}_{16} \), because \( \overrightarrow{T}_{16} \) has the properties \( P_c(2,3) \) and \( P_c(1,7) \). Now, suppose that \( i > 1 \) and the rows from 1 to \( i - 1 \) are already colored. To color the vertex \((1,i)\) we choose a color which is compatible
- with the color of vertex \((2,i - 1)\) in the star \(\{(2,i),(1,i),(2,i - 1)\}\),
- with the color of vertex \((m,i - 1)\) in the star \(\{(m,i),(1,i),(m,i - 1)\}\),
which is always possible using the property \( P_c(1,7) \). Using the property \( P_c(2,3) \) it is always possible to color vertex \((2,i)\) by the color compatible with color of the vertex \((3,i - 1)\) in the star \(\{(3,i),(2,i),(3,i - 1)\}\). Then we continue this method to color vertices \((3,i),\ldots,(m - 2,i)\). To color the vertex \((m - 1,i)\) we choose a color which is compatible with the colors of vertices \((m,i - 1)\) and \((1,i)\) in the star \(\{(m,i),(1,i),(m,i - 1),(m - 1,i)\}\). This is possible, because the colors of vertices \((1,i)\) and \((m,i - 1)\) are compatible in the star \(\{(m,i),(1,i),(m,i - 1)\}\) Finally we color the vertex \((m,i)\) using the property \( P_c(3,1) \). Similarly we can color the following rows.

Theorem 17. Every orientation of stacked prism graph \( Y_{m,n} = C_m \square P_n \) with \( 3 \leq m \leq 7 \) can be colored by the coloring graph \( \overrightarrow{H}_{10} \).
Proof. The proof of the theorem is similar to the proof of Theorem 15 and follows from Lemma 20.

Definition 18. For \( m \geq 3 \), the \( m \)-sunlet graph \( \text{Sun}_m \) is an undirected graph with the set of vertices \( V(\text{Sun}_m) = \{(1,1), \ldots, (m,1), (1,2), \ldots, (m,2)\} \) and edges of the form \( \{(i,1),(i,2)\} \) for \( 1 \leq i \leq m \), or \( \{(i,2),(i+1,2)\} \) for \( 1 \leq i < m \), or \( \{(m,2),(1,2)\} \); see Figure 6.

![Figure 6. m-sunlet graph.](image)

Definition 19. A set \( S \subseteq (V(\overrightarrow{H}_{10}))^m \) is cycle-closed under extension if

(a) for every orientation \( \overrightarrow{C} \) of the cycle \( C_m = (v_1, \ldots, v_m) \), there exists a coloring \( c: \overrightarrow{C} \to \overrightarrow{H}_{10} \) such that \( (c(v_1), \ldots, c(v_m)) \in S \),

(b) for every orientation \( \overrightarrow{\text{Sun}} \) of the \( m \)-sunlet graph \( \text{Sun}_m \) and for every sequence \( (c_1, \ldots, c_m) \in S \), there exists a coloring \( c: \overrightarrow{\text{Sun}} \to \overrightarrow{H}_{10} \) and an automorphism \( h_a \) of \( \overrightarrow{H}_{10} \) such that

1. \( (c(1,1), \ldots, c(m,1)) = (c_1, \ldots, c_m) \), and
2. \( h_a(c(1,2), \ldots, c(m,2)) \in S \).

Lemma 20. For each \( m = 3, 4, 5, 6, 7 \), there exists a nonempty set \( S_m \subseteq (V(\overrightarrow{H}_{10}))^m \), which is cycle-closed under extension.

Proof. In order to proof the lemma we use a computer. We have designed an algorithm, similar to the Algorithm ComputeSet\( S \), that finds a set cycle-closed under extension. The algorithm, for a given \( m \), uses the \( m \)-sunlet \( \text{Sun}_m \) instead of a comb \( R_8 \). Using the algorithm we have found that for each \( m = 3, \ldots, 7 \), there exists a nonempty set cycle-closed under extension.
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