A CLASSIFICATION OF CACTUS GRAPHS
ACCORDING TO THEIR DOMINATION NUMBER

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Abstract

A set $S$ of vertices in a graph $G$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to some vertex in $S$. The domination number, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$. The authors proved in [A new lower bound on the domination number of a graph, J. Comb. Optim. 38 (2019) 721–738] that if $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then $\gamma(G) \geq \left\lceil \frac{n - \ell + 2 - 2k}{3} \right\rceil$. As a consequence of the above bound, $\gamma(G) = \frac{n - \ell + 2(1 - k) + m}{3}$ for some integer $m \geq 0$. In this paper, we characterize the class of cactus graphs achieving equality here, thereby providing a classification of all cactus graphs according to their domination number.

Keywords: domination number, lower bounds, cycles, cactus graphs.

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1. Introduction

A dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ has a neighbor in $S$, where two vertices are neighbors in $G$ if they are adjacent. The minimum cardinality of a dominating set is the domination number of $G$, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. As remarked in [5], the notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. For fundamentals of domination theory in graphs we refer the reader to the so-called domination books by Haynes, Hedetniemi, and Slater [6, 7]. An updated glossary of domination parameters can be found in [4].

Two vertices $u$ and $v$ in a graph $G$ are connected if there exists a $(u,v)$-path in $G$. The graph $G$ is connected if every two vertices in $G$ are connected. A block of $G$ is a maximal connected subgraph of $G$ which has no cut-vertex of its own. A cactus is a connected graph in which every edge belongs to at most one cycle. Equivalently, a (nontrivial) cactus is a connected graph in which every block is an edge or a cycle. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the minimum length of a $(u,v)$-path in $G$. The diameter, $\text{diam}(G)$, of $G$ is the maximum distance among pairs of vertices in $G$.

For notation and graph theory terminology we generally follow [8]. In particular, the order of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is given by $n(G) = |V(G)|$ and its size by $m(G) = |E(G)|$. A neighbor of a vertex $v$ in $G$ is a vertex adjacent to $v$, and the open neighborhood of $v$ is the set of neighbors of $v$, denoted $N_G(v)$. The closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ in $G$ is given by $d_G(v) = |N_G(v)|$.

For a set $S$ of vertices in a graph $G$, the subgraph induced by $S$ is denoted by $G[S]$. Further, the subgraph obtained from $G$ by deleting all vertices in $S$ and all edges incident with vertices in $S$ is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. A leaf of a graph $G$ is a vertex of degree 1 in $G$, and its unique neighbor is called a support vertex. The set of all leaves of $G$ is denoted by $L(G)$, and we let $\ell(G) = |L(G)|$ be the number of leaves in $G$. We denote the set of support vertices of $G$ by $S(G)$. We call a vertex of degree at least 2 a non-leaf.

Following our notation in [5], we denote the path and cycle on $n$ vertices by $P_n$ and $C_n$, respectively. A complete graph on $n$ vertices is denoted by $K_n$, while a complete bipartite graph with partite sets of size $n$ and $m$ is denoted by $K_{n,m}$. A star is the graph $K_{1,k}$, where $k \geq 1$. Further if $k > 1$, the vertex of degree $k$ is called the center vertex of the star, while if $k = 1$, arbitrarily designate either vertex of $P_2$ as the center. A double star is a tree with exactly two (adjacent) non-leaf vertices.

A rooted tree $T$ distinguishes one vertex $r$ called the root. For each vertex
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Let \( v \neq r \) of \( T \), the parent of \( v \) is the neighbor of \( v \) on the unique \((r,v)\)-path, while a child of \( v \) is any other neighbor of \( v \). A descendant of \( v \) is a vertex \( u \neq v \) such that the unique \((r,u)\)-path contains \( v \). In particular, every child of \( v \) is a descendant of \( v \). We let \( D(v) \) denote the set of descendants of \( v \), and we define \( D[v] = D(v) \cup \{v\} \). The maximal subtree at \( v \) is the subtree of \( T \) induced by \( D[v] \), and is denoted by \( T_v \). We use the standard notation \([k] = \{1, \ldots, k\}\).

2. Main Result

Our aim in this paper is to provide a classification of all cactus graphs according to their domination number. For this purpose, we shall use a result of the authors in [5] (which we present in Section 4) that establishes a lower bound on the domination number of a graph in terms of its order, number of vertices of degree 1, and number of cycles. From this result, we prove our desired characterization below, where \( G_{m,k} \) is a family of graphs defined in Section 3.

Theorem 1. Let \( m \geq 0 \) be an integer. If \( G \) is a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves, then \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \), if and only if \( G \in G_{m,k} \).

We proceed as follows. In Section 3 we define the families \( G_{m,k} \) of graphs for each integer \( k \geq 0 \) and \( m \geq 0 \). Known results on the domination number are given in Section 4. In Section 5 we present a proof of our main result.

3. The Families \( G_{m,k} \) for \( m \geq 0 \) and \( k \geq 0 \)

In this section, we define the families \( G_{m,k} \) of graphs for each integer \( k \geq 0 \) and \( m \geq 0 \). The families \( G_{0}^{0}, G_{0}^{1}, G_{0}^{2}, T_{0}^{1,1}, T_{0}^{2,1} \) of graphs were defined by the authors in [5]. For completeness, we include these definitions in Sections 3.1 and 3.2. We first define the families \( G_{k}^{0}, G_{k}^{1} \) and \( G_{k}^{2} \) of graphs in the special case when \( k = 0 \).

3.1. The families \( G_{0}^{0}, G_{0}^{1} \) and \( G_{0}^{2} \)

Hajian et al. [5] defined the class of trees \( G_{0}^{0}, G_{0}^{1} \) and \( G_{0}^{2} \) as follows.

- Let \( G_{0}^{0} \) be the class of all trees \( T \) that can be obtained from a sequence \( T_1, \ldots, T_k \) of trees where \( k \geq 1 \) such that \( T_1 \) is a star with at least three vertices, \( T = T_k \), and, if \( k \geq 2 \), then the tree \( T_{i+1} \) can be obtained from the tree \( T_i \) by applying Operation \( \mathcal{O} \) defined below for all \( i \in [k - 1] \).

Operation \( \mathcal{O} \). Add a vertex disjoint copy of a star \( Q_i \) with at least three vertices to the tree \( T_i \) and add an edge joining a leaf of \( Q_i \) and a leaf of \( T_i \).
Let $T_{0}^{1,1}$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_{0}^{0}$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in $T'$. Now, let $\mathcal{G}_{0}^{1}$ be the class of all trees $T$ that can be obtained from a sequence $T_{1}, \ldots, T_{k}$ of trees where $k \geq 1$ such that $T_{1} \in T_{0}^{1,1} \cup \{P_{2}\}$, $T = T_{k}$, and, if $k \geq 2$, then the tree $T_{i+1}$ can be obtained from the tree $T_{i}$ by applying Operation $\mathcal{O}$ for all $i \in [k-1]$.

Let $T_{0}^{2,1}$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_{0}^{0}$ by adding a vertex disjoint copy of a star (with at least two vertices) and adding an edge from the center of the added star to a non-leaf in $T'$. Let $T_{0}^{2,2}$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_{0}^{1}$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in $T'$. Now, let $\mathcal{G}_{0}^{2}$ be the class of all trees $T$ that can be obtained from a sequence $T_{1}, \ldots, T_{k}$ of trees, where $k \geq 1$, such that $T_{1} \in T_{0}^{2,1} \cup T_{0}^{2,2} \cup \{P_{4}\}$, $T = T_{k}$, and, if $k \geq 2$, then the tree $T_{i+1}$ can be obtained from the tree $T_{i}$ by applying Operation $\mathcal{O}$ for all $i \in [k-1]$.

3.2. The families $\mathcal{G}_{k}^{0}, \mathcal{G}_{k}^{1}$ and $\mathcal{G}_{k}^{2}$ when $k \geq 1$

For $k \geq 1$, Hajian et al. [5] defined the families of graphs $\mathcal{G}_{k}^{0}, \mathcal{G}_{k}^{1}$ and $\mathcal{G}_{k}^{2}$ as follows.

- For $k \geq 1$, they recursively defined the family $\mathcal{G}_{i}^{0}$ of graphs for each $i \in [k]$ by the following procedure.

**Procedure A.** For $i \in [k]$, a graph $G_{i}$ belongs to the family $\mathcal{G}_{i}^{0}$ if it contains an edge $e = xy$ such that the graph $G_{i} - e$ belongs to the family $\mathcal{G}_{i-1}^{0}$ and the vertices $x$ and $y$ are leaves in $G_{i} - e$ that are connected by a unique path in $G_{i} - e$.

- For $k \geq 1$, they recursively defined the family $\mathcal{G}_{i}^{1}$ of graphs for each $i \in [k]$ by the following two procedures.

**Procedure B.** For $i \in [k]$, a graph $G_{i}$ belongs to the family $\mathcal{G}_{i}^{1}$ if it contains an edge $e = xy$ such that the graph $G_{i} - e$ belongs to the family $\mathcal{G}_{i-1}^{1}$ and the vertices $x$ and $y$ are leaves in $G_{i} - e$ that are connected by a unique path in $G_{i} - e$.

**Procedure C.** For $i \in [k]$, a graph $G_{i}$ belongs to the family $\mathcal{G}_{i}^{1}$ if it contains an edge $e = xy$ such that the graph $G_{i} - e$ belongs to the family $\mathcal{G}_{i-1}^{2}$ and the vertices $x$ and $y$ are connected by a unique path in $G_{i} - e$. Further, exactly one of $x$ and $y$ is a leaf in $G_{i} - e$.

- For $k \geq 1$, they recursively defined the family $\mathcal{G}_{i}^{2}$ of graphs for each $i \in [k]$ by the following four procedures.

**Procedure D.** For $i \in [k]$, a graph $G_{i}$ belongs to the family $\mathcal{G}_{i}^{2}$ if it contains an edge $e = xy$ such that the graph $G_{i} - e$ belongs to the family $\mathcal{G}_{i-1}^{2}$ and the vertices $x$ and $y$ are leaves in $G_{i} - e$ that are connected by a unique path in $G_{i} - e$. 

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**Procedure E.** For $i \in [k]$, a graph $G_i$ belongs to the family $G_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_i^1$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$. Further, exactly one of $x$ and $y$ is a leaf in $G_i - e$.

**Procedure F.** For $i \in [k]$, a graph $G_i$ belongs to the family $G_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_i^1$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$. Further, both $x$ and $y$ are non-leaves in $G_i - e$.

**Procedure G.** For $2 \leq i \in [k]$, a graph $G_i$ belongs to the family $G_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_i^1$ and the vertices $x$ and $y$ are connected by exactly two paths in $G_i - e$. Further, both $x$ and $y$ are leaves in $G_i - e$.

### 3.3. The family $G_0^m$ when $m \geq 3$

In this section, we define a family of graphs $G_0^m$ for each integer $m \geq 3$ as follows. We call a non-leaf $x$ in a tree $T$ a **special vertex** if $\gamma(T - x) \geq \gamma(T)$. For $m \geq 3$, we first recursively define the class $T_0^{m,1}$ and $T_0^{m,2}$ of trees as follows.

- Let $T_0^{m,1}$ be the class of all trees $T$ that can be obtained from a tree $T' \in G_0^{m-2}$ by adding a vertex disjoint copy of a star $Q$ and joining the center of $Q$ to a special vertex in $T'$.
- Let $T_0^{m,2}$ be the class of all trees $T$ that can be obtained from a tree $T' \in G_0^{m-1}$ by adding a vertex disjoint copy of a star $Q$ with at least three vertices and joining a leaf of $Q$ to a non-leaf in $T'$.

For $m \geq 3$, we next recursively define the family $G_0^m$ of graphs constructed from the families $G_0^{m-1}$ and $G_0^{m-2}$ as follows.

- Let $G_0^m$ be the class of all trees $T$ that can be obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in T_0^{m,1} \cup T_0^{m,2}$ and the tree $T = T_q$. Further, if $q \geq 2$, then for each $i \in [q] \setminus \{1\}$, the tree $T_i$ can be obtained from the tree $T_{i-1}$ by applying the Operation $O$ defined in Section 3.1.

**Operation $O$.** Add a vertex disjoint copy of a star $Q_i$ with at least three vertices to the tree $T_i$ and add an edge joining a leaf of $Q_i$ and a leaf of $T_i$.

### 3.4. The family $G_k^m$ when $m \geq 3$ and $k \geq 1$

For $m \geq 3$ and $k \geq 1$, we construct the family $G_k^m$ from $G_{k-1}^{m-2}$, $G_{k-1}^{m-1}$, and $G_k^{m-1}$, recursively, as follows.

**Procedure H.** For $i \in [k]$, a graph $G_i$ belongs to the family $G_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_{k-1}^{m-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both $x$ and $y$ are leaves in $G_i - e$. 
Procedure I. For \( i \in [k] \), a graph \( G_i \) belongs to the family \( G^m_i \) if it contains an edge \( e = xy \) such that the graph \( G_i - e \) belongs to the family \( G^m_{i-1} \) and the vertices \( x \) and \( y \) are connected by a unique path in \( G_i - e \) and \( \gamma(G_i) = \gamma(G_i - e) \). Further, exactly one of \( x \) and \( y \) is a leaf in \( G_i - e \).

Procedure J. For \( i \in [k] \), a graph \( G_i \) belongs to the family \( G^m_i \) if it contains an edge \( e = xy \) such that the graph \( G_i - e \) belongs to the family \( G^m_{i-2} \) and the vertices \( x \) and \( y \) are connected by a unique path in \( G_i - e \) and \( \gamma(G_i) = \gamma(G_i - e) \). Further, both \( x \) and \( y \) are non-leaves in \( G_i - e \).

4. Known Results

In this section, we present some preliminary observations and known results. We begin with the following properties of graphs that belong to the families \( G^0_k \), \( G^1_k \) and \( G^2_k \) for \( k \geq 0 \).

Observation 1. The following properties hold in a graph \( G \in G^0_k \cup G^1_k \cup G^2_k \), where \( k \geq 0 \).

(a) The graph \( G \) contains exactly \( k \) cycles.

(b) The graph \( G \in G^0_k \cup G^1_k \) is a cactus graph.

We shall also need the following elementary property of a dominating set in a graph.

Observation 2. If \( G \) is a connected graph of order at least 3, then there exists a \( \gamma \)-set of \( G \) that contains no leaf of \( G \).

The following lemma is established in [5].

Lemma 2 [5]. If \( G \) is a connected graph and \( C \) is an arbitrary cycle in \( G \), then there is an edge \( e \) of \( C \) such that \( \gamma(G - e) = \gamma(G) \).

Several authors obtained bounds on the domination number in terms of different variants of graphs, see for example [1, 2, 3, 6, 9]. Let \( R \) be the family of all trees in which the distance between any two distinct leaves is congruent to 2 modulo 3. Lemańska [9] established the following lower bound on the domination number of a tree in terms of its order and number of leaves.

Theorem 3 [9]. If \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves, then \( \gamma(T) \geq (n - \ell + 2)/3 \), with equality if and only if \( T \in R \).

Hajian et al. [5] showed that the family \( R \) is precisely the family \( G^0_0 \); that is, \( R = G^0_0 \).

As a consequence of Theorem 3, we have the following result.
Corollary 4 [9]. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ for some integer $m \geq 0$.

Hajian et al. [5] strengthened the result in Theorem 3 as follows.

Theorem 5 [5]. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then the following holds.

(a) $\gamma(T) \geq \frac{1}{3}(n - \ell + 2)$, with equality if and only if $T \in G^0_0$.
(b) $\gamma(T) = \frac{1}{3}(n - \ell + 3)$ if and only if $T \in G^1_0$.
(c) $\gamma(T) = \frac{1}{3}(n - \ell + 4)$ if and only if $T \in G^2_0$.

The result of Theorem 5 was generalized in [5] to connected graphs as follows.

Theorem 6 [5]. If $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then the following holds.

(a) $\gamma(G) \geq \frac{1}{3}(n - \ell + 2(1 - k))$, with equality if and only if $G \in G^0_k$.
(b) $\gamma(G) = \frac{1}{3}(n - \ell + 3 - 2k)$ if and only if $G \in G^1_k$.
(c) $\gamma(G) = \frac{1}{3}(n - \ell + 4 - 2k)$ if and only if $G \in G^2_k$.

As a consequence of Theorem 6(a), we have the following.

Corollary 7 [5]. If $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$ for some integer $m \geq 0$.

5. Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 1. For this purpose, we first prove Theorem 1 in the special case when $k = 0$, that is, when the cactus is a tree.

Theorem 8. Let $m \geq 0$ be an integer. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ if and only if $T \in G^m_0$.

Proof. Let $T$ be a tree of order $n \geq 2$ with $\ell$ leaves. We proceed by induction on $m \geq 0$, namely first-induction, to show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in G^m_0$. For the base step of the first-induction let $m = 0$. If $m = 0$, then the result follows by Theorem 5(a). If $m = 1$, then the result follows by Theorem 5(b). If $m = 2$, then the result follows by Theorem 5(c). This establishes the base step of the induction. Let $m \geq 3$ and assume that the result holds for all trees $T_0$ of order $n_0$ with $\ell_0$ leaves, for $m_0 < m$. Let $T$ be a tree of order $n$ and with $\ell$ leaves. We will show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in G^m_0$.

($\Rightarrow$) Assume that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in G^m_0$. We show that $T \in G^m_0$. If $T = P_2$, then by the definition of the family
If $T \in G^1_0$, then by Theorem 5(b), $\gamma(T) = \frac{1}{3}(n - \ell + 2 + 1)$, and so $m = 1$, a contradiction. Hence we may assume that $\text{diam}(T) \geq 2$, for otherwise the desired result follows. If $\text{diam}(T) = 2$, then $T$ is a star, and by the definition of the family $G^0_0$, we have $T \in G^0_0$. Thus by Theorem 5(a), $\gamma(T) = \frac{1}{3}(n - \ell + 2 + 1)$, and so $m = 0$, a contradiction. If $\text{diam}(T) = 2$, then $T$ is a double star, and by definition of the family $G^0_0$ we have $T \in G^0_0$. Thus by Theorem 5(c), $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, and so $m = 2$, a contradiction. Hence, $\text{diam}(T) \geq 4$ and $n \geq 5$.

We now root the tree $T$ at a vertex $r$ at the end of a longest path $P$ in $T$. Let $u$ be a vertex at maximum distance from $r$, and so $d_T(u, r) = \text{diam}(T)$. Necessarily, $r$ and $u$ are leaves. Let $v$ be the parent of $u$, let $w$ be the parent of $v$, let $x$ be the parent of $w$, and let $y$ be the parent of $x$. Possibly, $y = r$. Since $u$ is a vertex at maximum distance from the root $r$, every child of $v$ is a leaf. By Observation 2, there exists a $\gamma$-set, say $S$, of $T$ that contains no leaf of $T$; that is, $L(T) \cap S = \emptyset$. In particular, we note that $|S| = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. In order to dominate the vertex $u$, we note therefore that $v \in S$. Let $d_T(v) = t$. We note that $t \geq 2$.

**Claim 1.** If $d_T(w) \geq 3$, then $T \in G^m_0$.

**Proof.** Suppose that $d_T(w) \geq 3$. In this case, we consider the tree $T' = T - V(T_v)$, where $T_v$ is the maximal subtree at $v$. Let $T'$ have order $n'$ and let $T'$ have $\ell'$ leaves. We note that $n' = n - t$. Since $w$ is not a leaf in $T'$, we have $\ell' = \ell - (t - 1) = \ell - t + 1$. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. If a child of $w$ is a leaf in $T'$, then since the dominating set $S$ contains no leaves, we have that $w \in S$. If no child of $w$ is a leaf in $T$, then every child of $w$ is a support vertex and therefore belongs to the set $S$. In both cases, we note that the set $S \setminus \{v\}$ is a dominating set of $T'$, implying that $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every $\gamma$-set of $T'$ can be extended to a dominating set of $T$ by adding to it the vertex $v$, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$. Thus,

\[
\begin{align*}
\gamma(T') &= \gamma(T) - 1 \\
&= \frac{1}{3}(n - \ell + 2 + m) - 1 \\
&= \frac{1}{3}(n - \ell + m - 1) \\
&= \frac{1}{3}(n' + t) - (\ell' + t - 1) + m - 1 \\
&= \frac{1}{3}(n' - \ell' + m).
\end{align*}
\]

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. Thus, $m' = m - 2$. Applying the inductive hypothesis to the tree $T'$, we have $T' \in G^{m-2}_0$. Let $v'$ be a child of $w$ different from $v$. We note that the tree $T_{v'}$ is a component of $T' - w$ and this component is dominated by the vertex $v'$. We
can therefore choose a $\gamma$-set of $T' - w$ to contain the vertex $v'$. Such a $\gamma$-set of $T' - w$ is also a dominating set of $T'$, implying that $\gamma(T') \leq \gamma(T' - w)$; that is, the vertex $w$ is a special vertex of $T'$. Thus, the tree $T$ is obtained from the tree $T' \in G_{0}^{m-2}$ by adding a vertex disjoint copy of a star $T_{v}$ and joining the center $v$ of $T_{v}$ to a special vertex $w$ in $T'$. Thus $T \in T_{0}^{m,1}$. Consequently, $T \in G_{0}^{m}$. This completes the proof of Claim 1.

By Claim 1, we may assume that $d_{T}(w) = 2$, for otherwise $T \in G_{0}^{m}$ as desired. We now consider the tree $T' = T - V(T_{w})$, where $T_{w}$ is the maximal subtree at $w$. Let $T'$ have order $n'$ and let $T'$ have $\ell'$ leaves. We note that $n' = n - t - 1$. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$.

As observed earlier, the vertex $v$ belongs to the dominating set $S$. If $w \in S$, then we can replace $w$ in $S$ with the vertex $x$ to produce a new $\gamma$-set of $T$ that contains no leaf of $T$. Hence we may assume that $w \notin S$, implying that the set $S \setminus \{v\}$ is a dominating set of $T'$ and therefore $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every $\gamma$-set of $T'$ can be extended to a dominating set of $T$ by adding to it the vertex $v$, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$.

**Claim 2.** If $d_{T}(x) \geq 3$, then $T \in G_{0}^{m}$.

**Proof.** Suppose that $d_{T}(x) \geq 3$. In this case, the vertex $x$ is not a leaf of $T'$, implying that $\ell' = \ell - (t - 1) = \ell - t + 1$. Thus,

$$
\gamma(T') = \gamma(T) - 1
= \frac{1}{3}(n - \ell + m - 1)
= \frac{1}{3}(n' + t + 1) - (\ell' + t - 1) + m - 1
= \frac{1}{3}(n' - \ell' + m + 1).
$$

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. Thus, $m' = m - 1$. Applying the inductive hypothesis to the tree $T'$, we have $T' \in G_{0}^{m-1}$. Thus, the tree $T$ is obtained from the tree $T' \in G_{0}^{m-1}$ by adding a vertex disjoint copy of a star $T_{v}$ with at least three vertices and joining a leaf of the star $T_{v}$ to the non-leaf $x$ of $T'$. Thus $T \in T_{0}^{m,2}$. Consequently, $T \in G_{0}^{m}$.

By Claim 2, we may assume that $d_{T}(x) = 2$, for otherwise $T \in G_{0}^{m}$ as desired. In this case, the vertex $x$ is a leaf of $T'$, implying that $\ell' = \ell - (t - 1) + 1 = \ell - t + 2$. Thus,

$$
\frac{1}{3}(n' - \ell' + 2 + m') = \gamma(T') = \gamma(T) - 1
= \frac{1}{3}(n - \ell + m - 1)
= \frac{1}{3}(n' + t + 1) - (\ell' + t - 2) + m - 1
= \frac{1}{3}(n' - \ell' + m + 2),
$$
and so \( m = m' \). Applying the inductive hypothesis to the tree \( T' \), we have \( T' \in G_0^m \). Thus, the tree \( T \) is obtained from the tree \( T' \in G_0^m \) by adding a vertex disjoint copy of a star \( T_v \) with at least three vertices and adding the edge \( xw \) joining a leaf \( w \) of \( T_v \) and a leaf \( x \) of \( T' \); that is, \( T \) is obtained from \( T' \) by Operation \( O \). Hence, by definition of the family \( G_0^m \), we have \( T \in G_0^m \), as desired. This completes the necessity part of the proof of Theorem 8.

\((\Leftarrow\Rightarrow)\) Conversely, assume that \( T \in G_0^m \), where \( m \geq 0 \). Recall that \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves. Thus, \( T \) is obtained from a sequence \( T_1, \ldots, T_q \) of trees, where \( q \geq 1 \) and where the tree \( T_1 \in T_0^{m,1} \cup T_0^{m,2} \), and the tree \( T = T_q \).

Further, if \( q \geq 2 \), then for each \( i \in [q] \setminus \{1\} \), the tree \( T_i \) can be obtained from the tree \( T_{i-1} \) by applying the following Operation \( O \). We proceed by induction on \( q \geq 1 \), namely second-induction, to show that \( \gamma_i(T) = \frac{1}{3}(n - \ell + 2 + m) \).

Claim 3. If \( q = 1 \), then \( \gamma_1(T) = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Proof.** Suppose that \( q = 1 \). Thus, \( T_1 \in T_0^{m,1} \cup T_0^{m,2} \). We consider the two possibilities in turn, and in both cases we will show that the tree \( T \in G_0^m \) satisfies \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

Claim 3.1. If \( T \in T_0^{m,1} \), then \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Proof.** Suppose that \( T \in T_0^{m,1} \). Thus, \( T \) is obtained from a tree \( T' \in G_0^{m-2} \) by adding a vertex disjoint copy of a star \( Q \) with \( t \geq 2 \) vertices and joining the center of \( Q \), say \( y \), to a special vertex \( x \) in \( T' \). Let \( T' \) have order \( n' \), and so \( n' = n - t \). Further, let \( T' \) have \( \ell' \) leaves. Since \( x \) is a non-leaf of \( T' \), we have \( \ell' = \ell - (t - 1) \). Applying the first-induction hypothesis to the tree \( T' \in G_0^{m-2} \), we have \( \gamma_i(T') = \frac{1}{3}(n' - \ell' + 2 + (m - 2)) = \frac{1}{3}(n' - \ell' + m) \).

We show next that \( \gamma(T) = \gamma(T') + 1 \). Since \( x \) is a special vertex of \( T' \), we note that \( \gamma(T' - x) \geq \gamma(T') \). Every \( \gamma \)-set of \( T' \) can be extended to a dominating set of \( T \) by adding to it the vertex \( y \), implying that \( \gamma(T) \leq \gamma(T') + 1 \). Conversely, we can choose a \( \gamma \)-set, say \( D \), of \( T \) to contain the vertex \( y \) which dominates the star \( Q \). If \( x \in D \), then \( D \setminus \{y\} \) is a dominating set of \( T' \), and so \( \gamma(T') \leq |D| - 1 \). If \( x \notin D \), then \( D \setminus \{y\} \) is a dominating set of \( T' - x \), and so \( \gamma(T') \leq |D| - 1 \). In both cases, \( \gamma(T') \leq |D| - 1 = \gamma(T) - 1 \). Consequently, \( \gamma(T) = \gamma(T') + 1 \). Thus,

\[
\gamma(T) = \gamma(T') + 1 \\
= \frac{1}{3}(n' - \ell' + m) + 1 \\
= \frac{1}{3}((n - t) - (\ell - t + 1) + m) + 1 \\
= \frac{1}{3}(n - \ell + 2 + m).
\]

This completes the proof of Claim 3.1. \( \square \)
Claim 3.2. If $T \in \mathcal{T}_0^{m,2}$, then $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that $T \in \mathcal{T}_0^{m,2}$. Thus, $T$ is obtained from a tree $T' \in G_0^{m-1}$ by adding a vertex disjoint copy of a star $Q$ with $t \geq 3$ vertices and joining a leaf, say $v$, of $Q$ to a non-leaf, say $w$, in $T'$. Let $u$ be the center of the star $Q$. Let $T'$ have order $n'$, and so $n' = n - t$. Further, let $T'$ have $\ell'$ leaves. Since $w$ is a non-leaf of $T'$, we have $\ell' = \ell - (t - 2)$. Applying the first-induction hypothesis to the tree $T' \in G_0^{m-1}$, we have $\gamma_t(T') = \frac{1}{3}(n' - \ell' + 2 + m) = \frac{1}{3}(n' - \ell' + m + 1)$.

We show next that $\gamma(T) = \gamma_t(T') + 1$. Every $\gamma$-set of $T'$ can be extended to a dominating of $T$ by adding to it the vertex $u$, implying that $\gamma(T) \leq \gamma(T') + 1$. By Observation 2, there exists a $\gamma$-set $D$ of $T$ that contains no leaf of $Q$. Thus, $u \in D$. If $v \in D$, then we can replace $v$ in $D$ with the vertex $w$. Hence we may assume that $v \notin D$, implying that $D \setminus \{u\}$ is a dominating set of $T'$, and so $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus,

$$
\gamma(T) = \gamma(T') + 1
= \frac{1}{3}(n' - \ell' + m + 1) + 1
= \frac{1}{3}((n - t) - (\ell - t + 2) + m + 1) + 1
= \frac{1}{3}(n - \ell + 2 + m).
$$

This completes the proof of Claim 3.2. \qed

By Claims 3.1 and 3.2, if $T \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Claim 3. \qed

By Claim 3, if $q = 1$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This establishes the base step of the second-induction. Let $q \geq 2$ and assume that if $q'$ is an integer where $1 \leq q' < q$ and if $T' \in G_0^m$ is a tree of order $n' \geq 2$ with $\ell'$ leaves obtained from a sequence of $q'$ trees, then $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m)$. Recall that $T$ is obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, and the tree $T = T_q$. Further for each $i \in [q] \setminus \{1\}$, the tree $T_i$ can be obtained from the tree $T_{i-1}$ by applying the Operation $\mathcal{O}$.

We now consider the tree $T' = T_{q-1}$. Thus, the tree $T \in G_0^m$ is obtained from the tree $T'$ by adding a vertex disjoint copy of a star $Q$ with $t \geq 3$ vertices and adding an edge joining a leaf of $Q$ to a leaf of $T'$. Let $T'$ have order $n'$ and let $T'$ have $\ell'$ leaves. We note that $n' = n - t$ and $\ell' = \ell - (t - 2) + 1 = \ell - t + 3$. Applying the second-induction hypothesis to the tree $T' \in G_0^m$, we have $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m)$. Analogous arguments as before show that $\gamma(T) = \gamma_t(T') + 1$. Thus,

$$
\gamma(T) = \gamma(T') + 1
= \frac{1}{3}(n' - \ell' + 2 + m) + 1
= \frac{1}{3}((n - t) - (\ell - t + 3) + 2 + m) + 1
= \frac{1}{3}(n - \ell + 2 + m).
$$
Hence we have shown that if $T \in G_{k}^{m}$, where $m \geq 0$ and where $T$ has order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Theorem 8.

We are now in a position to prove our main result, namely Theorem 1. Recall its statement.

**Theorem 1.** Let $m \geq 0$ be an integer. If $G$ is a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$, if and only if $G \in G_{k}^{m}$.

**Proof.** Let $m \geq 0$ be an integer, and let $G$ be a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves. We proceed by induction on $k$ to show that $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$ if and only if $G \in G_{k}^{m}$. If $k = 0$, then the result follows from Theorem 8. This establishes the base case. Let $k \geq 1$ and assume that if $G'$ is a cactus graph of order $n' \geq 2$ with $k'$ cycles and $\ell'$ leaves where $0 \leq k' < k$, then $\gamma(G') = \frac{1}{3}(n' - \ell' + 2(1 - k') + m')$ if and only if $G' \in G_{k'}^{m'}$. Let $G$ be a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves. We will show that $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$, if and only if $G \in G_{k}^{m}$.

If $m = 0$, then the result follows by Theorem 6(a). If $m = 1$, then the result follows by Theorem 6(b). If $m = 2$, then the result follows by Theorem 6(c). Thus, we may assume that $m \geq 3$, for otherwise the desired result follows.

($\Rightarrow$) Assume that $\gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k)$ (where we recall that here $m \geq 3$). We will show that $T \in G_{k}^{m}$. By Lemma 2, the graph $G$ contains a cycle edge $e$ such that $\gamma(G - e) = \gamma(G)$. Let $e = uv$, and consider the graph $G' = G - e$. Let $G'$ have order $n'$ with $k' \geq 0$ cycles and $\ell'$ leaves. We note that $n' = n$. Further, since $G$ is a cactus graph, $k' = k - 1$. Removing the cycle edge $e$ from $G$ produces at most two new leaves, namely the ends of the edge $e$, implying that $\ell' - 2 \leq \ell \leq \ell'$. By Corollary 7, we have $\gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k')$ for some integer $m' \geq 0$. Applying the inductive hypothesis to the cactus graph $G'$, we have that $G' \in G_{k'}^{m'} = G_{k-1}^{m}$. Our earlier observations imply that

\[
\frac{1}{3}(n - \ell + 2 + m - 2k) = \gamma(G) = \gamma(G')
= \frac{1}{3}(n' - \ell' + 2 + m' - 2k')
= \frac{1}{3}(n - \ell' + 2 + m' - 2(k - 1)),
\]

and so $m - \ell = m' - \ell' + 2$. Since $G$ is a cactus, the vertices $u$ and $v$ are connected in $G' = G - e$ by a unique path. As observed earlier, $\ell' - 2 \leq \ell \leq \ell'$.

Suppose that $\ell = \ell'$. In this case, neither $u$ nor $v$ is a leaf of $G'$, implying that both $u$ and $v$ have degree at least 2 in $G'$. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m - 2$. Thus, $G' \in G_{k-1}^{m-2}$. Hence, the graph $G$ is obtained from $G'$ by Procedure J and therefore $G \in G_{k}^{m}$.
Suppose that $\ell = \ell' - 1$. In this case, exactly one of $u$ and $v$ is a leaf of $G'$. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m - 1$. Thus, $G' \in G^{m-1}_{k-1}$. Hence, the graph $G$ is obtained from $G'$ by Procedure I, and therefore $G \in G^{m}_{k}$. 

Suppose that $\ell = \ell' - 2$. In this case, both $u$ and $v$ are leaves in $G'$. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m$. Thus, $G' \in G^{m}_{k-1}$. Hence, the graph $G$ is obtained from $G'$ by Procedure H, and therefore $G \in G^{m}_{k}$. This completes the necessity part of the proof of Theorem 1.

$(\Leftarrow)$ Conversely, assume that $G \in G^{m}_{k}$. Recall that by our earlier assumptions, $m \geq 3$ and $k \geq 1$. Thus, the graph $G$ is obtained from either a graph $G' \in G^{m}_{k-1}$ by Procedure H or from a graph $G' \in G^{m-1}_{k-1}$ by Procedure I or from a graph $G' \in G^{m-2}_{k-1}$ by Procedure J. In all three cases, let $G'$ have order $n'$ with $k' \geq 0$ cycles and $\ell'$ leaves. Further, in all cases we note that $n' = n$ and $k' = k - 1$. We consider the three possibilities in turn.

Suppose firstly that $G$ is obtained from a graph $G' \in G^{m}_{k-1}$ by Procedure H. In this case, $\ell = \ell' - 2$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in G^{m}_{k-1}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m - 2(k - 1)) = \frac{1}{3}(n - (\ell + 2) + 4 + m - 2k)$.

Suppose next that $G$ is obtained from a graph $G' \in G^{m-1}_{k-1}$ by Procedure I. In this case, $\ell = \ell' - 1$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in G^{m-1}_{k-1}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m - 1) - 2(k - 1)) = \frac{1}{3}(n - (\ell + 1) + 3 + m - 2k)$.

Suppose finally that $G$ is obtained from a graph $G' \in G^{m-2}_{k-1}$ by Procedure J. In this case, $\ell = \ell'$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in G^{m-2}_{k-1}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m - 2) - 2(k - 1)) = \frac{1}{3}(n - \ell + 2 + m - 2k)$. In all three cases, $\gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k)$. This completes the proof of Theorem 1. 

\section*{References}


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