ON THE OPTIMALITY OF 3-RESTRICTED ARC CONNECTIVITY FOR DIGRAPHS AND BIPARTITE DIGRAPHS

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Abstract

Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ contains at least $k$ such that $D \setminus V(D')$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists in $D$, then $D$ is called $\lambda^k$-connected. For a $\lambda^k$-connected digraph $D$, the $k$-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all $k$-restricted arc cuts of $D$. It is known that for many digraphs $\lambda^k(D) \leq \xi^k(D)$, where $\xi^k(D)$ denotes the minimum $k$-degree of $D$. $D$ is called $\lambda^k$-optimal if $\lambda^k(D) = \xi^k(D)$. In this paper, we will give some sufficient conditions for digraphs and bipartite digraphs to be $\lambda^3$-optimal.

Keywords: restricted arc-connectivity, bipartite digraph, optimality, digraph, network.

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1. Introduction

It is well-known that the network can be modelled as a digraph $D$ with vertices $V(D)$ representing sites and arcs $A(D)$ representing links between sites of the network. Let $v \in V(D)$, the out-neighborhood of $v$ is the set $N^+(v) = \{x \in V(D) : vx \in A(D)\}$ and the out-degree of $v$ is $d^+(v) = |N^+(v)|$. The in-neighborhood of $v$ is the set $N^-(v) = \{x \in V(D) : xv \in A(D)\}$ and the in-degree of $v$ is $d^-(v) = |N^-(v)|$. The neighborhood of $v$ is $N(v) = N^+(v) \cup N^-(v)$.

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Let \( \delta^+(D) \), \( \delta^-(D) \) and \( \delta(D) \) denote, respectively, the minimum out-degree, the minimum in-degree and the minimum degree of \( D \).

For a pair nonempty vertex sets \( X \) and \( Y \) of \( D \), \( [X,Y] = \{xy \in A(D) : x \in X, y \in Y\} \). Specially, if \( Y = \overline{X} \), where \( \overline{X} = V(D) \setminus X \), then we write \( \partial^+(X) \) or \( \partial^-(Y) \) instead of \( [X,Y] \). For \( X \subseteq V(D) \), the subdigraph of \( D \) induced by \( X \) is denoted by \( D[X] \). The underlying graph \( U(D) \) of \( D \) is the unique graph obtained from \( D \) by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. \( D \) is connected if \( U(D) \) is connected and \( D \) is strongly connected (or, just, strong) if there exists a directed \((x,y)\)-path and a directed \((y,x)\)-path for any \( x,y \in V(D) \). We define a digraph with one vertex to be strong. A connected (strong) component of \( D \) is a maximal induced subdigraph of \( D \) which is connected (strong). If \( D \) has \( p \) strong components, then these strong components can be labeled \( D_1, \ldots, D_p \) such that there is no arc from \( D_j \) to \( D_i \) unless \( j < i \). We call such an ordering an acyclic ordering of the strong components of \( D \).

In a strong digraph \( D \), we often use arc connectivity of \( D \) to measure the reliability. An arc set \( S \) is a \emph{arc cut} of \( D \) if \( D - S \) is not strong. The arc connectivity \( \lambda(D) \) is the minimum cardinality over all arc cuts of \( D \). The arc cut \( S \) of \( D \) with cardinality \( \lambda(D) \) is called a \( \lambda \)-cut. Whitney’s inequality shows \( \lambda(D) \leq \delta(D) \). A strong digraph \( D \) with \( \lambda(D) = \delta(D) \) is called \( \lambda \)-optimal. However, only using arc connectivity to measure the reliability is not enough. In [12], Volkmann introduced the concept of restricted arc connectivity. An arc subset \( S \) of \( D \) is a \emph{restricted arc cut} if \( D - S \) has a strong component \( D' \) with order at least \( k \) such that \( D \setminus V(D') \) contains an arc. If such an arc cut exists in \( D \), then \( D \) is called \( \lambda' \)-connected. For a \( \lambda' \)-connected digraph \( D \), the restricted arc connectivity, denoted by \( \lambda'(D) \), is the minimum cardinality over all restricted arc cuts of \( D \). The restricted arc cut \( S \) of \( D \) with cardinality \( \lambda'(D) \) is called a \( \lambda' \)-cut. In [13], Wang and Lin introduced the notion of \emph{minimum arc degree}. Let \( xy \in A(D) \). Then

\[
\Omega(\{x,y\}) = \{\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^+(\{x\}) \cup \partial^-(\{y\}), \partial^+(\{y\}) \cup \partial^-(\{x\})\}.
\]

The arc degree of \( xy \) is \( \xi'(xy) = \min\{\{S\} : S \in \Omega(\{x,y\})\} \) and the minimum arc degree of \( D \) is \( \xi'(D) = \min\{\xi'(xy) : xy \in A(D)\} \).

It was proved in [3, 13] that for many \( \lambda' \)-connected digraphs, \( \xi'(D) \) is an upper bound of \( \lambda'(D) \). In [13], Wang and Lin introduced the concept of \( \lambda' \)-optimality. A \( \lambda' \)-connected digraph \( D \) with \( \xi'(D) = \lambda'(D) \) is called \( \lambda' \)-optimal. As a generalization of restricted arc connectivity, in [10], Lin et al. introduced the concept of \( k \)-restricted arc connectivity.

**Definition** [10]. Let \( D \) be a strong digraph. An arc subset \( S \) is a \( k \)-restricted arc cut of \( D \) if \( D - S \) has a strong component \( D' \) with order at least \( k \) such that
$D \setminus V(D')$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists in $D$, then $D$ is called $\lambda^k$-\emph{connected}. For a $\lambda^k$-connected digraph $D$, the $k$-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all $k$-restricted arc cuts of $D$. The $k$-restricted arc cut $S$ of $D$ with cardinality $\lambda^k(D)$ is called a $\lambda^k$-\emph{cut}.

**Definition** [10]. Let $D$ be a strong digraph. For any $X \subseteq V(D)$, let $\Omega(X) = \{\partial^+(X_1) \cup \partial^-(X \setminus X_1) : X_1 \subseteq X\}$ and $\xi(X) = \min\{|S| : S \in \Omega(X)\}$. Define the minimum $k$-degree of $D$ to be

$$\xi^k(D) = \min\{\xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected}\}.$$ 

Clearly, $\lambda^1(D) = \lambda(D), \lambda^2(D) = \lambda'(D), \xi^1(D) = \delta(D)$ and $\xi^2(D) = \xi'(D)$.

Let $D$ be a $\lambda^k$-connected digraph, where $k \geq 2$. Then $D$ is $\lambda^{k-1}$-connected and $\lambda^{k-1}(D) \leq \lambda^k(D)$. It was shown in [10] that $\xi^k(D)$ is an upper bound of $\lambda^k(D)$ for many digraphs. And a $\lambda^k$-connected digraph $D$ with $\lambda^k(D) = \xi^k(D)$ is called $\lambda^k$-\emph{optimal}.

The research on the $\lambda^k$-optimality of digraph $D$ is considered to be a hot issue. In [11], Hellwig and Volkmann concluded many sufficient conditions for digraphs to be $\lambda$-optimal. Besides, sufficient conditions for digraphs to be $\lambda$-optimal were also given by several authors, for example by Balbuena et al. [1–4], Chen et al. [5,6], Grüter and Guo [7,8], Liu and Zhang [9], Volkmann [12] and Wang and Lin [13]. However, closely related conditions for $\lambda^3$-optimal digraphs have received little attention until recently. In [10], Lin et al. gave some sufficient conditions for digraphs to be $\lambda^3$-optimal. In this paper, we will give some sufficient conditions for digraphs to be $\lambda^3$-optimal. As corollaries, degree conditions or degree sum conditions for a digraph or a bipartite digraph to be $\lambda^3$-optimal are given. The main contributions in this paper are as following.

**Theorem 1.** Let $D$ be a digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then $D$ is $\lambda^3$-\emph{optimal}.

**Theorem 2.** Let $D = (X, Y, A(D))$ be a bipartite digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite, then $D$ is $\lambda^3$-\emph{optimal}.

2. **Proof of Theorem 1**

We first introduce three useful lemmas.

**Lemma 3** (Theorem 1.4 in [10]). Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$. Then $D$ is $\lambda^k$-\emph{connected} and $\lambda^k(D) \leq \xi^k(D)$.
Lemma 4. Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$, and let $S = \partial^+(X)$ be a $\lambda^k$-cut of $D$, where $X$ is a subset of $V(D)$. If $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$ or $D[\overline{X}]$ contains a connected subdigraph $C$ with order $k$ such that $|N^-(y) \cap X| \geq k$ for any $y \in \overline{X} \setminus V(C)$, then $D$ is $\lambda^k$-optimal.

Proof. By Lemma 3, $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$. By reason of symmetry, we only prove the case that $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$. The hypotheses imply that

$$\xi^k(D) \leq |\partial^+(V(B))| = |(V(B), X \setminus V(B))| + |(V(B), \overline{X})|$$

$$\leq k|X \setminus V(B)| + |(V(B), \overline{X})| \leq \sum_{x \in X \setminus V(B)} |N^+(x) \cap X| + |(V(B), \overline{X})|$$

$$= |(X \setminus V(B), \overline{X})| + |(V(B), \overline{X})| = |X, \overline{X}| = 3 = \lambda^k(D).$$

Thus $\lambda^k(D) = \xi^k(D)$ and $D$ is $\lambda^k$-optimal.

Lemma 5 (Lemma 4.1 in [10]). Let $D$ be a strong digraph with $|V(D)| \geq 6$ and $\delta(D) \geq 4$, and let $S$ be a $\lambda^3$-cut of $D$. If $D$ is not $\lambda^3$-optimal, then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Proof of Theorem 1. Clearly, $D$ is a strong digraph with $\delta(D) \geq 5$. By Lemma 3, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that $D$ is not $\lambda^3$-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let $S$ be a $\lambda^3$-cut of $D$. By Lemma 5, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $Y = \overline{X}$, and let $X_i = \{x \in X : |N^+(x) \cap Y| = i\}, Y_i = \{y \in Y : |N^-(y) \cap X| = i\}, i = 0, 1, 2, 3$, and let $X_3 = \{x \in X : |N^+(x) \cap Y| \geq 3\}, Y_3 = \{y \in Y : |N^-(y) \cap X| \geq 3\}$.

Claim 1. $\min\{|X|, |Y|\} \geq 4$.

Proof. Suppose that $|X| = 3$. Then $\lambda^3(D) = |S| = |\partial^+(X)| \geq \xi(X) \geq \xi^3(D)$, contrary to the assumption. Suppose that $|Y| = 3$. Then $\lambda^3(D) = |S| = |\partial^-(Y)| \geq \xi(Y) \geq \xi^3(D)$, contrary to the assumption. Claim 1 follows.

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. For the reason of symmetry, we only prove that $X_0 = \emptyset$ by contradiction. Suppose $X_0 \neq \emptyset$ and let $x \in X_0$. Then for any $\overline{x} \in Y$, $x\overline{x} \notin A(D)$ and we have that $5 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X| + |N^+(x) \cap N^-(\overline{x}) \cap Y| \leq$
\(|N^-(x) \cap X| + |N^+(x) \cap Y| = |N^-(x) \cap X|\). It implies that \(|N^-(x) \cap X| \geq 5\). Therefore \(Y \subseteq Y_3\). So \(D\) is \(\lambda^3\)-optimal by Lemma 4, a contradiction to our assumption.

Combining Claim 2 with Lemma 4, we have that \(Y_1 \cup Y_2 \neq \emptyset\) and \(X_1 \cup X_2 \neq \emptyset\). Otherwise we will obtain that \(D\) is \(\lambda^3\)-optimal, which is a contradiction. Next, we consider two cases.

**Case 1.** \(X_1 \neq \emptyset\). Let \(x' \in X_1\) and suppose \(N^+(x') \cap Y = \{y'\}\). Then for any \(y \in Y \setminus \{y'\}\), \(x'y \notin A(D)\), so we have that \(5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 1\). So \(|N^-(y) \cap X| \geq 4\) and \(Y \setminus \{y'\} \subseteq Y_3\). On the other hand, since \(Y_1 \cup Y_2 \neq \emptyset\), \(y' \in Y_1 \cup Y_2\). Besides, \(5 \leq \delta(D) \leq \delta^-(y') = |N^-(y')| = |N^-(y') \cap Y| + |N^-(y') \cap X| \leq |N^-(y) \cap Y| + 2\), thus \(|N^-(y') \cap Y| \geq 3\). Let \(y_1, y_2 \in N^-(y') \cap Y\), then \(D[y', y_1, y_2]\) is connected and \(|N^-(y) \cap X| \geq 4\) for any \(y \in Y \setminus \{y', y_1, y_2\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

**Case 2.** \(X_2 \neq \emptyset\). Let \(x' \in X_2\) and suppose \(N^+(x') \cap Y = \{y', y''\}\). Then for any \(y \in Y \setminus \{y', y''\}\), \(x'y \notin A(D)\), thus \(5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 2\). So \(|N^-(y) \cap X| \geq 3\) and \(Y \setminus \{y', y''\} \subseteq Y_3\). On the other hand, since \(Y_1 \cup Y_2 \neq \emptyset\), \(y' \in Y_1 \cup Y_2\) or \(y'' \in Y_1 \cup Y_2\). If \(|Y_1 \cup Y_2| = 1\), then we can prove that \(D\) is \(\lambda^3\)-optimal by a proof similar to Case 1, which is a contradiction. If \(Y_1 \cup Y_2 = \{y', y''\}\), then we consider two subcases.

**Subcase 2.1.** \(y'y'' \in A(D)\) or \(y'y'' \notin A(D)\). Since \(y'' \in Y_1 \cup Y_2\) and \(\delta(D) \geq 5\), then there exists \(y_1 \in N^-(y'') \cap Y\) such that \(y_1 \neq y'\). Therefore \(D[y', y'', y_1]\) is connected and \(|N^-(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_1\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

**Subcase 2.2.** \(y'y'' \notin A(D)\) and \(y'y'' \notin A(D)\). Since \(y'y'' \notin A(D)\) and \(y'y'' \notin A(D)\), then \(5 \leq |N^+(y') \cap N^-(y'')| = |N^+(y') \cap N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq |N^-(y) \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq 2 + |N^+(y') \cap N^-(y') \cap Y|\). Therefore \(|N^+(y') \cap N^-(y'') \cap Y| \geq 3\). Let \(y_1 \in N^+(y') \cap N^-(y'') \cap Y\). Then \(D[y', y'', y_1]\) is connected and \(|N^-(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_1\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete.

**Corollary 6.** Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(d^+(u) + d^-(v) \geq |V(D)| + 3\) for any \(u, v \in V(D)\) with \(uv \notin A(D)\), then \(D\) is \(\lambda^3\)-optimal.

**Corollary 7** (Theorem 1.7 in [10]). Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(|V(D)| \geq \frac{|V(D)| + 3}{2}\), then \(D\) is \(\lambda^3\)-optimal.
Remark 8. To show the condition that “$|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Theorem 1 is sharp, we give a class of digraphs. Let $m, k$ be positive integers with $m \geq 3$, and let $D$ be a digraph with $|V(D)| = 4m + 4$. Define the vertex set of $D$ as $V(D) = B \cup C$, where $B = \{x_0, \ldots, x_m, w_0, \ldots, w_m\}$ and $C = \{y_0, \ldots, y_m, z_0, \ldots, z_m\}$. And define the arc set of $D$ as $A(D) = A(D[B]) \cup A(D[C]) \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $A(D[B]) \cup A(D[C]) = \{uv : \text{for any } u, v \in B \text{ or } C\}$, $M_1 = \{x_iy_{k \mod m + 1} : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 1\}$, $M_2 = \{w_iz_{k \mod m + 1} : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$, $M_3 = \{y_ix_{k \mod m + 1} : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$ and $M_4 = \{z_iw_{k \mod m + 1} : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$.

Clearly, $D$ is strong and there exists $0 \leq i, j \leq m$ such that $|N^+(x_i) \cap N^-(y_j)| = 4$ and $x_iy_j \notin A(D)$. And $\partial^+(B)$ is a 3-restricted edge cut with $|\partial^+(B)| = (2 + 3) \cdot (m + 1) = 5m + 5$. On the other hand, $\xi^3(D) = \xi(\{x_i, x_p, x_q\}) = |\partial^+(\{x_i, x_p, x_q\})| = 3 \cdot (2m + 3) - 6 = 6m + 3$, where $0 \leq i, p, q \leq m$. So $\lambda^3(D) \leq |\partial^+(B)| = 5m + 5 \leq 6m + 3 = \xi^3(D)$ for $m \geq 3$. Thus $D$ is not $\lambda^3$-optimal.

Besides, in $D$, there exists $0 \leq i, j \leq m$ such that $x_iy_j \notin A(D)$ and $d^+(x_i) + d^-(y_j) = 2 \cdot (2m + 3) = |V(D)| + 2 < |V(D)| + 3$, and $\delta(D) = 2m + 3 = \frac{|V(D)| + 3}{2}$. So this example also shows that the conditions that “$d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Corollary 6 and “$\delta(D) \geq \frac{|V(D)| + 3}{2}$” in Corollary 7 are sharp.

3. Proof of Theorem 2

We first introduce several useful lemmas.

Lemma 9 (Lemma 2.1 in [10]). Let $D$ be a strong digraph and $X_1, Y_1$ disjoint subsets of $V(D)$. If $D[X_1]$ contains a connected subdigraph with order at least $k$ and $D[Y_1]$ contains a strong subdigraph with order at least $k$, then $D$ is $\lambda^k$-connected and each arc set in $\{\partial^-(Y_1), \partial^+(Y_1)\} \cup \Omega(X_1)$ is a $k$-restricted arc cut of $D$.

Lemma 10. Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$.

Proof. By reason of symmetry, we only consider the case that $\delta^-(D) \geq 3$. Let $X'$ be a subset of $V(D)$ with $|X'| = 3$ such that $D[X']$ is connected and $\xi^3(D) = \xi(X')$. Without loss of generality, assume that $|X' \cap X| = 1$ and $|X' \cap Y| = 2$.

Let $X' \cap X = \{x\}$ and $X' \cap Y = \{y, z\}$. Let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D[X']$.

First, we claim that $V(D_1) \cap Y \neq \emptyset$. Otherwise, we have that $V(D_1) \subseteq X$ and $|V(D_1)| = 1$. Let $V(D_1) = \{u\}$. Then $N^-(u) \subseteq \{y, z\}$. So $3 \leq \delta^-(D) \leq d^-(u) = |N^-(u)| \leq |\{y, z\}| = 2$, a contradiction. Next, we aim to prove $|V(D_1)| \geq 3$. 


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Since \( N^-(v) \subseteq \{ x \} \cup (V(D_1) \cap X) \) for any \( v \in V(D_1) \cap Y \), we have \( 3 \leq \delta^-(D) \leq d^-(v) = |N^-(v)| \leq |\{x\} \cup (V(D_1) \cap X)| = |\{x\}| + |V(D_1) \cap X| = 1 + |V(D_1) \cap X| \). Thus \( |V(D_1) \cap X| \geq 2 \) and \( |V(D_1)| = |V(D_1) \cap X| + |V(D_1) \cap Y| \geq 2 + 1 = 3 \). It follows that \( |V(D_1)| \geq 3 \). Since \( D[X'] \) is connected and \( D[X'] \subseteq D \setminus V(D_1) \), by Lemma 9, each arc set in \( \Omega(X') \) is a 3-restricted arc cut of \( D \). Therefore, \( D \) is \( \lambda^3 \)-connected and \( \lambda^3(D) \leq \xi(X') = \xi^3(D) \).

**Lemma 11.** Let \( D = (X,Y,A(D)) \) be a strong bipartite digraph with \( \delta(D) \geq 3 \) or \( \delta^-(D) \geq 3 \), and let \( S = \partial^+(X') \) be a \( \lambda^3 \)-cut of \( D \), where \( X' \) is a subset of \( V(D) \). If \( D[X'] \) contains a connected subdigraph \( B \) with order 3 such that \( |N^+(x) \cap \overline{X}| \geq 2 \) for any \( x \in X' \setminus V(B) \) or \( D[\overline{X}] \) contains a connected subdigraph \( C \) with order 3 such that \( |N^-(y) \cap X'| \geq 2 \) for any \( y \in \overline{X} \setminus V(C) \), then \( D \) is \( \lambda^3 \)-optimal.

**Proof.** By Lemma 10, \( D \) is \( \lambda^3 \)-connected and \( \lambda^3(D) \leq \xi^3(D) \). By reason of symmetry, we only prove the case that \( D[X'] \) contains a connected subdigraph \( B \) with order 3 such that \( |N^+(x) \cap \overline{X}| \geq 2 \) for any \( x \in X' \setminus V(B) \). The hypotheses imply that
\[
\lambda^3(D) \leq |\partial^+(V(B))| = |\{V(B),X' \setminus V(B)\}| + |\{V(B),\overline{X}\}|
\leq 2|X' \setminus V(B)| + |\{V(B),\overline{X}\}| \leq \sum_{x \in X' \setminus V(B)} |N^+(x) \cap \overline{X}| + |\{V(B),\overline{X}\}|
= |\{X' \setminus V(B),\overline{X}\}| + |\{V(B),\overline{X}\}| = |X',\overline{X}| = |S| = \lambda^3(D).
\]

Thus \( \lambda^3(D) = \xi^3(D) \) and \( D \) is \( \lambda^3 \)-optimal.

By a proof similar to that of Lemma 4.1 shown in [10], we can get the following Lemma 12.

**Lemma 12.** Let \( D = (X,Y,A(D)) \) be a strong bipartite digraph with \( \delta(D) \geq 3 \), and let \( S \) be a \( \lambda^3 \)-cut of \( D \). If \( D \) is not \( \lambda^3 \)-optimal, then there exists a subset of vertices \( X' \subseteq V(D) \) such that \( S = \partial^+(X') \) and both induced subdigraphs \( D[X'] \) and \( D[\overline{X}] \) contain a connected subdigraph with order 3.

**Proof of Theorem 2.** Since \( |V(D)| \geq 6 \), for any \( u,v \in V(D) \) in the same partite, \( |N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3 \). Therefore \( D \) is strong and \( \delta(D) \geq 3 \).

By Lemma 10, \( D \) is \( \lambda^3 \)-connected and \( \lambda^3(D) \leq \xi^3(D) \). Suppose, on the contrary, that \( D \) is not \( \lambda^3 \)-optimal, that is, \( \lambda^3(D) < \xi^3(D) \). Let \( S \) be a \( \lambda^3 \)-cut of \( D \). Then by Lemma 12, there exists a subset of vertices \( X' \subseteq V(D) \) such that \( S = \partial^+(X') \) and both induced subdigraphs \( D[X'] \) and \( D[\overline{X}] \) contain a connected subdigraph with order 3.

Let \( \overline{X} = X'' \), and let \( X'_X = X' \cap X, \ X'_Y = X' \cap Y, \ X''_X = X'' \cap X \) and \( X''_Y = X'' \cap Y \). And let \( X'_{X_i} = \{ x \in X'_X : |N^+(x) \cap X''_i| = i \}, \ X'_{Y_i} = \{ y \in X'_Y : |N^+(y) \cap X''_i| = i \}, \ X''_{X_i} = \{ x \in X''_X : |N^-(x) \cap X'_i| = i \}, \ X''_{Y_i} = \{ y \in X''_Y : |N^-(y) \cap X'_i| = i \}, \ X''_{X_i} = \{ x \in X''_X : |N^-(x) \cap X'_i| = i \}, \ X''_{Y_i} = \{ y \in X''_Y : |N^-(y) \cap X'_i| = i \}, \ X''_{X_i} = \{ x \in X''_X : |N^-(x) \cap X'_i| = i \}, \ X''_{Y_i} = \{ y \in X''_Y : |N^-(y) \cap X'_i| = i \} \).
\(|N^-(y) \cap X'_i| = i\), \(i = 0, 1\), and \(X'_{X_2} = \{x \in X'_X : |N^+(x) \cap X''_Y| \geq 2\}\),
\(X'_{X_2} = \{y \in X'_X : |N^+(y) \cap X''_Y| \geq 2\}\), \(X''_Y = \{x \in X''_X : |N^-(x) \cap X'_Y| \geq 2\}\),
\(X''_Y = \{y \in X''_X : |N^-(y) \cap X'_Y| \geq 2\}\).

**Claim 1.** \(\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2\).

**Proof.** If, on the contrary \(|X'_X| = 1\), let \(X'_X = \{v\}\). Then \(|N(v) \cap X'_Y| \geq 2\) for \(D[X']\) contains a connected subgraph with order 3. Let \(y_1, y_2 \in N(v) \cap X'_Y\). Then \(D[y, y_1, y_2]\) is connected, and for any \(x' \in X'\setminus\{v, y_1, y_2\}\), \(N^+(x') \subseteq \{v\} \cup (N^+(x') \cap X''_Y)\), we have \(3 \leq \delta(D) \leq d^+(x') = |N^+(x')| \leq |\{v\}| + |N^+(x') \cap X''_Y| = 1 + |N^+(x') \cap X''_Y|\). Therefore \(|N^+(x') \cap X''_Y| \geq 2\). By Lemma 11, \(D\) is \(\lambda^3\)-optimal, a contradiction to our assumption. Thus \(|X'_X| \geq 2\). Similarly, we can prove that \(\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2\). \(\square\)

**Claim 2.** Either \(X''_{X_0} = \emptyset\) or \(X''_{X_0} = \emptyset\) and either \(X''_{Y_0} = \emptyset\) or \(X''_{Y_0} = \emptyset\).

**Proof.** If \(X''_{X_0} \neq \emptyset\) and \(X''_{X_0} \neq \emptyset\), then there exists \(x \in X''_{X_0} \subseteq X\) and \(\pi \in X''_{X_0} \subseteq X\) such that \(|N^+(x) \cap N^-(\pi)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1\). On the other hand, since \(x \in X''_{X_0}\) and \(\pi \in X''_{Y_0}\), \(N^+(x) \subseteq X'_Y\) and \(N^-(\pi) \subseteq X''_Y\), which implies that \(|N^+(x) \cap N^-(\pi)| = 0\), a contradiction. Thus either \(X''_{X_0} = \emptyset\) or \(X''_{X_0} = \emptyset\). Similarly, we can obtain that either \(X''_{Y_0} = \emptyset\) or \(X''_{Y_0} = \emptyset\). \(\square\)

We consider the following two cases.

**Case 1.** \(X''_{X_0} = X''_{Y_0} = \emptyset\) or \(X''_{X_0} = X''_{Y_0} = \emptyset\). By reason of symmetry, we only prove the case that \(X''_{X_0} = X''_{Y_0} = \emptyset\).

**Claim 1.1.** Either \(X'_{X_1} = \emptyset\) and \(X'_{Y_1} \neq \emptyset\) or \(X'_{X_1} \neq \emptyset\) and \(X'_{Y_1} = \emptyset\).

**Proof.** Since \(D\) is not \(\lambda^3\)-optimal, by Lemma 11, we have that \(X'_{X_1} \cup X'_{Y_1} \neq \emptyset\). Suppose \(X'_{X_1} \neq \emptyset\) and \(X'_{Y_1} \neq \emptyset\). Then \(x_1 \in X'_{X_1}\). Then for any \(\pi \in X''_{X_0}\), we have that \(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(\pi)| = |N^+(x_1) \cap N^-(\pi) \cap X'_Y| + |N^+(x_1) \cap N^-(\pi) \cap X''_Y| \leq |N^-(\pi) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^-(\pi) \cap X'_Y| + 1\). It implies that \(|N^-(\pi) \cap X'_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2\). So \(X'_Y \subseteq X''_{Y_2}\). By a similar proof, we can also prove that \(X'_X \subseteq X''_{X_2}\). Therefore \(D\) is \(\lambda^3\)-optimal by Lemma 11, a contradiction. The proof of Claim 1.1 is complete. \(\square\)

Without loss of generality, let \(X'_{X_1} \neq \emptyset\) and \(X'_{Y_1} = \emptyset\).

**Case 1.1.** \(|X'_{X_1}| = 1\). Let \(x_1 \in X'_{X_1}\). Then \(3 \leq \delta(D) \leq d^+(x_1) = |N^+(x_1) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap X'_Y| + 1\), therefore \(|N^+(x_1) \cap X'_Y| \geq 2\). Let \(y_1, y_2 \in N^+(x_1) \cap X'_Y\). Then \(D[x_1, y_1, y_2]\) is connected, and for any \(v \in X'\setminus\{x_1, y_1, y_2\}\), \(|N^+(v) \cap X''_Y| \geq 2\). By Lemma 11, \(D\) is \(\lambda^3\)-optimal, a contradiction.

**Case 1.2.** \(|X'_{X_1}| \geq 2\). Let \(x_1, x_2 \in X'_{X_1}\). Then \(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(x_2)| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + |N^+(x_1) \cap N^-(x_2) \cap X''_Y| \leq |N^+(x_1) \cap N^-(x_2) \cap X'_Y| \leq|
\[ N^-(x_2) \cap X'_Y + |N^+(x_1) \cap X''_Y| = \left| N^+(x_1) \cap N^-(x_2) \cap X'_Y \right| + 1. \] So \( |N^+(x_1) \cap N^-(x_2) \cap X'_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2. \) Let \( y_1 \in N^+(x_1) \cap N^-(x_2) \cap X'_Y. \) Then

\[
\xi^3(D) \leq \xi(\{x_1, x_2, y_1\}) \leq |\partial^+(\{x_1, x_2, y_1\})| = |\{x_1, x_2, y_1\}| = |\{x_1\}, X'_Y \{y_1\}| + |\{x_2\}, X'_Y \{y_1\}| + |\{x_2, x_1, y_1\}| + |\{y_1\}, X'_Y| \leq 2 \cdot (|X'_Y| - 1) + 2 \cdot |X'_X| - 2 + |\{y_1\}, X''_Y| \leq |S| = \lambda^3(D).
\]

Thus \( D \) is \( \lambda^3 \)-optimal, a contradiction.

**Case 2.** \( X'_{X_0} = X''_{Y_0} = \emptyset \) or \( X''_{X_0} = X'_Y = \emptyset. \) By reason of symmetry, we only prove the case that \( X'_{X_0} = X''_{Y_0} = \emptyset. \) Without loss of generality, we may assume that \( X'_{X_0} \neq \emptyset \) and \( X''_{X_0} \neq \emptyset. \) Otherwise, by Case 1, \( D \) is \( \lambda^3 \)-optimal, a contradiction. On the other hand, since for any \( u \in X'_{Y_0}, N^+(u) \subseteq X'_X, \) we have

\[
\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq \delta(D) \leq d^+(u) = |N^+(u)| \leq |X'_X|. \text{ Therefore } |X'_X| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1.
\]

Similarly, we can also prove that \( |X''_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1. \) Thus

\[
|X'_Y| + |X''_Y| = |V(D)| - |X'_X| - |X''_Y|
\]

\[
\leq |V(D)| - 2 \cdot \left( \left\lceil \frac{|V(D)|}{4} \right\rceil \right) = \frac{|V(D)|}{4} - 2.
\]

**Claim 2.1.** \( |X'_X| \geq |X'_Y| + 1 \) or \( |X''_Y| \geq |X''_X| + 1. \)

**Proof.** Otherwise, we have that \( |X'_Y| + |X''_Y| \geq |X'_X| + |X''_X| \geq 2 \cdot \left( \left\lceil \frac{|V(D)|}{4} \right\rceil \right) \geq \frac{|V(D)|}{2} + 2, \) a contradiction to (1).

Without loss of generality, we assume that \( |X'_X| \geq |X'_Y| + 1 \) in the following discussion.

**Claim 2.2.** \( |N^+(x) \cap X'_Y| \geq 3 \) and \( |N^-(y) \cap X'_X| \geq 3 \) for any \( x \in X'_X \) and \( y \in X'_Y. \)

**Proof.** By reason of symmetry, we only prove that for any \( x \in X'_X, |N^+(x) \cap X''_Y| \geq 3. \) Since \( X'_{X_0} \neq \emptyset, \) for any \( x \in X'_X \) and \( y \in X''_{X_0}, \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x) \cap N^-(y)| = |N^+(x) \cap N^-(y) \cap X'_Y| + |N^+(x) \cap N^-(y) \cap X''_Y| \leq |N^-(y) \cap X'_Y| + |N^+(x) \cap X''_Y| = |N^+(x) \cap X''_Y|, \) so \( |N^+(x) \cap X''_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3. \)

**Claim 2.3.** \( X'_{Y_2} = X''_{X_2} = \emptyset. \)

**Proof.** Here, we only prove that \( X'_{Y_2} = \emptyset. \) The proof of the statement that \( X''_{X_2} = \emptyset \) is similar. Suppose, by a contradiction, there exists \( y \in X'_{Y_2}. \) Let
$x_1, x_2 \in N^+(y) \cap X'_X$. Then

$$\xi^3(D) \leq \xi([x_1, x_2, y]) \leq |\partial^+([y]) \cup \partial^-(\{x_1, x_2\})|$$

$$= |\partial^+([y])| + |\partial^-(\{x_1, x_2\})| - 2 = |\{y\}, X'_X| + |\{y\}, X''_X|$$

$$+ |[X'_Y, \{x_1\}]| + |[X''_Y, \{x_1\}]| + |[X'_Y, \{x_2\}]| + |[X''_Y, \{x_2\}]| - 2$$

$$\leq |X'_X| + |\{y\}, X'_X| + |[X'_Y, \{x_1\}]| + 2|X''_Y| + |[X'_Y, \{x_2\}]| - 2$$

$$\leq 3 \max \{|X'_X|, |X''_X|\} + |\{y\}, X'_X| + |[X'_Y, \{x_1\}]|$$

$$+ |[X'_Y, \{x_2\}]| - 2 \leq |S| = \lambda^3(D).$$

So $D$ is $\lambda^3$-optimal, a contradiction.

\[\square\]

Claim 2.4. For any $x \in X'_X$, $|N(X) \cap X'_X| \geq 2$.

**Proof.** Let $X'_Y = \{y_1, y_2, \ldots, y_p\}$ and let $S^* = \{s^* : s^* \in N^+(y_i) \cap N^-(y_j) \cap X'_X$, where $i, j \in \{1, \ldots, p\}$ and $i \neq j\}$. Then $D[S^* \cup X'_Y]$ is strong. Besides, by Claim 2.3, we have that for any $i, j \in \{1, \ldots, p\}$ and $i \neq j$, $y_i, y_j \in X'_X \cup X'_Y$.

Therefore $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(y_i) \cap N^-(y_j)| = |N^+(y_i) \cap N^-(y_j) \cap X'_X| + |N^+(y_i) \cap X''_X| \leq |N^+(y_i) \cap N^-(y_j) \cap X'_X| + |N^+(y_i) \cap X'_X| \leq |N^+(y_i) \cap N^-(y_j) \cap X'_X| + 1$. So $|N^+(y_i) \cap N^-(y_j) \cap X'_X| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2$. Similarly, we can prove that $|N^+(y_j) \cap N^-(y_i) \cap X'_X| \geq 2$. On the other hand, since $|X'_Y| \geq 2$, we have $|S^* \cup X'_Y| \geq 3$. For any $x \in S^*$, clearly, $|N(x) \cap X'_X| \geq 2$. Next, we claim that for any $x \in X'_X \setminus S^*$, $|N(x) \cap X'_X| \leq |[X'_Y, \{x\}]|$. Suppose there exists $x^* \in X'_X \setminus S^*$ such that $|N^+(x^*) \cap X'_X| > |[X'_Y, \{x^*\}]|$. Since $D[S^* \cup X'_Y]$ is strong and $|S^* \cup X'_Y| \geq 3$, we have $X'_Y \setminus \{x^*\}$ is a 3-restricted edge cut. Therefore $|\partial^+(X'_Y \setminus \{x^*\})| = |S| - |N^+(x^*) \cap X'_X| + |X'_Y \setminus \{x^*\}| < |S|$, a contradiction to the minimality of $S$. Thus $|[X'_Y, \{x\}]| \geq |N^+(x) \cap X'_X|$. By Claim 2.2, we have that $|[X'_Y, \{x\}]| \geq 3$. The proof of Claim 2.4 is complete. \[\square\]

Let $x_1 \in X'_X$ such that $|N^+(x_1) \cap X''_X| \leq |N^+(x) \cap X'_Y|$ for any $x \in X'_X$, and let $y_1, y_2 \in N(x_1) \cap X'_Y$. Then

$$\xi^3(D) \leq |\partial^+([x_1, y_1, y_2])| = |\{x_1, y_1, y_2\}, X'_X \setminus \{x_1, y_1, y_2\}| + |\{x_1, y_1, y_2\}, X''_X|$$

$$\leq 2(|X'_X| - 1) + |X'_X| - 2 + |\{x_1\}, X''_X| + |\{y_1\}, X''_X| + |\{y_2\}, X''_X|$$

$$\leq 3|X'_X| - 5 + |\{x_1\}, X''_X| + |\{y_1\}, X''_X|$$

$$+ |\{y_2\}, X''_X|(|X'_X| \geq |X'_X| + 1) \leq |\{x_1\}, X''_X| \times |X'_X|$$

$$+ |\{y_1\}, X''_X| + |\{y_2\}, X''_X| \leq |S| = \lambda^3(D).$$

So $D$ is $\lambda^3$-optimal, a contradiction.

The proof is complete. \[\blacksquare\]
From Theorem 2, we have following corollaries.

**Corollary 13.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$. If for any $u, v \in V(D)$ in the same partite, $d^+(u) + d^-(v) \geq |V(D)| - 1$, then $D$ is $\lambda^3$-optimal.

**Corollary 14.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \left\lceil \frac{|V(D)|}{2} \right\rceil$, then $D$ is $\lambda^3$-optimal.

![Figure 1. The example from Remark 15.](image)

(Unordered edges represent two arcs with the same end-vertices and opposite directions.)

**Remark 15.** To show that the condition “$|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite” in Theorem 2 is sharp, we consider the digraph $T$ shown in Figure 1. Clearly, $|V(D)| \geq 6$ and $D$ is strong. There exists $x_1, y_1$ in the same partite such that $|N^+(x_1) \cap N^-(y_1)| = 2 < 3 = \left\lceil \frac{|V(T)|}{4} \right\rceil + 1$. Clearly, $\partial^+\{x_1, x_2, x_3, x_4\}$ is a 3-restricted edge cut and $\xi^3(T) = |\partial^+\{x_1, x_2, x_3\}| = 5$. Therefore, $\lambda^3(T) \leq |\partial^+\{x_1, x_2, x_3, x_4\}| = 4 < 5 = \xi^3(T)$ and $T$ is not $\lambda^3$-optimal.

Besides, since $d^+(x_3) + d^-(y_4) = 6 < 7 = |V(T)| - 1$ and $\delta(T) = 3 < 4 = \left\lceil \frac{|V(D)|}{2} \right\rceil$, this example also shows that the conditions “$d^+(u) + d^-(v) \geq |V(D)| - 1$ for any $u, v \in V(D)$ in the same partite” in Corollary 13 and “$\delta(D) \geq \left\lceil \frac{|V(D)|}{2} \right\rceil$ ” in Corollary 14 are sharp.

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**References**


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