ON THE OPTIMALITY OF 3-RESTRICTED ARC CONNECTIVITY FOR DIGRAPHS AND BIPARTITE DIGRAPHS

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Abstract

Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ has a strong component $D'$ with order at least $k$ such that $D\setminus V(D')$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists in $D$, then $D$ is called $\lambda^k$-connected. For a $\lambda^k$-connected digraph $D$, the $k$-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all $k$-restricted arc cuts of $D$. It is known that for many digraphs $\lambda^k(D) \leq \xi^k(D)$, where $\xi^k(D)$ denotes the minimum $k$-degree of $D$. $D$ is called $\lambda^k$-optimal if $\lambda^k(D) = \xi^k(D)$. In this paper, we will give some sufficient conditions for digraphs and bipartite digraphs to be $\lambda^3$-optimal.

Keywords: restricted arc-connectivity, bipartite digraph, optimality, digraph, network.

2010 Mathematics Subject Classification: 05C40, 68R10.

1. Introduction

It is well-known that the network can be modelled as a digraph $D$ with vertices $V(D)$ representing sites and arcs $A(D)$ representing links between sites of the network. Let $v \in V(D)$, the out-neighborhood of $v$ is the set $N^+(v) = \{x \in V(D) : vx \in A(D)\}$ and the out-degree of $v$ is $d^+(v) = |N^+(v)|$. The in-neighborhood of $v$ is the set $N^-(v) = \{x \in V(D) : xv \in A(D)\}$ and the in-degree of $v$ is $d^-(v) = |N^-(v)|$. The neighborhood of $v$ is $N(v) = N^+(v) \cup N^-(v)$.

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Let $\delta^+(D), \delta^-(D)$ and $\delta(D)$ denote, respectively, the minimum out-degree, the minimum in-degree and the minimum degree of $D$.

For a pair nonempty vertex sets $X$ and $Y$ of $D$, $[X,Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Specially, if $Y = \overline{X}$, where $\overline{X} = V(D) \setminus X$, then we write $\partial^+(X)$ or $\partial^-(Y)$ instead of $[X,Y]$. For $X \subseteq V(D)$, the subdigraph of $D$ induced by $X$ is denoted by $D[X]$. The underlying graph $U(D)$ of $D$ is the unique graph obtained from $D$ by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. $D$ is connected if $U(D)$ is connected and $D$ is strongly connected (or, just, strong) if there exists a directed $(x,y)$-path and a directed $(y,x)$-path for any $x, y \in V(D)$. We define a digraph with one vertex to be strong. A connected (strong) component of $D$ is a maximal induced subdigraph of $D$ which is connected (strong). If $D$ has $p$ strong components, then these strong components can be labeled $D_1, \ldots, D_p$ such that there is no arc from $D_j$ to $D_i$ unless $j < i$. We call such an ordering an acyclic ordering of the strong components of $D$.

In a strong digraph $D$, we often use arc connectivity of $D$ to measure the reliability. An arc set $S$ is a arc cut of $D$ if $D - S$ is not strong. The arc connectivity $\lambda(D)$ of $D$ is the minimum cardinality over all arc cuts of $D$. The arc cut $S$ of $D$ with cardinality $\lambda(D)$ is called a $\lambda$-cut. Whitney’s inequality shows $\lambda(D) \leq \delta(D)$. A strong digraph $D$ with $\lambda(D) = \delta(D)$ is called $\lambda$-optimal. However, only using arc connectivity to measure the reliability is not enough. In [12], Volkmann introduced the concept of restricted arc connectivity. An arc subset $S$ of $D$ is a restricted arc cut if $D - S$ has a strong component $D'$ with order at least 2 such that $D \setminus V(D')$ contains an arc. If such an arc cut exists in $D$, then $D$ is called $\lambda'$-connected. For a $\lambda'$-connected digraph $D$, the restricted arc connectivity, denoted by $\lambda'(D)$, is the minimum cardinality over all restricted arc cuts of $D$. The restricted arc cut $S$ of $D$ with cardinality $\lambda'(D)$ is called a $\lambda'$-cut. In [13], Wang and Lin introduced the notion of minimum arc degree. Let $xy \in A(D)$. Then

$$\Omega(\{x,y\}) = \{\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^+(\{x\}) \cup \partial^-(\{y\}), \partial^+(\{y\}) \cup \partial^-(\{x\})\}. $$

The arc degree of $xy$ is $\xi'(xy) = \min\{\{S : S \in \Omega(\{x,y\})\}$ and the minimum arc degree of $D$ is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$.

It was proved in [3, 13] that for many $\lambda'$-connected digraphs, $\xi'(D)$ is an upper bound of $\lambda'(D)$. In [13], Wang and Lin introduced the concept of $\lambda'$-optimality. A $\lambda'$-connected digraph $D$ with $\xi'(D) = \lambda'(D)$ is called $\lambda'$-optimal. As a generalization of restricted arc connectivity, in [10], Lin et al. introduced the concept of $k$-restricted arc connectivity.

**Definition** [10]. Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ has a strong component $D'$ with order at least $k$ such that
$D \setminus V(D')$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists in $D$, then $D$ is called $\lambda^k$-connected. For a $\lambda^k$-connected digraph $D$, the $k$-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all $k$-restricted arc cuts of $D$. The $k$-restricted arc cut $S$ of $D$ with cardinality $\lambda^k(D)$ is called a $\lambda^k$-cut.

**Definition** [10]. Let $D$ be a strong digraph. For any $X \subseteq V(D)$, let $\Omega(X) = \{\partial^+(X_1) \cup \partial^-(X \setminus X_1) : X_1 \subseteq X\}$ and $\xi(X) = \min\{|S| : S \in \Omega(X)\}$. Define the minimum $k$-degree of $D$ to be

$$\xi^k(D) = \min\{|\xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected}\}.$$  

Clearly, $\lambda^1(D) = \lambda(D)$, $\lambda^2(D) = \lambda'(D)$, $\xi^1(D) = \delta(D)$ and $\xi^2(D) = \xi'(D)$. Let $D$ be a $\lambda^k$-connected digraph, where $k \geq 2$. Then $D$ is $\lambda^{k-1}$-connected and $\lambda^{k-1}(D) \leq \lambda^k(D)$. It was shown in [10] that $\xi^k(D)$ is an upper bound of $\lambda^k(D)$ for many digraphs. And a $\lambda^k$-connected digraph $D$ with $\lambda^k(D) = \xi^k(D)$ is called $\lambda^k$-optimal.

The research on the $\lambda^k$-optimality of digraph $D$ is considered to be a hot issue. In [11], Hellwig and Volkmann concluded many sufficient conditions for digraphs to be $\lambda$-optimal. Besides, sufficient conditions for digraphs to be $\lambda^k$-optimal were also given by several authors, for example by Balbuena *et al.* [1–4], Chen *et al.* [5,6], Grüter and Guo [7,8], Liu and Zhang [9], Volkmann [12] and Wang and Lin [13]. However, closely related conditions for $\lambda^3$-optimal digraphs have received little attention until recently. In [10], Lin *et al.* gave some sufficient conditions for digraphs to be $\lambda^3$-optimal. In this paper, we will give some sufficient conditions for digraphs to be $\lambda^3$-optimal. As corollaries, degree conditions or degree sum conditions for a digraph or a bipartite digraph to be $\lambda^3$-optimal are given. The main contributions in this paper are as following.

**Theorem 1.** Let $D$ be a digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then $D$ is $\lambda^3$-optimal.

**Theorem 2.** Let $D = (X, Y, A(D))$ be a bipartite digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite, then $D$ is $\lambda^3$-optimal.

2. Proof of Theorem 1

We first introduce three useful lemmas.

**Lemma 3** (Theorem 1.4 in [10]). Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$. Then $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$.
Lemma 4. Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$, and let $S = \partial^+(X)$ be a $\lambda_k$-cut of $D$, where $X$ is a subset of $V(D)$. If $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$ or $D[\overline{X}]$ contains a connected subdigraph $C$ with order $k$ such that $|N^-(y) \cap X| \geq k$ for any $y \in \overline{X} \setminus V(C)$, then $D$ is $\lambda_k$-optimal.

Proof. By Lemma 3, $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$. By reason of symmetry, we only prove the case that $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$. The hypotheses imply that

\[
\xi^k(D) \leq |\partial^+(V(B))| = |(V(B), X \setminus V(B))| + |(V(B), \overline{X})|\]

\[
\leq k|X \setminus V(B)| + |(V(B), \overline{X})| \leq \sum_{x \in X \setminus V(B)} |N^+(x) \cap X| + |(V(B), \overline{X})|\]

\[
= |(X \setminus V(B), \overline{X})| + |(V(B), \overline{X})| = ||X, \overline{X}|| = |S| = \lambda^k(D).
\]

Thus $\lambda^k(D) = \xi^k(D)$ and $D$ is $\lambda^k$-optimal.

Lemma 5 (Lemma 4.1 in [10]). Let $D$ be a strong digraph with $|V(D)| \geq 6$ and $\delta(D) \geq 4$, and let $S$ be a $\lambda^3$-cut of $D$. If $D$ is not $\lambda^3$-optimal, then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Proof of Theorem 1. Clearly, $D$ is a strong digraph with $\delta(D) \geq 5$. By Lemma 3, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that $D$ is not $\lambda^3$-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let $S$ be a $\lambda^3$-cut of $D$. By Lemma 5, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $Y = \overline{X}$, and let $X_i = \{x \in X : |N^+(x) \cap Y| = i\}, Y_i = \{y \in Y : |N^-(y) \cap X| = i\}, i = 0, 1, 2$, and let $X_3 = \{x \in X : |N^+(x) \cap Y| \geq 3\}, Y_3 = \{y \in Y : |N^-(y) \cap X| \geq 3\}$.

Claim 1. $\min\{|X|, |Y|\} \geq 4$.

Proof. Suppose that $|X| = 3$. Then $\lambda^3(D) = |S| = |\partial^+(X)| \geq \xi(X) \geq \xi^3(D)$, contrary to the assumption. Suppose that $|Y| = 3$. Then $\lambda^3(D) = |S| = |\partial^-(Y)| \geq \xi(Y) \geq \xi^3(D)$, contrary to the assumption. Claim 1 follows.

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. For the reason of symmetry, we only prove that $X_0 = \emptyset$ by contradiction. Suppose $X_0 \neq \emptyset$ and let $x \in X_0$. Then for any $\overline{x} \in Y$, $x\overline{x} \notin A(D)$ and we have that $5 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X| + |N^+(x) \cap N^-(\overline{x}) \cap Y| \leq$
\[ |N^{-}(x) \cap X| + |N^{+}(x) \cap Y| = |N^{-}(\pi) \cap X| \]. It implies that \(|N^{-}(\pi) \cap X| \geq 5\). Therefore \(Y \subseteq Y_{3}\). So \(D\) is \(\lambda^{3}\)-optimal by Lemma 4, a contradiction to our assumption.

Combining Claim 2 with Lemma 4, we have that \(Y_{1} \cup Y_{2} \neq \emptyset\) and \(X_{1} \cup X_{2} \neq \emptyset\). Otherwise we will obtain that \(D\) is \(\lambda^{3}\)-optimal, which is a contradiction. Next, we consider two cases.

Case 1. \(X_{1} \neq \emptyset\). Let \(x' \in X_{1}\) and suppose \(N^{+}(x') \cap Y = \{y'\}\). Then for any \(y \in Y \setminus \{y'\}\), \(x'y \notin A(D)\), so we have that \(5 \leq |N^{+}(x') \cap N^{-}(y)| = |N^{+}(x') \cap N^{-}(y) \cap X| + |N^{+}(x') \cap N^{-}(y) \cap Y| \leq |N^{-}(y) \cap X| + |N^{+}(x') \cap Y| = |N^{-}(y) \cap X| + 1\). So \(|N^{-}(y) \cap X| \geq 4\) and \(Y \setminus \{y'\} \subseteq Y_{3}\). On the other hand, since \(Y_{1} \cup Y_{2} \neq \emptyset\), so \(y' \in Y_{1} \cup Y_{2}\). Besides, \(5 \leq \delta(D) \leq \delta^{-}(y') = |N^{-}(y')| = |N^{-}(y') \cap Y| + |N^{-}(y') \cap X| \leq |N^{-}(y') \cap Y| + 2\), thus \(|N^{-}(y') \cap Y| \geq 3\). Let \(y_{1}, y_{2} \in N^{-}(y') \cap Y\), then \(D[y', y_{1}, y_{2}]\) is connected and \(|N^{-}(y) \cap X| \geq 4\) for any \(y \in Y \setminus \{y', y_{1}, y_{2}\}\). By Lemma 4, we have that \(D\) is \(\lambda^{3}\)-optimal, a contradiction.

Case 2. \(X_{2} \neq \emptyset\). Let \(x' \in X_{2}\) and suppose \(N^{+}(x') \cap Y = \{y', y''\}\). Then for any \(y \in Y \setminus \{y', y''\}\), \(x'y \notin A(D)\), thus \(5 \leq |N^{+}(x') \cap N^{-}(y)| = |N^{+}(x') \cap N^{-}(y) \cap X| + |N^{+}(x') \cap N^{-}(y) \cap Y| \leq |N^{-}(y) \cap X| + |N^{+}(x') \cap Y| = |N^{-}(y) \cap X| + 2\). So \(|N^{-}(y) \cap X| \geq 3\) and \(Y \setminus \{y', y''\} \subseteq Y_{3}\). On the other hand, since \(Y_{1} \cup Y_{2} \neq \emptyset\), \(y' \in Y_{1} \cup Y_{2}\) or \(y'' \in Y_{1} \cup Y_{2}\). If \(|Y_{1} \cup Y_{2}| = 1\), then we can prove that \(D\) is \(\lambda^{3}\)-optimal by a proof similar to Case 1, which is a contradiction. If \(Y_{1} \cup Y_{2} = \{y', y''\}\), then we consider two subcases.

Subcase 2.1. \(y'y'' \in A(D)\) or \(y''y' \in A(D)\). Since \(y'' \in Y_{1} \cup Y_{2}\) and \(\delta(D) \geq 5\), then there exists \(y_{1} \in N^{-}(y'') \cap Y\) such that \(y_{1} \neq y'\). Therefore \(D[y', y'', y_{1}]\) is connected and \(|N^{-}(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_{1}\}\). By Lemma 4, we have that \(D\) is \(\lambda^{3}\)-optimal, a contradiction.

Subcase 2.2. \(y'y'' \notin A(D)\) and \(y''y' \notin A(D)\). Since \(y'y'' \notin A(D)\) and \(y''y' \notin A(D)\), then \(5 \leq |N^{+}(y') \cap N^{-}(y'') \cap X| + |N^{+}(y') \cap N^{-}(y'') \cap Y| \leq |N^{-}(y'') \cap X||N^{+}(y') \cap N^{-}(y'') \cap Y| \leq 2 + |N^{+}(y') \cap N^{-}(y'') \cap Y|\). Therefore \(|N^{+}(y') \cap N^{-}(y'') \cap Y| \geq 3\). Let \(y_{1} \in N^{+}(y') \cap N^{-}(y'') \cap Y\). Then \(D[y', y'', y_{1}]\) is connected and \(|N^{-}(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_{1}\}\). By Lemma 4, we have that \(D\) is \(\lambda^{3}\)-optimal, a contradiction.

The proof is complete.

From Theorem 1, we have following corollaries.

**Corollary 6.** Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(d^{+}(u) + d^{-}(v) \geq |V(D)| + 3\) for any \(u, v \in V(D)\) with \(uv \notin A(D)\), then \(D\) is \(\lambda^{3}\)-optimal.

**Corollary 7** (Theorem 1.7 in [10]). Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(\delta(D) \geq \frac{|V(D)|+3}{2}\), then \(D\) is \(\lambda^{3}\)-optimal.
Remark 8. To show the condition that “$|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Theorem 1 is sharp, we give a class of digraphs. Let $m, k$ be positive integers with $m \geq 3$, and let $D$ be a digraph with $|V(D)| = 4m + 4$. Define the vertex set of $D$ as $V(D) = B \cup C$, where $B = \{x_0, \ldots, x_m, w_0, \ldots, w_m\}$ and $C = \{y_0, \ldots, y_m, z_0, \ldots, z_m\}$. And define the arc set of $D$ as $A(D) = A(D[B]) \cup A(D[C]) \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $A(D[B]) \cup A(D[C]) = \{uv: \text{for any } u, v \in B \text{ or } C\}$, $M_1 = \{x_0 y_{i (\text{mod } m + 1)}: 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 1\}$, $M_2 = \{w_i z_{i (\text{mod } m + 1)}: 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$, $M_3 = \{y_i x_{i (\text{mod } m + 1)}: 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$, and $M_4 = \{z_i w_{i (\text{mod } m + 1)}: 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$.

Clearly, $D$ is strong and there exists $0 \leq i, j \leq m$ such that $|N^+(x_i) \cap N^-(y_j)| = 4$ and $x_i y_j \notin A(D)$. And $\delta^+(B)$ is a 3-restricted edge cut with $|\delta^+(B)| = (2 + 3)(m + 1) = 5m + 5$. On the other hand, $\xi^3(D) = \xi(\{x_1, x_p, x_q\}) = |\delta^+(\{x_1, x_p, x_q\})| = 3 \cdot (2m + 3) - 6 = 6m + 3$, where $0 \leq l, p, q \leq m$. So $\lambda^3(D) \leq |\delta^+(B)| = 5m + 5 < 6m + 3 = \xi^3(D)$ for $m \geq 3$. Thus $D$ is not $\lambda^3$-optimal.

Besides, in $D$, there exists $0 \leq i, j \leq m$ such that $x_i y_j \notin A(D)$ and $d^+(x_i) + d^-(y_j) = 2 \cdot (2m + 3) = |V(D)| + 2 < |V(D)| + 3$, and $\delta(D) = 2m + 3 = \frac{|V(D)| + 3}{2}$. So this example also shows that the conditions that “$d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Corollary 6 and “$\delta(D) \geq \frac{|V(D)| + 3}{2}$” in Corollary 7 are sharp.

3. Proof of Theorem 2

We first introduce several useful lemmas.

Lemma 9 (Lemma 2.1 in [10]). Let $D$ be a strong digraph and $X_1, Y_1$ disjoint subsets of $V(D)$. If $D[X_1]$ contains a connected subdigraph with order at least $k$ and $D[Y_1]$ contains a strong subdigraph with order at least $k$, then $D$ is $\lambda^k$-connected and each arc set in $\{\delta^+(Y_1), \delta^+(Y_1)\} \cup \Omega(X_1)$ is a $k$-restricted arc cut of $D$.

Lemma 10. Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$.

Proof. By reason of symmetry, we only consider the case that $\delta^-(D) \geq 3$. Let $X'$ be a subset of $V(D)$ with $|X'| = 3$ such that $D[X']$ is connected and $\xi^3(D) = \xi(X')$. Without loss of generality, assume that $|X' \cap X| = 1$ and $|X' \cap Y| = 2$. Let $X' \cap X = \{x\}$ and $X' \cap Y = \{y, z\}$. Let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D[X']$.

First, we claim that $V(D_1) \cap Y \neq \emptyset$. Otherwise, we have that $V(D_1) \subseteq X$ and $|V(D_1)| = 1$. Let $V(D_1) = \{u\}$. Then $N^-(u) \subseteq \{y, z\}$. So $3 \leq \delta^-(D) \leq d^-(u) = |N^-(u)| \leq |\{y, z\}| = 2$, a contradiction. Next, we aim to prove $|V(D_1)| \geq 3$. 


Since \( N^-(v) \subseteq \{ x \} \cup (V(D_1) \cap X) \) for any \( v \in V(D_1) \cap Y \), we have \( 3 \leq \delta^-(D) \leq d^-(w) = |N^-(w)| \leq |\{x\} \cup (V(D_1) \cap X)| = |\{x\}| + |V(D_1) \cap X| = 1 + |V(D_1) \cap X| \). Thus \(|V(D_1) \cap X| \geq 2\) and \(|V(D_1)| = |V(D_1) \cap X| + |V(D_1) \cap Y| \) \( \geq 2 + 1 = 3\). It follows that \(|V(D_1)| \geq 3\). Since \( D[X'] \) is connected and \( D[X'] \subseteq D \setminus V(D_1) \), by Lemma 9, each arc set in \( \Omega(X') \) is a 3-restricted arc cut of \( D \). Therefore, \( D \) is \( \lambda^3 \)-connected and \( \lambda^3(D) \leq \xi(X') = \xi^3(D) \). \( \blacksquare \)

**Lemma 11.** Let \( D = (X, Y, A(D)) \) be a strong bipartite digraph with \( \delta^+(D) \geq 3 \) or \( \delta^-(D) \geq 3 \), and let \( S = \partial^+(X') \) be a \( \lambda^3 \)-cut of \( D \), where \( X' \) is a subset of \( V(D) \). If \( D[X'] \) contains a connected subdigraph \( B \) with order 3 such that \( |N^+(x) \cap \overline{X}| \geq 2 \) for any \( x \in X' \setminus V(B) \) or \( D[\overline{X}] \) contains a connected subdigraph \( C \) with order 3 such that \( |N^-(y) \cap X'| \geq 2 \) for any \( y \in \overline{X} \cap V(C) \), then \( D \) is \( \lambda^3 \)-optimal. 

**Proof.** By Lemma 10, \( D \) is \( \lambda^3 \)-connected and \( \lambda^3(D) \leq \xi^3(D) \). By reason of symmetry, we only prove the case that \( D[X'] \) contains a connected subdigraph \( B \) with order 3 such that \( |N^+(x) \cap \overline{X}| \geq 2 \) for any \( x \in X' \setminus V(B) \). The hypotheses imply that

\[
\xi^3(D) \leq |\partial^+(V(B))| = |\{V(B), X' \setminus V(B)\}| + |\{V(B), \overline{X}\}|
\]
\[
\leq 2|X' \setminus V(B)| + |\{V(B), \overline{X}\}| \leq \sum_{x \in X' \setminus V(B)} |N^+(x) \cap \overline{X}| + |\{V(B), \overline{X}\}|
\]
\[
= |\{X' \setminus V(B), \overline{X}\}| + |\{V(B), \overline{X}\}| = |X', \overline{X}| = |S| = \lambda^3(D).
\]

Thus \( \lambda^3(D) = \xi^3(D) \) and \( D \) is \( \lambda^3 \)-optimal. \( \blacksquare \)

By a proof similar to that of Lemma 4.1 shown in [10], we can get the following Lemma 12.

**Lemma 12.** Let \( D = (X, Y, A(D)) \) be a strong bipartite digraph with \( \delta(D) \geq 3 \), and let \( S \) be a \( \lambda^3 \)-cut of \( D \). If \( D \) is not \( \lambda^3 \)-optimal, then there exists a subset of vertices \( X' \subset V(D) \) such that \( S = \partial^+(X') \) and both induced subdigraphs \( D[X'] \) and \( D[\overline{X}] \) contain a connected subdigraph with order 3.

**Proof of Theorem 2.** Since \(|V(D)| \geq 6\), for any \( u, v \in V(D) \) in the same partite, \(|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3\). Therefore \( D \) is strong and \( \delta(D) \geq 3 \). By Lemma 10, \( D \) is \( \lambda^3 \)-connected and \( \lambda^3(D) \leq \xi^3(D) \). Suppose, on the contrary, that \( D \) is not \( \lambda^3 \)-optimal, that is, \( \lambda^3(D) \leq \xi^3(D) \). Let \( S \) be a \( \lambda^3 \)-cut of \( D \). Then by Lemma 12, there exists a subset of vertices \( X' \subset V(D) \) such that \( S = \partial^+(X') \) and both induced subdigraphs \( D[X'] \) and \( D[\overline{X}] \) contain a connected subdigraph with order 3.

Let \( \overline{X} = X'' \), and let \( X'_X = X' \cap X, X'_Y = X' \cap Y, X''_X = X'' \cap X \) and \( X''_Y = X'' \cap Y \). And let \( X'_X = \{x \in X'_X : |N^+(x) \cap X''_X| = i\} \), \( X'_Y = \{y \in X'_Y : |N^+(y) \cap X''_Y| = i\} \), \( X''_X = \{x \in X''_X : |N^-(x) \cap X''_Y| = i\} \), \( X''_Y = \{y \in X''_Y : |N^-(y) \cap X''_Y| = i\} \):
Claim 1. \[ \min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2. \]

**Proof.** If, on the contrary \(|X'_X| = 1\), let \(X'_X = \{v\} \). Then \(|N(v) \cap X'_V| \geq 2\) for \(D[X']\) contains a connected subdigraph with order 3. Let \(y_1, y_2 \in N(v) \cap X'_Y\). Then \(D[y_1, y_2]\) is connected, and for any \(x' \in X'(\{v, y_1, y_2\}, N^+(x') \subseteq \{v\} \cup (N^+(x') \cap X'')\), we have \(3 \leq |D| \leq |x'| = |N^+(x')| \leq |\{v\}| + |N^+(x') \cap X''| = 1 + |N^+(x') \cap X''| \geq 2\). By Lemma 11, \(D\) is \(\lambda^3\)-optimal, a contradiction to our assumption. Thus \(|X'_X| \geq 2\). Similarly, we can prove that \(\min\{|X'_Y|, |X''_X|, |X''_Y|\} \geq 2\). □

Claim 2. Either \(X''_X = \emptyset\) or \(X''_X = \emptyset\) and either \(X'_Y = \emptyset\) or \(X''_Y = \emptyset\).

**Proof.** If \(X''_X \neq \emptyset\) and \(X''_X \neq \emptyset\), then there exists \(x \in X''_X \subseteq X\) and \(x \in X''_X \subseteq X\) such that \(|N^+(x) \cap N^-(x)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1\). On the other hand, since \(x \in X''_X \subseteq X\) and \(x \in X''_X \subseteq X\) and \(N^+(x) \subseteq X'_Y\) and \(N^-(x) \subseteq X''_Y\), which implies that \(N^+(x) \cap N^-(x) = \emptyset\), a contradiction. Thus either \(X''_X = \emptyset\) or \(X''_X = \emptyset\).

Similarly, we can obtain that either \(X'_Y = \emptyset\) or \(X''_Y = \emptyset\). □

We consider the following two cases.

**Case 1.** \(X''_X = X''_X = \emptyset\) or \(X''_X = X''_X = \emptyset\). By reason of symmetry, we only prove the case that \(X''_X = X''_X = \emptyset\).

**Claim 1.1.** Either \(X'_{X_1} = \emptyset\) and \(X'_Y = \emptyset\) or \(X''_{X_1} = \emptyset\) and \(X''_Y = \emptyset\).

**Proof.** Since \(D\) is not \(\lambda^3\)-optimal, by Lemma 11, we have that \(X'_{X_1} \cup X'_Y \neq \emptyset\). Suppose \(X'_{X_1} \neq \emptyset\) and \(X'_Y \neq \emptyset\). Take \(x_1 \in X'_{X_1}\). Then for any \(\bar{x} \in X''_X\), we have that \(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(\bar{x})| = |N^+(x_1) \cap N^-(\bar{x}) \cap X'_Y| + |N^+(x_1) \cap N^-(\bar{x}) \cap X''_Y| \leq |N^-(\bar{x}) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^-(\bar{x}) \cap X'_Y| + 1\). It implies that \(|N^-(\bar{x}) \cap X'_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2\). So \(X'_Y \subseteq X''_Y\). By a similar proof, we can also prove that \(X''_X \subseteq X''_X\). Therefore \(D\) is \(\lambda^3\)-optimal by Lemma 11, a contradiction. The proof of Claim 1.1 is complete. □

Without loss of generality, let \(X'_{X_1} \neq \emptyset\) and \(X'_Y = \emptyset\).

**Case 1.1.** \(|X'_{X_1}| = 1\). Let \(x_1 \in X'_{X_1}\). Then \(3 \leq \delta(D) \leq d^+(x_1) = |N^+(x_1)| = |N^+(x_1) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap X'_Y| + 1\), therefore \(|N^+(x_1) \cap X'_Y| \geq 2\). Let \(y_1, y_2 \in N^+(x_1) \cap X'_Y\). Then \(D[y_1, y_2]\) is connected, and for any \(v \in X' \setminus \{x_1, y_1, y_2\}, |N^+(v) \cap X''_Y| \geq 2\). By Lemma 11, \(D\) is \(\lambda^3\)-optimal, a contradiction.

**Case 1.2.** \(|X'_{X_1}| \geq 2\). Let \(x_1, x_2 \in X'_{X_1}\). Then \(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(x_2)| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + |N^+(x_1) \cap N^-(x_2) \cap X''_Y| \leq |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + 1\). □
N^-(x_2) \cap X'_1 | + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + 1. So |N^+(x_1) \cap N^-(x_2) \cap X'_Y| \geq \left \lceil \frac{|V(D)|}{4} \right \rceil \geq 2. Let y_1 \in N^+(x_1) \cap N^-(x_2) \cap X'_Y. Then

\xi^3(D) \leq \xi(\{x_1, x_2, y_1\}) \leq |\partial^+(\{x_1, x_2, y_1\})|

= ||\{x_1\}, X'_Y \backslash \{y_1\}|| + ||\{x_1\}, X''_Y|| + ||\{x_2\}, X'_Y||

+ ||\{y_1\}, X'_X \backslash \{x_1, x_2\}|| + ||\{y_1\}, X''_X||

\leq 2 \cdot (|X'_Y| - 1) + 2 + |X'_X| - 2 + ||\{y_1\}, X''_X|| \leq |S| = \lambda^3(D).

Thus D is \lambda^3-optimal, a contradiction.

Case 2. X'_{X_0} = X''_{X_0} = 0 or X''_{X_0} = X'_Y = 0. By reason of symmetry, we only prove the case that X'_{X_0} = X''_{X_0} = 0. Without loss of generality, we may assume that X'_{X_0} \neq 0 and X''_{X_0} \neq 0. Otherwise, by Case 1, D is \lambda^3-optimal, a contradiction. On the other hand, since for any u \in X''_{X_0} \cap X'_{X_0}, N^+(u) \subseteq X'_X, we have \left \lceil \frac{|V(D)|}{4} \right \rceil + 1 \leq \delta(D) \leq d^+(u) = |N^+(u)| \leq |X'_X|. Therefore |X'_X| \geq \left \lceil \frac{|V(D)|}{4} \right \rceil + 1.

Similarly, we can also prove that |X''_Y| \geq \left \lceil \frac{|V(D)|}{4} \right \rceil + 1. Thus

|X'_X| + |X''_Y| = |V(D)| - |X'_X| - |X''_Y|

(1) \leq |V(D)| - 2 \cdot \left (\left \lceil \frac{|V(D)|}{4} \right \rceil + 1 \right) \leq \frac{|V(D)|}{2} - 2.

Claim 2.1. |X'_X| \geq |X'_Y| + 1 or |X''_Y| \geq |X''_X| + 1.

Proof. Otherwise, we have that |X'_X| + |X''_Y| \geq |X'_X| + |X''_Y| \geq 2 \cdot \left (\left \lceil \frac{|V(D)|}{4} \right \rceil + 1 \right) \geq \frac{|V(D)|}{2} + 2, a contradiction to (1). \qed

Without loss of generality, we assume that |X'_X| \geq |X'_Y| + 1 in the following discussion.

Claim 2.2. |N^+(x) \cap X''_Y| \geq 3 and |N^-(y) \cap X''_X| \geq 3 for any x \in X'_{X_0} and y \in X''_{X_2}.

Proof. By reason of symmetry, we only prove that for any x \in X'_{X_0}, |N^+(x) \cap X''_Y| \geq 3. Since X'_{X_0} \neq 0, for any x \in X'_{X_0} and x \in X''_{X_0}, \left \lceil \frac{|V(D)|}{4} \right \rceil + 1 \leq |N^+(x) \cap N^-(\bar{x})| = |N^+(x) \cap N^-(\bar{x}) \cap X'_Y| + |N^+(x) \cap N^-(\bar{x}) \cap X''_X| \leq |N^-(\bar{x}) \cap X'_Y| + \left |N^+(x) \cap X''_Y\right| = |N^+(x) \cap X''_Y|, so |N^+(x) \cap X''_Y| \geq \left \lceil \frac{|V(D)|}{4} \right \rceil + 1 \geq 3. \qed

Claim 2.3. X''_{X_2} = X'_{X_2} = 0.

Proof. Here, we only prove that X''_{X_2} = 0. The proof of the statement that X'_{X_2} = 0 is similar. Suppose, by a contradiction, there exists y \in X''_{X_2}. Let
\(x_1, x_2 \in N^+(y) \cap X''_X\). Then

\[
\xi^3(D) \leq \xi(\{x_1, x_2, y\}) \leq |\partial^+(\{y\}) \cup \partial^-(\{x_1, x_2\})| + |\{x_1, x_2\} , X''_X| + |\{y\} , X''_X| + |\{y\} , X''_X|
\]

\[
\leq |X'_X + |\{x_1, x_2\} \cup |X'_X, \{x_1\} + |X'_X, \{x_2\}| \leq 2|X'_X| + |X'_X, \{x_2\}|- 2
\]

\[
\leq 3 \max \{|X'_X, |X''_X| + |\{y\}, X''_X| + |\{y\} , X''_X|\} \leq |S| = \lambda^3(D).
\]

So \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete. □

Claim 2.4. For any \(x \in X'_X\), \(|N(X) \cap X''_X| \geq 2\).

Proof. Let \(X'_Y = \{y_1, y_2, \ldots, y_p\}\) and let \(S^* = \{s^*: s^* \in N^+(y_i) \cap N^-(y_j) \cap X'_X\), where \(i, j \in \{1, \ldots, p\}\) and \(i \neq j\). Then \(D[S^* \cup X'_Y]\) is strong. Besides, by Claim 2.3, we have that for any \(i, j \in \{1, \ldots, p\}\) and \(i \neq j, y_i, y_j \in X'_Y \cup X'_X\).

Therefore \(\left|\frac{|\lambda^3(D)|}{4}\right| + 1 \leq |N^+(y_i) \cap N^-(y_j)| = |N^+(y_i) \cap N^-(y_j) \cap X'_X| + |N^+(y_i) \cap N^-(y_j) \cap X'_X| \leq |N^+(y_i) \cap N^-(y_j) \cap X'_X| + 1. \) Therefore \(|N^+(y_i) \cap N^-(y_j) \cap X'_X| \geq \left|\frac{|\lambda^3(D)|}{4}\right| \geq 2\). Similarly, we can prove that \(|N^+(y_i) \cap N^-(y_j) \cap X'_X| \geq 2\). On the other hand, since \(|X'_Y| \geq 2\), we have \(|S^* \cup X'_Y| \geq 3\). For any \(x \in S^*, \) clearly, \(|N(x) \cap X''_X| \geq 2\). Next, we claim that for any \(x \in X'_X \setminus S^*, \) \(|N(x) \cap X'_X| \leq |\{X'_X, \{x\})|\).

Suppose there exists \(x^* \in X'_X \setminus S^*\) such that \(|N^+(x^*) \cap X''_X| > |\{X'_X, \{x^*\})|\).

Since \(D[S^* \cup X'_Y]\) is strong and \(|S^* \cup X''_X| \geq 3\), we have \(X'_X \setminus \{x^*\}\) is a 3-restricted edge cut. Therefore \(|\partial^+(X'_X \setminus \{x^*\})| = |\{x^*\} \cup |\{x^*\} \cup X''_X| + |X'_X, \{x^*\} | < |S|\), a contradiction to the minimality of \(S\). Thus \(|X'_X, \{x\})| \geq |N^+(x) \cap X''_X|\). By Claim 2.2, we have that \(|X'_X, \{x\})| \geq 3\). The proof of Claim 2.4 is complete. □

Let \(x_1 \in X'_X\) such that \(|N^+(x_1) \cap X''_X| \leq |N^+(u) \cap X''_X| \) for any \(u \in X'_X\), and let \(y_1, y_2 \in N(x_1) \cap X''_X\). Then

\[
\xi^3(D) \leq |\partial^+((x_1, y_1, y_2))| = |\{x_1, y_1, y_2\} , X''_X, \{x_1, y_1, y_2\} + |\{x_1, y_1, y_2\} , X''_X| + |\{y_1, y_2\} , X''_X| + |\{y_2\} , X''_X|
\]

\[
\leq 2|X'_X| - 2 + |\{x_1\} , X''_X| + |\{y_1\} , X''_X| + |\{y_2\} , X''_X|
\]

\[
\leq 3|X'_X| - 5 + |\{x_1\} , X''_X| + |\{y_1\} , X''_X|
\]

\[
+ |\{y_2\} , X''_X|(|X'_X| \geq |X'_X| + 1) \leq |\{x_1\} , X''_X| \times |X'_X|
\]

\[
+ |\{y_1\} , X''_X| + |\{y_2\} , X''_X| \leq |S| = \lambda^3(D).
\]

So \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete. □
From Theorem 2, we have following corollaries.

**Corollary 13.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$. If for any $u, v \in V(D)$ in the same partite, $d^+(u) + d^-(v) \geq |V(D)| - 1$, then $D$ is $\lambda^3$-optimal.

**Corollary 14.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \lceil \frac{|V(D)|}{2} \rceil$, then $D$ is $\lambda^3$-optimal.

(Unordered edges represent two arcs with the same end-vertices and opposite directions.)

Figure 1. The example from Remark 15.

**Remark 15.** To show that the condition “$|N^+(u) \cap N^-(v)| \geq \lceil \frac{|V(D)|}{4} \rceil + 1$ for any $u, v \in V(D)$ in the same partite” in Theorem 2 is sharp, we consider the digraph $T$ shown in Figure 1. Clearly, $|V(D)| \geq 6$ and $D$ is strong. There exists $x_1, y_1$ in the same partite such that $|N^+(x_1) \cap N^-(y_1)| = 2 < 3 = \lceil \frac{|V(D)|}{4} \rceil + 1$. Clearly, $\partial^+\{x_1, x_2, x_3, x_4\}$ is a 3-restricted edge cut and $\xi^3(T) = |\partial^+(\{x_1, x_2, x_3\})| = 5$. Therefore, $\lambda^3(T) \leq |\partial^+(\{x_1, x_2, x_3, x_4\})| = 4 < 5 = \xi^3(T)$ and $T$ is not $\lambda^3$-optimal.

Besides, since $d^+(x_3) + d^-(y_4) = 6 < 7 = |V(T)| - 1$ and $\delta(T) = 3 < 4 = \lceil \frac{|V(D)|}{2} \rceil$, this example also shows that the conditions “$d^+(u) + d^-(v) \geq |V(D)| - 1$” for any $u, v \in V(D)$ in the same partite” in Corollary 13 and “$\delta(D) \geq \lceil \frac{|V(D)|}{2} \rceil$” in Corollary 14 are sharp.

**Acknowledgements**

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions. This work is supported by National Natural Science Foundation of China (Nos. 11531011).

**References**


doi:10.1007/s10114-010-9313-y


Received 8 October 2018
Revised 5 October 2019
Accepted 5 October 2019