ON THE $\rho$-EDGE STABILITY NUMBER OF GRAPHS

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Abstract

For an arbitrary invariant $\rho(G)$ of a graph $G$ the $\rho$-edge stability number $es_\rho(G)$ is the minimum number of edges of $G$ whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$ or with $E(H) = \emptyset$.

In the first part of this paper we give some general lower and upper bounds for the $\rho$-edge stability number. In the second part we study the $\chi'$-edge stability number of graphs, where $\chi' = \chi'(G)$ is the chromatic index of $G$. We prove some general results for the so-called chromatic edge stability index $es_{\chi'}(G)$ and determine $es_{\chi'}(G)$ exactly for specific classes of graphs.

Keywords: edge stability number, line stability, invariant, chromatic edge stability index, chromatic index, edge coloring.

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1. Introduction

We consider in this paper finite simple graphs $G = (V(G), E(G))$. A graph is empty if $E(G) = \emptyset$.

Definition. A (graph) invariant $\rho(G)$ is a function $\rho : \mathcal{I} \rightarrow \mathbb{R}^+_0 \cup \{\infty\}$, where $\mathcal{I}$ is the class of finite simple graphs. An invariant $\rho(G)$ is integer valued if its image set consists of non-negative integers, that is, $\rho(\mathcal{I}) \subseteq \mathbb{N}_0$.

An invariant $\rho(G)$ is monotone increasing if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$, and monotone decreasing if $H \subseteq G$ implies $\rho(H) \geq \rho(G)$; $\rho(G)$ is monotone if it is monotone increasing or monotone decreasing.
If $H_1$ and $H_2$ are disjoint graphs, then an invariant is called additive if $\rho(H_1 \cup H_2) = \rho(H_1) + \rho(H_2)$ and maxing if $\rho(H_1 \cup H_2) = \max\{\rho(H_1), \rho(H_2)\}$.

**Example 1.** The chromatic number $\chi(G)$ is integer valued, monotone increasing, and maxing, and $\chi(G) - 1 \leq \chi(G-e) \leq \chi(G)$ holds for any edge $e$ of $G$.

**Example 2.** The domination number $\gamma(G)$ is integer valued, not monotone, and additive, and $\gamma(G) \leq \gamma(G-e) \leq \gamma(G)+1$ holds for any edge $e$ of $G$.

**Definition.** Let $\rho(G)$ be an arbitrary invariant of a graph $G$. We define the $\rho$-edge stability number $es_\rho(G)$ of $G$ as the minimum number of edges of $G$ whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$ or with $E(H) = \emptyset$.

In [2] the $\rho$-edge stability number is also defined and called $\rho$-line-stability. The $\rho$-edge stability number $es_\rho(G)$ is an integer valued invariant. Some easy observations follow directly from the definition. For example, $es_\rho(G) = 0$ if and only if $G$ is empty. If $G$ is not empty, then $1 \leq es_\rho(G) \leq |E(G)|$. If $\rho(G)$ does not change by any edge removal, for example, the order of the graph $G$, then $es_\rho(G) = |E(G)|$.

If $\rho(G)$ is monotone increasing, then the subgraph $H$ in the definition on the previous page fulfills $\rho(H) < \rho(G)$ or $H$ is empty. Conversely, if $\rho(G)$ is monotone decreasing, then this subgraph $H$ fulfills $\rho(H) > \rho(G)$ or $H$ is empty.

For some specific invariants $\rho(G)$ the problem of determining the $\rho$-edge stability number was already considered, for example for the chromatic number $\chi(G)$ and particularly for the domination number $\gamma(G)$. In this paper the attention is drawn to the $\chi'$-edge stability number where $\chi' = \chi'(G)$ is the chromatic index of $G$.

The $\chi$-edge stability number or chromatic edge stability number $es_\chi(G)$ was introduced in [1, 5] and also studied in [3].

The increase of the domination number $\gamma(G)$ with respect to edge removal was extensively studied (see e.g. [2] or [6] for a survey). The so-called bondage number $b(G)$ is equal to the $\gamma$-edge stability number $es_\gamma(G)$ if $G$ is not empty, and $b(G) = \infty$ if $G$ is empty.

In this paper we first consider the general case and give bounds for arbitrary $\rho$-edge stability numbers of graphs. Section 3 contains some examples of invariants $\rho(G)$ for which $es_\rho(G)$ can easily be determined. In Sections 4 and 5 we study the $\chi'$-edge stability number of graphs.

2. **General Results**

An easy observation for the bondage number $b(G)$ and some implications (see [6]) can be transferred to arbitrary edge stability numbers $es_\rho(G)$. 
Proposition 3. Let $H$ be a spanning subgraph of $G$ obtained from $G$ by removing $k$ edges. Then $es_{\rho}(G) \leq es_{\rho}(H) + k$. Moreover, if $\rho(G) \neq \rho(H)$, then $es_{\rho}(G) \leq k$.

**Proof.** Let $H = G - E'$ where $E' \subset E(G)$ with $|E'| = k$. If $\rho(G) \neq \rho(H)$, then $es_{\rho}(G) \leq |E'| = k \leq es_{\rho}(H) + k$.

Therefore, assume in the following that $\rho(G) = \rho(H)$. If $\rho(H)$ cannot be changed by edge removal, then $es_{\rho}(H) = |E(H)|$, and $es_{\rho}(G) \leq |E(G)| = |E(H)| + |E'| = es_{\rho}(H) + k$ follows.

Otherwise, let $E''$ be a set of edges of $H$ such that $|E''| = es_{\rho}(H)$ and $\rho(H - E'') \neq \rho(H)$. Set $E''' = E' \cup E''$ with $|E'''| = |E'| + |E''| = k + es_{\rho}(H)$. It follows that $\rho(G) = \rho(H) \neq \rho(H - E''') = \rho(G - E''')$, which implies $es_{\rho}(G) \leq |E'''| = es_{\rho}(H) + k$.

Upper bounds for $es_{\rho}(G)$ can be obtained by carefully selecting spanning subgraphs $H$ with a fixed $\rho$-edge stability number. The next result considers as an example the case $es_{\rho}(H) = 1$.

**Corollary 4.** If $H$ is a spanning subgraph of $G$ with $es_{\rho}(H) = 1$, then $es_{\rho}(G) \leq 1 + |E(G)| - |E(H)|$.

**Proof.** The result immediately follows from Proposition 3, since $H$ is obtained from $G$ by removing $k = |E(G)| - |E(H)|$ edges. ■

In [2] it was stated that if there is at least one vertex $v \in V(G)$ such that $\gamma(G - v) \geq \gamma(G)$, then $b(G) \leq d(v) \leq \Delta(G)$, where $d(v)$ is the degree of $v$ and $\Delta(G)$ the maximum degree of the graph $G$. This result can be generalized as follows.

**Proposition 5.** Let $\rho(G)$ be additive. If there is a vertex $v \in V(G)$ such that $\rho(G - v) > \rho(G)$, or $\rho(G - v) = \rho(G)$ and $\rho(K_1) > 0$, or $\rho(G - v) < \rho(G)$ and $\rho(K_1) = 0$, then $es_{\rho}(G) \leq d(v) \leq \Delta(G)$.

**Proof.** The given conditions imply $\rho(G) < \rho(G - v) \leq \rho(G - v) + \rho(K_1) = \rho(G - E_v)$, or $\rho(G) = \rho(G - v) \leq \rho(G - v) + \rho(K_1) = \rho(G - E_v)$, or $\rho(G) > \rho(G - v) = \rho(G - v) + \rho(K_1) = \rho(G - E_v)$, where $E_v$ is the set of edges incident to $v$. Therefore, $\rho(G) \neq \rho(G - E_v)$ and thus $es_{\rho}(G) \leq |E_v| = d(v) \leq \Delta(G)$ by Proposition 3. ■

Alternatively, the condition $\gamma(G - v) \geq \gamma(G)$ implies that there is a minimal dominating set of $G - v$ which contains a neighbor $w$ of $v$, that is, there is an induced subgraph $H = G - E_v + vw$, obtained from $G$ by removing all edges incident to $v$ except $vw$, with $b(H) = 1$, and the conclusion $b(G) \leq d(v)$ follows by Corollary 4 (see [6]). This second proof method leads to the following result.
Corollary 6. If there is an edge set $E' \subseteq E_v$ such that $\rho(G) \neq \rho(G - E')$ or $\rho(G - E_v) \neq \rho(G - E')$, where $E_v$ is the set of edges incident to $v$, then $es_\rho(G) \leq d(v) \leq \Delta(G)$.

**Proof.** If $\rho(G) \neq \rho(G - E')$, then $es_\rho(G) \leq |E'| \leq d(v) \leq \Delta(G)$. If $\rho(G - E_v) \neq \rho(G - E')$, then $\rho(G) \neq \rho(G - E_v)$ or $\rho(G) \neq \rho(G - E')$ and the result follows by Proposition 3.

If removing a pending edge always changes $\rho(G)$, then Corollary 6 implies that $es_\rho(G) \leq d(v)$ for each non-isolated vertex $v \in V(G)$. Thus the following holds.

Corollary 7. If $G$ is a graph without isolated vertices and if removing a pending edge always changes the invariant $\rho$, then $es_\rho(G) \leq \delta(G)$.

This holds for example for the number $k(G)$ of components of $G$, since a pending edge is a bridge (see also Section 3).

Another result from [2] can be generalized by requesting appropriate conditions for the considered invariant.

Proposition 8. If $\rho(G)$ is additive and $\rho(K_2) \neq \rho(2K_1)$, then $es_\rho(G) \leq \min\{d(u) + d(v) - 1 : uv \in E(G)\}$.

**Proof.** For an arbitrary edge $uv \in E(G)$ set $H = G - E_u - E_v + uv \cong G - \{u, v\} \cup K_2$, which is obtained from $G$ by removing $k = d(u) + d(v) - 2$ edges. Since $\rho(H) = \rho(G - \{u, v\}) + \rho(K_2) \neq \rho(G - \{u, v\}) + \rho(2K_1) = \rho(H - uv)$ implies $es_\rho(H) = 1$, the result follows from Corollary 4.

This result can be generalized by considering an arbitrary subgraph $S$ of $G$ instead of $K_2$. The additivity of $\rho(G)$ gives an upper bound on $es_\rho(G)$ which only depends on $S$ and the number of removed edges.

Theorem 9. Let $\rho(G)$ be additive and $S \subseteq G$ a subgraph for which $\rho(S)$ can be changed by edge deletions. Then $es_\rho(G) \leq es_\rho(S) + |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$, where $E(U, W)$ is the set of edges between vertex sets $U$ and $W$.

**Proof.** Consider the spanning subgraph $H = G - V(S) \cup S$ of $G$ which is obtained from $G$ by removing $k = |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$ edges, namely all edges between $V(S)$ and $V(G) \setminus V(S)$ as well all edges in $G[V(S)]$ not contained in $S$. By Proposition 3, $es_\rho(G) \leq es_\rho(H) + k$.

Let $E' \subseteq E(S)$ be an edge set such that $|E'| = es_\rho(S)$ and $\rho(S) \neq \rho(S - E')$. Then by the additivity of the invariant, $\rho(H) = \rho(G - V(S)) + \rho(S) \neq \rho(G - V(S)) + \rho(S - E') = \rho(H - E')$ which implies by Proposition 3 that $es_\rho(H) \leq |E'| = es_\rho(S)$. Thus, $es_\rho(G) \leq es_\rho(S) + k$.  

If $S$ is a spanning subgraph, then $V(S) = V(G)$, thus Theorem 9 gives the bound $es_\rho(G) \leq es_\rho(S) + |E(G)| - |E(S)|$, which follows directly by Proposition 3. If $S$ is an induced subgraph, then the bound of Theorem 9 simplifies to $es_\rho(G) \leq es_\rho(S) + |E(V(S), V(G) \setminus V(S))|$. An additional condition on the invariant $\rho$ is necessary to prove the corresponding result for maxing invariants.

**Theorem 10.** Let $\rho(G)$ be maxing and $S \subseteq G$ a subgraph for which $\rho(S)$ can be changed by edge deletions and $\rho(S) > \rho(G - V(S))$. Then $es_\rho(G) \geq es_\rho(S) + |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$.

**Proof.** As in the proof of Theorem 9, consider the spanning subgraph $H = G - V(S) \cup S$ of $G$ which is obtained by removing $k = |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$ edges from $G$. By Proposition 3, $es_\rho(G) = es_\rho(H) + k$.

Let $E' \subseteq E(S)$ be an edge set such that $|E'| = es_\rho(S)$ and $\rho(S) \neq \rho(S - E')$. Since the invariant is maxing and $\rho(S) > \rho(G - V(S))$ by assumption and $\rho(S) \neq \rho(S - E')$, it holds that $\rho(H) = \max\{\rho(G - V(S)), \rho(S)\} = \rho(S) \neq \max\{\rho(G - V(S)), \rho(S - E')\} = \rho(H - E')$. By Proposition 3, $es_\rho(H) \leq |E'| = es_\rho(S)$ and therefore $es_\rho(G) \leq es_\rho(S) + k$.

In the proofs of Theorems 9 and 10, the disjoint union of two graphs was considered. The proof idea can be transferred to the disjoint union of arbitrarily many graphs.

**Theorem 11.** Let $\rho(G)$ be additive, let $G = H_1 \cup \cdots \cup H_k$ be a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 0$ are defined such that $\rho(H_i)$ can be changed by edge deletion if and only if $1 \leq i \leq s$. Then $es_\rho(G) = |E(G)|$ if $s = 0$ and $es_\rho(G) = \min\{es_\rho(H_i) : 1 \leq i \leq s\}$ if $s \neq 0$.

**Proof.** If $s = 0$, then $\rho(H_i)$ cannot be changed by edge deletion for every subgraph $H_i$, which implies by the additivity that also $\rho(G) = \rho(H_1) + \cdots + \rho(H_k)$ cannot be changed by edge deletions, that is, $es_\rho(G) = |E(G)|$.

If $s \neq 0$, then let $H_j$ be a subgraph with $es_\rho(H_j) = \min\{es_\rho(H_i) : 1 \leq i \leq s\}$ and $E' \subseteq E(H_j)$ be an edge set with $|E'| = es_\rho(H_j)$ and $\rho(H_j - E') \neq \rho(H_j)$. By the additivity, $\rho(G - E') = \rho(H_1) + \cdots + \rho(H_{j-1}) + \rho(H_j - E') + \rho(H_{j+1}) + \cdots + \rho(H_k) \neq \rho(H_j) + \cdots + \rho(H_{j-1}) + \rho(H_j) + \rho(H_{j+1}) + \cdots + \rho(H_k) = \rho(G)$, which implies $es_\rho(G) \leq |E'| = es_\rho(H_j)$.

Let $E'' \subseteq E(G)$ be an edge set with $|E''| < es_\rho(H_j)$. By the minimality of $es_\rho(H_j)$, $\rho(H_i - E'') = \rho(H_i)$ for $i = 1, \ldots, k$, which implies $\rho(G - E'') = \rho(G)$ since $\rho(G)$ is additive. Therefore, $es_\rho(G) = es_\rho(H_j)$.

For maxing invariants we can prove the following result.

**Theorem 12.** Let $\rho(G)$ be maxing and monotone increasing, let $G = H_1 \cup \cdots \cup H_k$ be a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 1$ are defined
such that $\rho(H_i) = \rho(G)$ if and only if $1 \leq i \leq s$. Then $es_\rho(G) = |E(G)|$ if there is a subgraph $H_j$, $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge deletions, and $es_\rho(G) = \sum_{i=1}^{s} es_\rho(H_i)$ otherwise.

**Proof.** If there is a subgraph $H_j$, $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge deletions, then $\rho(G) = \rho(H_j) = \rho(G - E')$ for every $E' \subseteq E(G)$, since the invariant is maxing and monotone increasing (that is, removing edges does not increase the invariant). Therefore, $es_\rho(G) = |E(G)|$.

Otherwise, let $E' = E'_1 \cup \cdots \cup E'_s$ with $E'_i \subseteq E(H_i)$, $|E'_i| = es_\rho(H_i)$, and $\rho(H_i - E'_i) \neq \rho(H_i)$ for $i = 1, \ldots, s$. Since the invariant is maxing, $\rho(G - E') = \max\{\rho(H_i - E'_i) : 1 \leq i \leq s\} \cup \{\rho(H_i) : s + 1 \leq i \leq k\} \neq \rho(G)$ which implies $es_\rho(G) \leq |E'| = \sum_{i=1}^{s} es_\rho(H_i)$. If an edge set $E''$ with less than $|E'|$ edges is removed from $G$, then there is a subgraph $H_j$, $1 \leq j \leq s$, from which less than $es_\rho(H_j)$ edges are removed, which implies $\rho(H_j - E'') = \rho(H_j)$ and thus, since the invariant is maxing and monotone increasing, $\rho(G - E'') = \rho(H_j) = \rho(G)$. Therefore, $es_\rho(G) = |E'| = \sum_{i=1}^{s} es_\rho(H_i)$.

Theorems 11 and 12 imply that $\rho(G)$ can be computed by the $\rho$-edge stability numbers of the components of $G$ if the invariant is additive or if it is maxing and monotone increasing. Therefore, it is sufficient to consider connected graphs $G$ in these cases.

A lower bound for $es_\rho(G)$ given in [3] can be generalized as follows.

**Theorem 13.** Let $\rho(G)$ be monotone and let $G$ be a nonempty graph with $\rho(G) = k$. If $G$ contains $s$ nonempty subgraphs $G_1, \ldots, G_s$ with $\rho(G_1) = \cdots = \rho(G_s) = k$ such that $a \geq 0$ is the number of edges that occur in at least two of these subgraphs and $q \geq 1$ is the maximum number of these subgraphs with a common edge, then both $es_\rho(G) \geq \frac{1}{q} \sum_{i=1}^{s} es_\rho(G_i) \geq s/q$ and $es_\rho(G) \geq \sum_{i=1}^{s} es_\rho(G_i) - a(q - 1)$ hold.

**Proof.** Let $\rho(G)$ be monotone increasing. Let $E'$ be a set of edges of $G$ with $|E'| = es_\rho(G)$ such that $\rho(G - E') < k$ or $G - E'$ is empty. If $\rho(G - E') < k$, then the set $E'$ must contain at least $es_\rho(G_i)$ edges of each graph $G_i$, $1 \leq i \leq s$, since otherwise $k > \rho(G - E') \geq \rho(G_i - E(G_j)) = k$ for some $j$, $1 \leq j \leq s$, a contradiction. If $G - E'$ is empty, then $E' = E(G)$ contains all edges of $G_i$, $1 \leq i \leq s$. Therefore, $b = \sum_{i=1}^{s} |E' \cap E(G_i)| \geq \sum_{i=1}^{s} es_\rho(G_i) \geq s$.

On the other hand, at most $\bar{a} = \min\{a, |E'|\}$ edges of $E'$ are counted at most $q$ times in $b$, every other edge of $E'$ is counted at most once, so $b \leq \bar{a} - q + (|E'| - \bar{a}) \cdot 1 = |E'| + \bar{a}(q - 1)$.

Since $\bar{a} \leq |E'|$, $b \leq q \cdot |E'|$ and therefore $es_\rho(G) = |E'| \geq b/q \geq \frac{1}{q} \sum_{i=1}^{s} es_\rho(G_i) \geq s/q$. On the other hand, $\bar{a} \leq a$ implies $es_\rho(G) = |E'| \geq b - a(q - 1) \geq \sum_{i=1}^{s} es_\rho(G_i) - a(q - 1)$.

The proof for monotone decreasing $\rho(G)$ runs analogously.
Note that we do not require that the graphs $G_i$ are distinct in Theorem 13. The lower bound of the first inequality can be improved by considering additional subgraphs $G_i$ with $\rho(G_i) = k$ that do not increase the number $q$. A refinement of the latter inequality can be achieved if the number of occurrences of fixed edges in the subgraphs is taken into account.

**Corollary 14.** Let $\rho(G)$ be monotone and let $G$ be a nonempty graph with $\rho(G) = k$. If $G$ contains $s$ nonempty subgraphs $G_1, \ldots, G_s$ with $\rho(G_1) = \cdots = \rho(G_s) = k$ and pairwise disjoint edge sets, then $es_{\rho}(G) \geq \sum_{i=1}^{s} es_{\rho}(G_i) \geq s$.

**Proof.** Each edge of $G$ is contained in at most $q = 1$ of the given subgraphs since they are pairwise edge disjoint. The result follows from Theorem 13. 

**Corollary 15.** Let $\rho(G)$ be monotone. If $H \subseteq G$ and $\rho(H) = \rho(G)$, then $es_{\rho}(H) \leq es_{\rho}(G)$.

**Proof.** If $H$ is empty, then $es_{\rho}(H) = 0 \leq es_{\rho}(G)$; otherwise Corollary 14 with $s = 1$ implies the result. 

Note that in general $es_{\rho}(G)$ must not be monotone even if $\rho(G)$ is monotone.

### 3. Examples for Edge Stability Numbers

In this section the edge stability numbers for some well-known invariants are considered, beginning with the minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$.

**Proposition 16.** $es_{\delta}(G) = |E(G)|$ if $\delta(G) = 0$ and $es_{\delta}(G) = 1$ if $\delta(G) \neq 0$.

**Proof.** If $\delta(G) = 0$, that is, $G$ has isolated vertices, then the minimum degree cannot be decreased by edge removal, hence $es_{\delta}(G) = |E(G)|$ by definition. If $\delta(G) \neq 0$, then it suffices to remove one edge incident to a vertex of degree $\delta(G)$ in order to decrease the minimum degree, hence $es_{\delta}(G) = 1$.

**Proposition 17.** $es_{\Delta}(G) = 0$ if $G$ is empty and $es_{\Delta}(G) = |V_{\Delta} - \alpha'(G[V_{\Delta}])|$ if $G$ is not empty, where $V_{\Delta}$ is the set of vertices of $G$ of degree $\Delta(G)$ and $\alpha'(G)$ is the edge independence number or matching number of $G$.

**Proof.** If $G$ is empty, then $es_{\Delta}(G) = 0$ by definition. If $G$ is not empty, then $\Delta(G) \geq 1$. Let $E'$ be an edge set of $G$ with $\Delta(G - E') = \Delta(G) - 1 \geq 0$. Each vertex from $V_{\Delta}$ is incident with at least one edge from $E'$. At most $\alpha'(G[V_{\Delta}])$ edges from $E'$ connect two vertices each from $V_{\Delta}$ such that all these vertices are distinct. The remaining vertices of $V_{\Delta}$ need one additional incident edge from $E'$ each. Therefore, $es_{\Delta}(G) \geq \alpha'(G[V_{\Delta}]) + |V_{\Delta}| - 2\alpha'(G[V_{\Delta}]) = |V_{\Delta}| - \alpha'(G[V_{\Delta}])$. 

Note that we do not require that the graphs $G_i$ are distinct in Theorem 13. The lower bound of the first inequality can be improved by considering additional subgraphs $G_i$ with $\rho(G_i) = k$ that do not increase the number $q$. A refinement of the latter inequality can be achieved if the number of occurrences of fixed edges in the subgraphs is taken into account.
Equality holds by selecting an appropriate maximum matching in $G[V_\Delta]$ and an incident edge for each not matched vertex from $V_\Delta$.

If $G$ is regular and not empty, then $es_\Delta(G) = |V(G)| - \alpha'(G)$. For example, $es_\Delta(K_n) = \frac{1}{2}n$ if $n$ is even and $es_\Delta(K_n) = \frac{1}{2}(n + 1)$ if $n \geq 3$ is odd.

Let $k(G)$ be the number of components of a graph $G$ and $\lambda(G)$ the edge connectivity of $G$, that is, the minimum number of edges whose removal gives a disconnected graph or the singleton $K_1$. By the definitions it follows that if $G$ is connected, then $es_k(G) = \lambda(G)$. A direct implication of Theorem 11 is the following general result which also covers disconnected graphs.

**Proposition 18.** Let $G$ be a graph with $k(G)$ components $H_1, \ldots, H_{k(G)}$. Then $es_k(G) = 0$ if $G$ is empty and $es_k(G) = \min\{\lambda(H_i) : 1 \leq i \leq k(G), H_i \not\sim K_1\}$ if $G$ is not empty.

**Proof.** The number of components $k(H)$ is additive and can be increased by edge deletions for nonempty graphs. Let $H_1, \ldots, H_s$ be the nonempty components of $G$ and $H_{s+1}, \ldots, H_{k(G)}$ be singletons $K_1$, $0 \leq s \leq k(G)$. Then Theorem 11 gives $es_k(G) = 0$ if $s = 0$, that is, if $G$ is empty, and $es_k(G) = \min\{es_k(H_i) : 1 \leq i \leq s\} = \min\{\lambda(H_i) : 1 \leq i \leq s\}$ otherwise.

**Proposition 19.** $es_\chi(G) = 1$ if $G$ is connected and not a singleton, and $es_\chi(G) = |E(G)|$ otherwise.

**Proof.** If $G$ is connected and not a singleton, then let $E'$ be an edge set with $|E'| = \lambda(G) \geq 1$ such that $G - E'$ is disconnected. For any edge $e \in E'$, $\lambda(G - e) = \lambda(G) - 1$, hence $es_\chi(G) = 1$. If $G$ is disconnected or a singleton, then $\lambda(G) = 0$ and the invariant cannot be changed by edge removal, hence $es_\chi(G) = |E(G)|$ by definition.

4. General Results for the Chromatic Edge Stability Index

If $G = (V(G), E(G))$ is a graph, a function $c : E(G) \to \{1, \ldots, k\}$ such that $c(e_1) \neq c(e_2)$ for any two adjacent edges $e_1$ and $e_2$ is called a $k$-edge-coloring of $G$, and $G$ is called $k$-edge-colorable. The minimum $k$ for which $G$ is $k$-edge-colorable is the chromatic index $\chi'(G)$ of $G$. By Vizing’s Theorem, the chromatic index can only attain one of two values, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Graphs with $\chi'(G) = \Delta(G)$ are called class 1 graphs and graphs with $\chi'(G) = \Delta(G) + 1$ are called class 2 graphs. We define the invariant class$(G) = \chi'(G) - \Delta(G) + 1 \in \{1, 2\}$. A graph $G$ is called overfull if its order $n$ is odd and if it contains more than $\Delta(G)(n - 1)/2$ edges. Obviously, an overfull graph must be a class 2 graph.
Note that $\chi'(G)$ is an invariant which is monotone increasing, integer valued, and maxing, and it holds that $\chi'(G-e) \leq \chi'(G) \leq \chi'(G-e) + 1$ for any edge $e$ of $G$.

In this section we consider the $\chi'$-edge stability number $es_{\chi'}(G)$ which we also call chromatic edge stability index of $G$. Using Theorem 12 we can compute $es_{\chi'}(G)$ by the chromatic edge stability indexes of its components. Let $G = H_1 \cup \cdots \cup H_k(G)$ such that $\chi'(G) = \chi'(H_i)$ if and only if $1 \leq i \leq s$ for $s \leq k(G)$. Then $es_{\chi'}(G) = \sum_{i=1}^s es_{\chi'}(H_i)$. Therefore, we can assume without loss of generality in the following that $G$ is connected.

**Proposition 20.** $es_{\chi'}(G) \leq |\{E(G)|/\chi'(G)\| \leq \alpha'(G)$ if $G$ is nonempty, and $es_{\chi'}(G) = \alpha'(G) = 0$ if $G$ is empty.

**Proof.** Let $t'(G)$ be the minimum number of edges in a color class of the graph $G$ where the minimum is taken over all edge colorings of $G$ with $\chi'(G)$ colors.

If $G$ is nonempty, then removing any color class from $G$ reduces the chromatic index, thus $es_{\chi'}(G) \leq t'(G)$ follows. By the pigeonhole principle, any edge coloring of $G$ with $\chi'(G)$ colors has a color class with at most $|\{E(G)|/\chi'(G)\|$ edges, which implies $t'(G) \leq |\{E(G)|/\chi'(G)\|$. On the other hand, the lower bound $\chi'(G) \geq |E(G)|/\alpha'(G)$ implies the second inequality.

If $G$ is empty, then the result is obvious. \[\Box\]

**Lemma 21.** If $G$ is a class 1 graph, then $es_{\chi'}(G) \geq es_{\Delta}(G)$.

**Proof.** If $G$ is empty, then $es_{\chi'}(G) = es_{\Delta}(G) = 0$. If $G$ is nonempty, then there is a set $E'$ of edges of $G$ such that $|E'| = es_{\chi'}(G)$ and $\Delta(G - E') \leq \chi'(G - E') < \chi'(G) = \Delta(G)$. It follows that $\Delta(G - E') < \Delta(G)$ which implies $es_{\chi'}(G) = |E'| \geq es_{\Delta}(G)$. \[\Box\]

The following proposition gives a class of graphs for which equality always holds.

**Proposition 22.** If $G$ is a regular class 1 graph, then $es_{\chi'}(G) = es_{\Delta}(G) = \alpha'(G)$.

**Proof.** If $G$ is empty, then $es_{\chi'}(G) = es_{\Delta}(G) = \alpha'(G) = 0$. If $G$ is nonempty, then $es_{\Delta}(G) \leq es_{\chi'}(G) \leq \alpha'(G) = 1/2 |V(G)|$ by Lemma 21 and Proposition 20. Since $es_{\Delta}(G) = |V(G)| - \alpha'(G) = 1/2 |V(G)|$ by Proposition 17, $es_{\chi'}(G) = es_{\Delta}(G) = \alpha'(G) = 1/2 |V(G)|$ follows. \[\Box\]

More generally, we can characterize in a certain way all class 1 graphs with $es_{\chi'}(G) = es_{\Delta}(G)$.

**Proposition 23.** If $G$ is a class 1 graph, then $es_{\chi'}(G) = es_{\Delta}(G)$ if and only if $G$ is empty or if there is an edge set $E'$ such that $|E'| = es_{\Delta}(G)$, $\Delta(G - E') < \Delta(G)$, and $G - E'$ is in class 1.
Proof. Let \( G \) be a non-empty class 1 graph.
If \( es_{\chi'}(G) = es_{\Delta}(G) \), then let \( E' \) be an arbitrary edge set with \( |E'| = es_{\chi'}(G) \) and \( \chi'(G-E') < \chi'(G) \), which implies \( \chi'(G-E') = \chi'(G) - 1 \). Since \( \Delta(G-E') \leq \chi'(G-E') \) and \( \chi'(G) = \Delta(G) \), \( \Delta(G-E') < \Delta(G) \). Moreover, \( |E'| = es_{\Delta}(G) \), which implies that \( \Delta(G-E') = \Delta(G) - 1 \). Therefore, \( \chi'(G-E') = \Delta(G) - 1 \), that is, \( G-E' \) is in class 1.

The second assertion follows from Lemma 21, which states \( es_{\chi'}(G) \geq es_{\Delta}(G) \), and from the properties of the given set \( E' \), since \( \chi'(G-E') = \Delta(G-E') \leq \Delta(G) \) which implies \( |E'| = es_{\Delta}(G) \geq es_{\chi'}(G) \).

Proposition 22 follows from this characterization, since a regular class 1 graph is 1-factorable, and removing a 1-factor \( E' \) leaves a class 1 graph.

Proposition 23 and Lemma 21 imply that if \( G \) is in class 1 but \( G-E' \) is in class 2 for all sets \( E' \) with \( |E'| = es_{\Delta}(G) \) and \( \Delta(G-E') < \Delta(G) \), then \( es_{\chi'}(G) > es_{\Delta}(G) \). An example for such graphs is given in Theorem 31.

Theorem 24. If \( G \) is a class 2 graph, then \( es_{\chi'}(G) = \min\{es_{\Delta}(G), es_{\text{class}}(G)\} \).

Proof. Since \( G \) is in class 2, the graph \( G \) is not empty and the invariants \( \Delta(G) \), \( \text{class}(G) = 2 \), and \( \chi'(G) = \Delta(G) + 1 \) can be reduced by edge removal.

By removing \( es_{\Delta}(G) \) edges \( E' \) such that \( \Delta(G-E') < \Delta(G) \) we obtain \( \chi'(G-E') = \Delta(G) - 1 < \Delta(G) + 1 = \chi'(G) \) which implies \( |E'| \geq es_{\chi'}(G) \). By removing \( es_{\text{class}}(G) \) edges \( E'' \) such that \( \text{class}(G-E'') = 1 \) we obtain \( \chi'(G-E'') = \Delta(G) < \Delta(G) + 1 = \chi'(G) \) which implies \( |E''| \geq es_{\chi'}(G) \). It follows that \( \min\{es_{\Delta}(G), es_{\text{class}}(G)\} \geq es_{\chi'}(G) \).

Consider now a set of edges \( E''' \) such that \( \chi'(G-E''') < \chi'(G) = \Delta(G) + 1 \), that is, \( \chi'(G-E''') \leq \Delta(G) \). Then \( G-E''' \) cannot both be in class 2 and have the same maximum degree as \( G \) since this would imply \( \chi'(G-E''') = \Delta(G) + 1 \). Therefore, \( |E'''| \geq es_{\Delta}(G) \) or \( |E'''| \geq es_{\text{class}}(G) \) which implies \( es_{\chi'}(G) \geq \min\{es_{\Delta}(G), es_{\text{class}}(G)\} \).

For overfull graphs we can give a lower bound.

Corollary 25. If \( G \) is an overfull graph, then \( es_{\chi'}(G) \geq |E(G)| - \Delta(G)(|V(G)| - 1)/2 \).

Proof. Since \( G \) is overfull, \( G \) is in class 2, \( |E(G)| > \Delta(G)(n-1)/2 \), and the invariants \( \Delta(G) \) and \( \text{class}(G) \) can be reduced by edge deletions.

Let \( E' \) be an edge set such that \( |E'| = es_{\Delta}(G) \) and \( \Delta(G-E') < \Delta(G) \). By the handshake lemma, \( G-E' \) may contain at most \( \Delta(G-E')n/2 \leq (\Delta(G)-1)n/2 \) edges which implies \( es_{\Delta}(G) = |E'| \geq |E(G)|-\Delta(G)(n-1)/2 > |E(G)|-\Delta(G)(n-1)/2 \), since \( n > \Delta(G) \).

Let \( E'' \) be an edge set such that \( |E''| = es_{\text{class}}(G) \) and \( \text{class}(G-E'') = 1 \). Then \( G-E'' \) may contain at most \( \Delta(G-E'')(n-1)/2 \leq \Delta(G)(n-1)/2 \) edges.
(otherwise $G - E'$ would be still overfull) which implies $es_{\text{class}}(G) = |E'| \geq |E(G)| - \Delta(G)(n - 1)/2$.

By Theorem 24, $es_{\chi'}(G) = \min\{es_{\Delta}(G), es_{\text{class}}(G)\} \geq |E(G)| - \Delta(G)(n - 1)/2$.  

5. Chromatic Edge Stability Index for Specific Graph Classes

In this section we use general results of the previous section to determine the chromatic edge stability index of some well-known graph classes.

Theorem 26. If $G$ is bipartite, then $es_{\chi'}(G) = es_{\Delta}(G)$.

Proof. The result follows from Proposition 23, since every subgraph $G - E'$ of $G$ is bipartite and thus in class 1.

Theorem 26 and Proposition 17 imply the following results for complete bipartite graphs and paths.

Corollary 27. $es_{\chi'}(K_{m,n}) = es_{\Delta}(K_{m,n}) = \min\{m, n\}$.

Corollary 28. $es_{\chi'}(P_n) = es_{\Delta}(P_n) = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ \lceil (n - 2)/2 \rceil & \text{if } n \geq 3. \end{cases}$

Proposition 29. $es_{\chi'}(C_n) = n/2$ if $n$ is even, $es_{\chi'}(C_n) = 1$ if $n$ is odd, and $es_{\Delta}(C_n) = \lceil n/2 \rceil$ if $n \geq 2$.

Proof. By Proposition 17, $es_{\Delta}(C_n) = \lceil n/2 \rceil$. If $n$ is even, then $C_n$ is bipartite, and the result follows from Theorem 26. If $n$ is odd, then $\chi'(C_n) = 3$ and removing one edge from the cycle gives a 2-edge-colorable path $P_n$, which implies $es_{\chi'}(C_n) = 1$.

This proposition shows that the difference between the two invariants $es_{\Delta}(G)$ and $es_{\chi'}(G)$ may be arbitrarily large, since $es_{\Delta}(C_{2s+1}) - es_{\chi'}(C_{2s+1}) = s$. Moreover, Lemma 21 does not necessarily hold for class 2 graphs.

Next we consider complete graphs and complete graphs with an additional vertex.

Proposition 30. $es_{\chi'}(K_n) = \lfloor n/2 \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n - 1)/2 & \text{if } n \text{ is odd,} \end{cases}$ and $es_{\Delta}(K_n) = \lceil n/2 \rceil$ if $n \geq 2$. 

Proof. If \( n \geq 2 \), then \( es_\Delta(K_n) = \lceil n/2 \rceil \) follows from Propositions 17.

If \( n = 1 \) or if \( n \) is even, then \( K_n \) is a regular class 1 graph, and the result is an implication of Proposition 22.

If \( n \geq 3 \) is odd, then \( K_n \) is overfull and Corollary 25 implies \( es_{\chi'}(K_n) \geq \binom{n}{2} - (n-1)^2/2 = (n-1)/2 \). On the other hand, \( es_{\chi'}(K_n) \leq \alpha'(K_n) = (n-1)/2 \) follows from Proposition 20, that is, equality holds.

Theorem 31. Let \( G \) be a graph which consists of a complete graph \( K_n \), \( n \geq 2 \), and an additional vertex \( w \) connected to \( d = d(w) \) vertices of \( K_n \). Then we have the following.

1. \( es_\Delta(G) = \lfloor n/2 \rfloor \) if \( d = 0 \), \( es_\Delta(G) = \lceil d/2 \rceil \) if \( 1 \leq d \leq n - 1 \), \( es_\Delta(G) = \lfloor (n + 1)/2 \rfloor \) if \( d = n \).
2. If \( n \geq 3 \) odd, then \( es_{\chi'}(G) = \lfloor n/2 \rfloor \) if \( 0 \leq d \leq n - 1 \), \( es_{\chi'}(G) = (n + 1)/2 \) if \( d = n \).
3. If \( n \) even, then \( es_{\chi'}(G) = \lfloor n/2 \rfloor \) if \( d = 0 \), \( es_{\chi'}(G) = d \) if \( 1 \leq d \leq n/2 \), \( es_{\chi'}(G) = d - n/2 \) if \( n/2 < d \leq n \).

Proof. If \( d = 0 \), then \( G \cong K_n \cup K_1 \) and if \( d = n \), then \( G \cong K_{n+1} \), therefore the results follow from Propositions 17 and 30. Let \( 1 \leq d \leq n - 1 \) in the following and denote the vertices of \( K_n \) by \( v_1, \ldots, v_n \) such that \( w \) is adjacent to \( v_1, \ldots, v_d \).

1. If \( 1 \leq d \leq n - 1 \), then \( G \) has \( d \) vertices of maximum degree \( \Delta(G) = n \), namely the neighbors of \( w \). Then, by Proposition 17, \( es_\Delta(G) = d - \alpha'(K_d) = d - \lfloor d/2 \rfloor = \lfloor d/2 \rfloor \).

2. Since \( n \geq 3 \), \( n \) is odd, and \( K_n \subseteq G \subseteq K_{n+1} \), \( \chi'(K_n) = \chi'(K_{n+1}) = n \). By Corollary 15 and Proposition 30, \( es_{\chi'}(G) \geq es_{\chi'}(K_n) = \lfloor n/2 \rfloor \). Since \( d < n \), there is a color class with \( \lfloor n/2 \rfloor \) edges in every proper \( n \)-edge-coloring of \( G \), whose removal reduces the chromatic index. Therefore, \( es_{\chi'}(G) = \lfloor n/2 \rfloor \).

3. If \( n \) even, then we consider two cases.

Case 3(a): If \( 1 \leq d \leq n/2 \), then \( G \) is in class 1. Consider the natural edge coloring of \( K_n \) with \( n - 1 \) colors where the vertices are in order \( v_1, v_{d+1}, v_2, v_{d+2}, \ldots, v_{d-1}, v_{2d-1}, v_d, v_{2d}, v_{2d+1}, \ldots, v_n \). Then the edges \( v_1v_{d+1}, \ldots, v_{d-1}v_{2d-1} \) are colored pairwise differently. Color these edges as well as edge \( wu_i \) with the new color \( n \) and then color \( uv_i \) with the old color of \( v_iv_{d+i}, i = 1, \ldots, d-1 \). This implies \( \chi'(G) = \Delta(G) = n \).

Let \( E' \) be a set of edges of \( G \) with \( |E'| = es_{\chi'}(G) \) and \( \chi'(G - E') < \chi'(G) = \Delta(G) = n \). Then \( \Delta(G - E') \leq n - 1 \). If \( \Delta(G - E') \leq n - 2 \), then the degree of the \( d \) vertices of maximum degree must be reduced by 2, which implies \( |E'| \geq d \). If \( \Delta(G - E') = n - 1 \), then \( G - E' \) cannot be overfull since otherwise \( \chi'(G - E') = \Delta(G - E') + 1 = \Delta(G) = \chi'(G) \), a contradiction. Hence \( |E(G - E')| = \)
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\[
\binom{n}{2} + d - |E| \leq \Delta(G - E')((|V(G)| - 1)/2 = n(n-1)/2 = \binom{n}{2}
\]
which again implies $|E'| \geq d$. Therefore, $es_{\chi'}(G) \geq d$.

On the other hand, removing all $d$ edges incident to $w$ gives a graph $G - E' \cong K_n \cup K_1$ with $\chi'(G - E') = n - 1$, that is, $es_{\chi'}(G) \leq d$, and equality follows.

Case 3(b): If $n/2 < d \leq n - 1$, then $G$ is overfull since $|E(G)| = \binom{n}{2} + d > n(n-1)/2 + n/2 = n^2/2 = \Delta(G)(|V(G)| - 1)/2$. Therefore, $es_{\chi'}(G) \geq d - n/2$ by Corollary 25. On the other hand, removing $d - n/2$ edges incident to $w$ gives a class 1 graph (see Case 3(a)), which implies $es_{\chi'}(G) \leq d - n/2$ and therefore $es_{\chi'}(G) = d - n/2$.

Parts of Proposition 30 and Theorem 31 follow also from a result by Plantholt [4], which states the following. If $G$ is a graph of odd order $n \geq 3$ with a spanning star, then $G$ is in class 1 if and only if it has at most $(n-1)^2/2$ edges. This implies for example that if $K_n$ is a complete graph of odd order $n \geq 3$ and $E' \subseteq E(K_n)$, then $\chi'(K_n - E') = n$ if and only if $|E'| \leq (n-3)/2$.

The result of Theorem 31 implies that the difference between $es_{\chi'}(G)$ and $es_{\Delta}(G)$ may be arbitrarily large for class 1 graphs.

**Theorem 32.** For every pair of positive integers $a, b$ there is a graph $G$ with $es_{\Delta}(G) = a$ and $es_{\chi'}(G) = b$.

**Proof.** If $a \leq b$, then for $d = 2a$ and $n = 2b + 1$ it holds that $n \geq 3$, $n$ is odd, and $1 \leq d \leq n - 1$, and the class 1 graph $G$ of Theorem 31 fulfills $es_{\Delta}(G) = \left\lfloor d/2 \right\rfloor = a$ and $es_{\chi'}(G) = (n - 1)/2 = b$.

If $a > b$, then for $d = 2a$ and $n = 2d - 2b$ (n even) it holds that $n/2 = d - b < d < 2d - 2b = n$, and the class 2 graph $G$ of Theorem 31 fulfills again $es_{\Delta}(G) = \lfloor d/2 \rfloor = a$ and $es_{\chi'}(G) = d - n/2 = b$.

Note that a graph $G$ with $es_{\Delta}(G) = a$, $es_{\chi'}(G) = b$, and $a > b$ is a class 2 graph by Lemma 21.

A wheel $W_n$ with $n \geq 3$ is the join of a cycle $C_n$, say with consecutive vertices $v_1, \ldots, v_n$, and a single vertex $w$. Wheels are class 1 graphs.

**Proposition 33.** $es_{\Delta}(W_3) = 2$, $es_{\Delta}(W_n) = 1$ for $n \geq 4$, $es_{\chi'}(W_n) = 2$ for $n \in \{3, 4\}$, $es_{\chi'}(W_n) = 1$ for $n \geq 5$.

**Proof.** If $n = 3$, then $W_3 \cong K_4$ and $es_{\Delta}(W_3) = es_{\chi'}(W_3) = 2$ follows from Propositions 17 and 30.

The wheel $W_n$ has only one vertex of maximum degree for $n \geq 4$, hence $es_{\Delta}(W_n) = 1$ for $n \geq 4$.

If $n = 4$, then $G \cong W_4 - wv_i$ ($i \in \{1, \ldots, 4\}$) has 5 vertices, 7 edges, and maximum degree 3. Since $7 = |E(G)| > \Delta(G)(|V(G)| - 1)/2 = 6$, the graph $G$ is
overfull and thus $\chi'(G) = \Delta(G) + 1 = 4 = \chi'(W_4)$, which implies $es_{\chi'}(W_4) \geq 2$. On the other hand, $es_{\chi'}(W_4) \leq \alpha'(W_4) = 2$, hence equality follows.

Let $n \geq 5$ and consider the $n$-edge-coloring of $W_n$ which assigns color $i \in \{1, \ldots, n\}$ to edges $wv_i$ and $v_{i+1}v_{i+2}$ (indices modulo $n$), and recolor edge $v_1v_2$ with color 3. Removing color class $n$ with only one edge $wv_n$ reduces the chromatic index, which implies $es_{\chi'}(W_n) = 1$ if $n \geq 5$.

It would be an interesting task to determine the chromatic edge stability index for some other classes of graphs. For example, $es_{\chi'}(P) = 2$ and $es_{\Delta}(P) = 5$ hold for the Petersen graph $P$.

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