HEREDITARY EQUALITY OF DOMINATION AND
EXPONENTIAL DOMINATION IN SUBCUBIC GRAPHS

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Abstract

Let \( \gamma(G) \) and \( \gamma_e(G) \) denote the domination number and exponential domination number of graph \( G \), respectively. Henning et al., in [Hereditary equality of domination and exponential domination, Discuss. Math. Graph Theory 38 (2018) 275–285] gave a conjecture: There is a finite set \( \mathcal{F} \) of graphs such that a graph \( G \) satisfies \( \gamma(H) = \gamma_e(H) \) for every induced subgraph \( H \) of \( G \) if and only if \( G \) is \( \mathcal{F} \)-free. In this paper, we study the conjecture for subcubic graphs. We characterize the class \( \mathcal{F} \) by minimal forbidden induced subgraphs and prove that the conjecture holds for subcubic graphs.

Keywords: dominating set, exponential dominating set, subcubic graphs.

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1. Introduction

Graph theory terminology not presented here can be found in [3]. Let \( G \) be a simple and undirected graph. The vertex set and the edge set of \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. The degree, neighborhood and closed neighborhood of a vertex \( v \) in the graph \( G \) are denoted by \( d_G(v) \), \( N_G(v) \) and \( N_G[v] = N_G(v) \cup \{v\} \), respectively. If the graph \( G \) is clear from context, we simply write \( d(v), N(v) \) and \( N[v] \), respectively. The minimum degree and maximum degree of the graph \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. Let \( S \subseteq V(G) \); \( N(S) = \bigcup_{v \in S} N(v) \).
and \( N[S] = N(S) \cup S \). The graph induced by \( S \subseteq V \) is denoted by \( G[S] \). The distance \( dist_G(X,Y) \) between two sets \( X \) and \( Y \) of vertices in \( G \) is the minimum length of a path in \( G \) between a vertex in \( X \) and a vertex in \( Y \). If no such path exists, then let \( dist_G(X,Y) = \infty \). Let \( P_n, C_n \) and \( K_n \) denote the path, cycle and complete graph with order \( n \), respectively. Let \( l(G) \) denote the maximum length of an induced cycle in \( G \). If \( \Delta(G) \leq 3 \), then \( G \) is called a subcubic graph.

A set \( D \subseteq V \) in a graph \( G \) is called a dominating set if every vertex outside \( D \) is adjacent to at least one vertex in \( D \). The domination number \( \gamma(G) \) equals the minimum cardinality of a dominating set in \( G \). The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [3] and [4].

Let \( D \) be a set of vertices of a graph \( G \). For two vertices \( u \) and \( v \) of \( G \), let \( dist_{(G,D)}(u,v) \) be the minimum length of a path \( P \) in \( G \) between \( u \) and \( v \) such that \( D \) contains exactly one endvertex of \( P \) but no internal vertex of \( P \). If no such path exists, then let \( dist_{(G,D)}(u,v) = \infty \). Note that, if \( u \) and \( v \) are distinct vertices in \( D \), then \( dist_{(G,D)}(u,u) = 0 \) and \( dist_{(G,D)}(u,v) = \infty \). For a vertex \( u \) of \( G \), let \( \omega_{(G,D)}(u) = \sum_{v \in D} \left( \frac{1}{2} \right)^{dist_{(G,D)}(u,v) - 1} \), where \( \left( \frac{1}{2} \right)^\infty = 0 \).

Dankelmann et al. [2] define a set \( D \) to be an exponential dominating set of \( G \) if \( \omega_{(G,D)}(u) \geq 1 \) for every vertex \( u \) of \( G \), and the exponential domination number \( \gamma_e(G) \) of \( G \) as the minimum size of an exponential dominating set of \( G \). Note that \( \omega_{(G,D)}(u) \geq 2 \) for \( u \in D \), and that \( \omega_{(G,D)}(u) \geq 1 \) for every vertex \( u \) that has a neighbor in \( D \), which implies \( \gamma_e(G) \leq \gamma(G) \).

Bessy et al. [1] show that computing the exponential domination number is APX-hard for subcubic graphs. It is not even known how to decide efficiently for a given tree \( T \) whether its exponential domination number \( \gamma_e(T) \) equals its domination number \( \gamma(T) \). The difficulty to decide whether \( \gamma_e(G) = \gamma(G) \) for a given graph \( G \) motivates the study of the hereditary class \( \mathcal{G} \) of graphs that satisfy this equality, that is, \( \mathcal{G} \) is the set of those graphs \( G \) such that \( \gamma_e(H) = \gamma(H) \) for every induced subgraph \( H \) of \( G \).

Henning et al. [5] proved the following results.

**Proposition 1** [5]. If \( G \) is a \( \{B,D,K_4,K_{2,3},P_3 \square P_3\} \)-free graph, then \( \gamma(H) = \gamma_e(H) \) for every induced subgraph \( H \) of \( G \) if and only if \( G \) is \( \{P_7,C_7,F_1,F_2,F_3,F_4,F_5\} \)-free.

**Proposition 2** [5]. If \( T \) is a tree, then \( \gamma(H) = \gamma_e(H) \) for every induced subgraph \( H \) of \( T \) if and only if \( T \) is \( \{P_7,F_1\} \)-free.

Furthermore, they gave the following conjecture.

**Conjecture 1** [5]. There is a finite set \( \mathcal{F} \) of graphs such that graph \( G \) satisfies \( \gamma(H) = \gamma_e(H) \) for every induced subgraph \( H \) of \( G \) if and only if \( G \) is \( \mathcal{F} \)-free.
In this paper, we study the conjecture for subcubic graphs. We characterize the class $\mathcal{F}$ by minimal forbidden induced subgraphs. Our main result is the following.

**Theorem 1.** Let $G$ be a subcubic graph. Then $\gamma(H) = \gamma_e(H)$ for every induced subgraph $H$ of $G$ if and only if $G$ is $\mathcal{F}$-free, where $\mathcal{F} = \{P_7, C_7, F_1, F_2, F_3, F_6, F_7, F_8, F_9, F_{10}, F_{11}\}$.

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**Figure 1.** The graphs $K_{2,3}$, $P_2 \square P_3$, $B$ and $D$.

**Figure 2.** The graphs $F_1$, $F_2$, $F_3$, $F_4$ and $F_5$.

**Figure 3.** The graphs $F_6, \ldots, F_{11}$.
2. Proof of Theorem 1

Proof. Since $\gamma(H) > \gamma_e(H)$ for every graph $H$ in $\mathcal{F}$, necessity follows. In order to prove sufficiency, suppose that $G$ is an $\mathcal{F}$-free graph with $\gamma(G) > \gamma_e(G)$ of minimum order. By the choice of $G$, we have $\gamma(H) = \gamma_e(H)$ for every proper induced subgraph $H$ of $G$. Clearly, $G$ is connected. Since $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$, we obtain $\gamma_e(G) \geq 2$ and $\gamma(G) \geq 3$. Since $G$ is $\{P_t, C_7\}$-free, either $G$ is a tree or $G$ is a subcubic graph with $3 \leq l(G) \leq 6$.

By Proposition 2, $G$ is not a tree. Then $G$ is a connected subcubic graph with $3 \leq l(G) \leq 6$. Let $C : x_1x_2x_3 \cdots x_{l(G)}x_1$ be a longest induced cycle of $G$. Let $R = V(G) \setminus V(C)$.

Case 1. $l(G) = 6$. Assume some vertex $z$ has distance 2 from a vertex on $V(C)$ in $G$ and $x_1yz$ is a path in $G$. If $y$ is adjacent to $x_2$, then $G[\{x_1, x_2, x_3, x_6, y, z\}] = F_6$, which is a contradiction. If $y$ is adjacent to $x_3$, then $G[\{x_1, x_3, x_4, x_6, y, z\}] = F_1$, which is a contradiction. By symmetry, we can assume without loss of generality that $y$ is adjacent to neither $x_5$ nor $x_6$. Then $G[\{x_1, x_2, x_5, x_6, y, z\}] = F_1$, which is a contradiction. So every vertex in $R$ has distance one from one vertex on $V(C)$. Since $G$ is $F_1$-free, every vertex in $R$ has at least two neighbors on $C$. Since $G$ is a subcubic graph and $\gamma(G) \geq 3$, $2 \leq |R| \leq 3$.

Case 1.1. $|R| = 3$. Say $R = \{u, v, w\}$. Then every vertex in $R$ is adjacent to exactly two vertices on $C$. Suppose that there exists one vertex in $R$ that is adjacent to two vertices on $C$ with distance three. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_4$. Then $G[\{x_1, x_2, x_3, x_5, x_6, u\}] = F_1$, which is a contradiction. Hence every vertex in $R$ is adjacent to two vertices on $C$ with distance at most two. Since $G$ is subcubic and the three vertices in $R$ can not all be adjacent to two vertices on $C$, there exists a vertex in $R$ that is adjacent to two adjacent vertices on $C$. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_2$. Assume that $x_3$ is adjacent to $v$. Then $v$ is adjacent to either $x_4$ or $x_5$.

If $v$ is adjacent to $x_4$, then $w$ is adjacent to $x_5$ and $x_6$. If $vw \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, x_5, v, w\}] = F_{10}$, which is a contradiction. If $vw \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, u, v, w\}] = F_{10}$, which is a contradiction.

If $v$ is adjacent to $x_5$, then $w$ is adjacent to $x_4$ and $x_6$. If $vw \in E(G)$, then $G[\{x_1, x_4, x_5, u, v, w\}] = F_8$, which is a contradiction. If $vw \notin E(G)$, then $G[\{x_1, x_2, x_5, v, w\}] = F_1$, which is a contradiction.

Case 1.2. $|R| = 2$. Say $R = \{u, v\}$. Suppose that there exists one vertex in $R$ such that it is adjacent to exactly two vertices on $C$ with distance three. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_4$. Then $G[\{x_1, x_2, x_3, x_5, x_6, u\}] = F_1$, which is a contradiction. Hence, we can assume that every vertex in $R$ is not adjacent to exactly two vertices on $C$ with distance three.

Proof. Since $\gamma(H) > \gamma_e(H)$ for every graph $H$ in $\mathcal{F}$, necessity follows. In order to prove sufficiency, suppose that $G$ is an $\mathcal{F}$-free graph with $\gamma(G) > \gamma_e(G)$ of minimum order. By the choice of $G$, we have $\gamma(H) = \gamma_e(H)$ for every proper induced subgraph $H$ of $G$. Clearly, $G$ is connected. Since $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$, we obtain $\gamma_e(G) \geq 2$ and $\gamma(G) \geq 3$. Since $G$ is $\{P_t, C_7\}$-free, either $G$ is a tree or $G$ is a subcubic graph with $3 \leq l(G) \leq 6$.

By Proposition 2, $G$ is not a tree. Then $G$ is a connected subcubic graph with $3 \leq l(G) \leq 6$. Let $C : x_1x_2x_3 \cdots x_{l(G)}x_1$ be a longest induced cycle of $G$. Let $R = V(G) \setminus V(C)$.

Case 1. $l(G) = 6$. Assume some vertex $z$ has distance 2 from a vertex on $V(C)$ in $G$ and $x_1yz$ is a path in $G$. If $y$ is adjacent to $x_2$, then $G[\{x_1, x_2, x_3, x_6, y, z\}] = F_6$, which is a contradiction. If $y$ is adjacent to $x_3$, then $G[\{x_1, x_3, x_4, x_6, y, z\}] = F_1$, which is a contradiction. By symmetry, we can assume without loss of generality that $y$ is adjacent to neither $x_5$ nor $x_6$. Then $G[\{x_1, x_2, x_5, x_6, y, z\}] = F_1$, which is a contradiction. So every vertex in $R$ has distance one from one vertex on $V(C)$. Since $G$ is $F_1$-free, every vertex in $R$ has at least two neighbors on $C$. Since $G$ is a subcubic graph and $\gamma(G) \geq 3$, $2 \leq |R| \leq 3$.

Case 1.1. $|R| = 3$. Say $R = \{u, v, w\}$. Then every vertex in $R$ is adjacent to exactly two vertices on $C$. Suppose that there exists one vertex in $R$ that is adjacent to two vertices on $C$ with distance three. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_4$. Then $G[\{x_1, x_2, x_3, x_5, x_6, u\}] = F_1$, which is a contradiction. Hence every vertex in $R$ is adjacent to two vertices on $C$ with distance at most two. Since $G$ is subcubic and the three vertices in $R$ can not all be adjacent to two vertices on $C$, there exists a vertex in $R$ that is adjacent to two adjacent vertices on $C$. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_2$. Assume that $x_3$ is adjacent to $v$. Then $v$ is adjacent to either $x_4$ or $x_5$.

If $v$ is adjacent to $x_4$, then $w$ is adjacent to $x_5$ and $x_6$. If $vw \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, x_5, v, w\}] = F_{10}$, which is a contradiction. If $vw \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, u, v, w\}] = F_{10}$, which is a contradiction.

If $v$ is adjacent to $x_5$, then $w$ is adjacent to $x_4$ and $x_6$. If $vw \in E(G)$, then $G[\{x_1, x_4, x_5, u, v, w\}] = F_8$, which is a contradiction. If $vw \notin E(G)$, then $G[\{x_1, x_2, x_5, v, w\}] = F_1$, which is a contradiction.

Case 1.2. $|R| = 2$. Say $R = \{u, v\}$. Suppose that there exists one vertex in $R$ such that it is adjacent to exactly two vertices on $C$ with distance three. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_4$. Then $G[\{x_1, x_2, x_3, x_5, x_6, u\}] = F_1$, which is a contradiction. Hence, we can assume that every vertex in $R$ is not adjacent to exactly two vertices on $C$ with distance three.
three. So there exists one vertex, say \( u \in R \), such that \( u \) is adjacent to two vertices on \( C \) with distance at most two.

Suppose that \( u \) is adjacent to \( x_1 \) and \( x_2 \). If \( v \) is adjacent to \( x_4 \), where \( i \in \{4, 5\} \), then \( \{x_1, x_4\} \) or \( \{x_2, x_5\} \) is a dominating set of \( G \) and \( \gamma(G) \leq 2 \), which is a contradiction. So \( v \) is adjacent to exactly two vertices \( x_3 \) and \( x_6 \) on \( C \) with distance three, which is a contradiction.

Suppose that \( u \) is adjacent to \( x_1 \) and \( x_3 \). If \( v \) is adjacent to \( x_4 \), where \( i \in \{4, 6\} \), then \( \{x_1, x_4\} \) or \( \{x_3, x_6\} \) is a dominating set of \( G \) and \( \gamma(G) \leq 2 \), which is a contradiction. So \( v \) is adjacent to exactly two vertices \( x_2 \) and \( x_5 \) on \( C \) with distance three, which is a contradiction.

Case 2. \( l(G) = 5 \). Assume some vertex \( z \) has distance 2 from \( V(C) \) in \( G \) and \( x_1yz \) is a path in \( G \). If \( y \) is adjacent to \( x_2 \), then \( G[\{x_1, x_2, x_3, x_5, y, z\}] = F_6 \), which is a contradiction. If \( y \) is adjacent to \( x_3 \), then \( G[\{x_2, x_3, x_4, x_5, y, z\}] = F_1 \), which is a contradiction. By symmetry, \( y \) has exactly one neighbor \( x_1 \) on \( C \). Then \( G[\{x_1, x_2, x_3, x_5, y, z\}] = F_1 \), which is a contradiction. So every vertex in \( R \) has distance one from one vertex on \( V(C) \). Since \( G \) is a subcubic graph and \( \gamma(G) \geq 3, 2 \leq |R| \leq 5 \).

Case 2.1. \( |R| = 5 \). Say \( R = \{y_i \mid x_iy_i \in E(G), i = 1, 2, \ldots, 5\} \). If \( y_iy_2 \notin E(G) \), then \( G[\{x_1, x_2, x_4, x_5, y_1, y_2\}] = F_1 \), which is a contradiction. Hence, \( y_1y_2 \in E(G) \). Similarly, \( y_iy_{i+1} \in E(G) \) for \( i = 1, 2, 3, 4 \). Then \( G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_2 \), which is a contradiction.

Case 2.2. \( |R| = 4 \). Say \( R = \{y_i \mid x_iy_i \in E(G), i = 1, 2, 3, 4\} \). If \( y_1y_2 \notin E(G) \), then \( G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1 \), which is a contradiction. If \( y_3y_4 \notin E(G) \), then \( G[\{x_1, x_2, x_3, x_4, y_3, y_4\}] = F_1 \), which is a contradiction. Hence, \( y_1y_2 \in E(G) \) and \( y_3y_4 \in E(G) \). Since \( x_5 \) is adjacent to at most one vertex in \( \{y_1, y_2, y_3, y_4\} \), either \( G[V(C) \cup \{y_1, y_2\}] = F_3 \) or \( G[V(C) \cup \{y_3, y_4\}] = F_3 \), which is a contradiction.

Case 2.3. \( |R| = 3 \). Let \( G' \) be a graph with \( V(G') = V(C) \cup \{y_1, y_2, y_3\} \) and \( E(G') = E(C) \cup \{x_1y_1, x_2y_2, x_3y_3, y_1y_2\} \). Suppose that \( G' \) is a subgraph of \( G \). If \( y_1x_5 \in E(G) \), then \( \{x_3, y_1\} \) is a dominating set of \( G \), which is a contradiction. Hence, \( y_1x_5 \notin E(G) \). It follows that \( y_1 \) is adjacent to at most one vertex in \( \{x_4, y_3\} \).

Suppose that \( y_1x_4 \in E(G) \). If \( y_2x_5 \in E(G) \), then \( G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_8 \), which is a contradiction. If \( y_3x_5 \in E(G) \), then \( G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_3 \) or \( G[V(C) \cup \{y_1, y_2, y_3\}] = F_1 \), which is a contradiction. If \( d_C(x_5) = 2 \), then \( G[\{x_2, x_3, x_4, x_5, y_2, y_3\}] = F_1 \) or \( G[V(C) \cup \{y_2, y_3\}] = F_3 \), which is a contradiction. Hence, \( y_1x_4 \notin E(G) \).

Suppose that \( y_1y_3 \in E(G) \). If \( y_3x_4 \in E(G) \), then \( G[\{x_2, x_3, x_4, x_5, y_1, y_3\}] = F_6 \), which is a contradiction. If \( y_3x_5 \in E(G) \), then \( G[V(C) \cup \{y_1, y_2\}] = F_3 \) or \( G[V(C) \cup \{y_1, y_2, y_3\}] = F_1 \), which is a contradiction. If \( y_3x_4 \notin E(G) \) and
$y_3x_5 \notin E(G)$, then $G[\{x_1, x_3, x_4, x_5, y_1, y_3\}] = C_6$ and $l(G) \geq 6$, which is a contradiction. Hence, $y_1y_3 \notin E(G)$.

So $d_G(y_1) = 2$. Since $G$ is $F_3$-free, $y_2x_4 \in E(G)$ or $y_2x_5 \in E(G)$. If $y_2x_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_5, y_2, y_3\}] = F_1$ or $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_3$, which is a contradiction. If $y_2x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction. Hence, we can assume that no subgraph in $G$ is isomorphic to $G'$.

By symmetry, we discuss it in the following cases.

Case 2.3.1. $R = \{y_i | x_iy_i \in E(G), i = 1, 2, 3\}$. If $E(G[\{y_1, y_2, y_3\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction. Hence, $E(G[\{y_1, y_2, y_3\}]) \neq \emptyset$. Since no subgraph in $G$ is isomorphic to $G'$, $y_1y_2, y_2y_3 \notin E(G)$ and $y_1y_3 \in E(G)$. Since no subgraph in $G$ is isomorphic to $G'$, $y_1x_4 \notin E(G)$. If $y_2x_4 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}] = F_1$, which is a contradiction. Hence, $y_2x_4 \in E(G)$. Then $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_3$, which is a contradiction.

Case 2.3.2. $R = \{y_i | x_iy_i \in E(G), i = 1, 2, 4\}$. If $E(G[\{y_1, y_2, x_3\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_1$, which is a contradiction. Hence, $E(G[\{y_1, y_2, x_3\}]) \neq \emptyset$. Suppose that $y_1x_3 \in E(G)$. Since $G$ is $F_1$-free and no subgraph in $G$ is isomorphic to $G'$, $y_1y_2, y_1y_4 \notin E(G)$ and $y_2y_4 \in E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_3$, which is a contradiction. Hence, $y_1x_3 \notin E(G)$.

Suppose that $y_2x_3 \in E(G)$. If $E(G[\{y_1, y_2, y_4\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_1$, which is a contradiction. Hence, $E(G[\{y_1, y_2, y_4\}]) \neq \emptyset$. If $y_1y_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_4\}] = C_6$, which is a contradiction. If $y_1y_2 \in E(G)$ or $y_2y_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_7$, which is a contradiction. Hence, $y_2x_3 \notin E(G)$.

So $y_1x_3 \notin E(G)$, $y_2x_3 \notin E(G)$ and $y_1y_2 \in E(G)$. Since no subgraph in $G$ is isomorphic to $G'$, $y_4x_3 \notin E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_2$ or $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_3$, which is a contradiction.

Case 2.4. $|R| = 2$. Say $R = \{y_1, y_2\}$ and $y_1x_1 \in E(G)$. If $y_2x_i \in E(G)$ for $i \in \{3, 4\}$, then $\{x_1, x_i\}$ is a dominating set of $G$, which is a contradiction. Hence, $y_2x_3 \notin E(G)$ and $y_2x_4 \notin E(G)$. Without loss of generality, we can assume that $y_2x_2 \in E(G)$. If $y_1x_i \in E(G)$ for $i \in \{4, 5\}$, then $\{x_2, x_i\}$ is a dominating set of $G$, which is a contradiction. Hence, $y_1x_4 \notin E(G)$ and $y_1x_5 \notin E(G)$.

If $y_1x_3 \notin E(G)$ and $y_1y_2 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $y_1x_3 \in E(G)$ or $y_1y_2 \in E(G)$.

Suppose that $y_1x_3 \in E(G)$. If $y_2x_5 \in E(G)$, then $\{x_3, x_5\}$ is a dominating set of $G$, which is a contradiction. Hence, $y_2x_5 \notin E(G)$. If $y_1y_2 \notin E(G)$, then $G[\{x_2, x_3, x_4, x_5, y_1, y_2\}] = F_1$, which is a contradiction. If $y_1y_2 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction. Hence, $y_1x_3 \notin E(G)$ and $y_1y_2 \in E(G)$. If $y_2x_5 \notin E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_3$, which is a contradiction. If $y_2x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction.
Case 3. \( l(G) = 4 \). Assume some vertex \( t \) has distance 3 from one vertex on \( V(C) \) in \( G \) and \( x_1yzt \) is a path in \( G \). If \( y \) is adjacent to \( x_2 \), then \( G[V(C) \cup \{y,z,t\}] = F_7 \), which is a contradiction. If \( y \) is adjacent to \( x_3 \), then \( G[V(C) \cup \{y,z,t\}] = F_8 \), which is a contradiction. If \( y \) is not adjacent to \( x_i \) for \( i = 2, 3, 4 \), then \( G[V(C) \cup \{y,z,t\}] = F_2 \), which is a contradiction. So every vertex in \( R \) has distance at most two from a vertex on \( V(C) \). If \( |N(V(C)) \cap R| = 1 \), say \( x_1y_1 \in E(G) \), then \( \{y_1, x_3\} \) is a dominating set of \( G \), which is a contradiction. Hence, \( 2 \leq |N(V(C)) \cap R| \leq 4 \).

Case 3.1. \( |N(V(C)) \cap R| = 4 \). Say \( N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1, 2, 3, 4\} \). If \( y_1y_3 \in E(G) \), then \( G[\{x_1, x_2, x_3, y_1, y_3\}] = C_5 \), which is a contradiction with \( l(G) = 4 \). By symmetry, \( y_1y_3 \notin E(G) \) and \( y_2y_4 \notin E(G) \).

If \( y_1y_2 \notin E(G) \) and \( y_2y_3 \notin E(G) \), then \( G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1 \), which is a contradiction. Hence, \( y_1y_2 \in E(G) \) or \( y_2y_3 \in E(G) \). Without loss of generality, we can assume that \( y_1y_2 \in E(G) \). If \( y_2y_3 \in E(G) \), then \( G[\{x_1, x_3, x_4, y_1, y_2, y_3, y_4\}] = F_2 \), which is a contradiction. If \( y_3y_4 \notin E(G) \), then \( G[\{x_2, x_3, x_4, y_2, y_3, y_4\}] = F_1 \), which is a contradiction.

Case 3.2. \( |N(V(C)) \cap R| = 3 \). Say \( N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1, 2, 3\} \). If \( y_1y_3 \in E(G) \), then \( G[\{x_1, x_2, x_3, y_1, y_3\}] = C_5 \), which is a contradiction. Hence, \( y_1y_3 \notin E(G) \). If \( y_1y_2 \notin E(G) \) and \( y_2y_3 \notin E(G) \), then \( G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1 \), which is a contradiction. Hence, \( y_1y_2 \in E(G) \) or \( y_2y_3 \in E(G) \). Without loss of generality, we can assume that \( y_1y_2 \in E(G) \). If \( y_1x_4 \in E(G) \), then \( G[\{x_2, x_3, x_4, y_1, y_2\}] = C_5 \), which is a contradiction.

Suppose that \( y_2y_3 \in E(G) \). If \( y_3x_4 \in E(G) \), then \( G[\{x_1, x_4, y_1, y_2, y_3\}] = C_5 \), which is a contradiction. Hence, \( y_1x_4 \notin E(G) \) and \( y_3x_4 \notin E(G) \). Then \( G[\{x_1, x_3, x_4, y_1, y_2, y_3\}] = C_6 \), which is a contradiction. Hence \( y_2y_3 \notin E(G) \).

Suppose that \( N(y_2) \setminus (V(C) \cup \{y_1, y_2, y_3\}) \neq \emptyset \), say \( t \in N(y_2) \setminus (V(C) \cup \{y_1, y_2, y_3\}) \). Since \( l(G) = 4 \) and \( y_2t \notin E(G) \), then \( G[\{x_1, x_2, x_3, y_1, y_2, y_3, t\}] = F_2 \), which is a contradiction. Hence \( N(y_3) \setminus (V(C) \cup \{y_1, y_2, y_3\}) = \emptyset \).

Suppose that \( N(y_2) \setminus (V(C) \cup N[y_1]) \neq \emptyset \), say \( t \in N(y_2) \setminus (V(C) \cup N[y_1]) \). Since \( l(G) = 4 \) and \( y_3t \notin E(G) \), then \( G[\{x_1, x_2, x_3, y_2, y_3, t\}] = F_1 \), which is a contradiction. Hence, \( N(y_2) \setminus (V(C) \cup N[y_1]) = \emptyset \). Then \( \{y_1, x_3\} \) is a dominating set of \( G \), which is a contradiction.

Case 3.3. \( |N(V(C)) \cap R| = 2 \).

Case 3.3.1. \( N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1, 2\} \). Since \( \{x_1, x_2\} \) is not a dominating set of \( G \), \( V(G) \setminus (V(C) \cup \{y_1, y_2\}) \neq \emptyset \). Say \( t_1 \in N(y_1) \setminus (V(C) \cup \{y_2\}) \).

Suppose that \( y_1y_2 \in E(G) \). If \( N(y_2) \setminus (V(C) \cup \{y_1, t_1\}) = \emptyset \), then \( \{y_1, x_3\} \) is a dominating set of \( G \), which is a contradiction. Hence, we can assume that \( t_2 = N(y_2) \setminus \{x_2, y_1\} \). If \( t_1t_2 \notin E(G) \), then \( G[\{x_2, x_3, y_2, t_1, t_2\}] = F_1 \), which
is a contradiction. If \( t_1 t_2 \in E(G) \), then \( G[x_2, x_3, x_4, y_1, y_2, t_1, t_2] = F_2 \), which is a contradiction. Hence, we can assume that \( y_1 y_2 \notin E(G) \).

Suppose that \( N(y_2) \setminus V(C) = \emptyset \). Since \( G[x_1, x_2, x_4, y_1, y_2, t_1] = F_1 \), \( x_4 y_1 \in E(G) \) or \( x_4 y_2 \in E(G) \). If \( x_4 y_1 \in E(G) \), then \( \{y_1, x_2\} \) is a dominating set of \( G \), which is a contradiction. Hence, \( x_4 y_2 \in E(G) \). If \( x_3 y_1 \in E(G) \) or \( x_3 y_2 \in E(G) \), then \( \{y_1, y_2\} \) is a dominating set of \( G \), which is a contradiction. Hence, \( x_3 y_1 \notin E(G) \) and \( x_3 y_2 \notin E(G) \). Then \( G[x_1, x_2, x_3, x_4, y_1, y_2, t_1] = F_8 \), which is a contradiction. Hence, \( N(y_2) \setminus V(C) \neq \emptyset \). Say \( t_2 \in N(y_2) \setminus V(C) \). Since \( G \) is \( F_1 \)-free, \( \{x_3, x_4\} \subseteq N(\{y_1, y_2\}) \). Then \( \{y_1, y_2\} \) is a dominating set of \( G \), which is a contradiction.

**Case 3.3.2.** \( N(C) = \{y_i \mid x_i y_i \in E(G) \}, i = 1, 3\). Since \( l = 4 \), \( y_1 y_3 \notin E(G) \). Since \( \{x_1, x_3\} \) is not a dominating set of \( G \), there is a vertex \( u \) at distance 2 from \( \{x_1, y_3\} \) in \( G \). If \( u x_i \in E(G) \) for \( i \in \{2, 3\} \), then \( G[u, x_i, x_1, y_1, y_2, y_3, y_4] = P_7 \), which is a contradiction. Suppose that \( u \) is adjacent to \( y_1 \). If \( u y_2 \notin E(G) \), then \( G[u, x_1, x_2, y_1, y_2, y_3] = F_1 \), which is a contradiction. If \( u y_2 \in E(G) \), then \( G[u, x_1, x_2, y_1, y_2, y_3, y_4] = F_{10} \), which is a contradiction. Hence, \( u y_1 \notin E(G) \). By a similar way, \( u y_2 \notin E(G) \) and \( u y_4 \notin E(G) \). Suppose that there exists a path \( y_3 u u \). If \( u y_2 \in E(G) \), then \( G[u, v, y_1, y_2, y_3, y_4] = F_6 \), which is a contradiction. Since \( l = 3 \), \( \{x_2, x_3, y_1\} \cap N(v) = \emptyset \). So \( G[u, v, x_1, x_2, y_1, y_2, y_3] = P_7 \), which is a contradiction. Hence, we can assume that every vertex in \( R \) has distance at most 3 from one vertex on \( V(C) \).

**Case 4.** \( l(G) = 3 \). Since \( G \) is \( P_7 \)-free, every vertex in \( R \) has distance at most 4 from one vertex on \( V(C) \). Assume vertex \( y_4 \) has distance 4 from one vertex on \( V(C) \) in \( G \) and \( x_1 y_1 y_2 y_3 y_4 \) is a path in \( G \). Since \( \{x_1, y_3\} \) is not a dominating set of \( G \), there is a vertex \( u \) at distance 2 from \( \{x_1, y_3\} \) in \( G \). If \( u x_i \in E(G) \) for \( i \in \{2, 3\} \), then \( G[u, x_i, x_1, y_1, y_2, y_3, y_4] = P_7 \), which is a contradiction. Suppose that \( u \) is adjacent to \( y_1 \). If \( u y_2 \notin E(G) \), then \( G[u, x_1, x_2, y_1, y_2, y_3] = F_1 \), which is a contradiction. Hence, \( u y_1 \notin E(G) \). By a similar way, \( u y_2 \notin E(G) \) and \( u y_4 \notin E(G) \). Suppose that there exists a path \( y_3 u u \). If \( u y_2 \in E(G) \), then \( G[u, v, y_1, y_2, y_3, y_4] = F_6 \), which is a contradiction. Since \( l = 3 \), \( \{x_2, x_3, y_1\} \cap N(v) = \emptyset \). So \( G[u, v, x_1, x_2, y_1, y_2, y_3] = P_7 \), which is a contradiction. Hence, we can assume that every vertex in \( R \) has distance at most 3 from one vertex on \( V(C) \).

**Case 4.1.** \( |N(V(C)) \cap R| = 3 \). Say \( |N(V(C)) \cap R| = \{y_i \mid x_i y_i \in E(G), i = 1, 2, 3\} \). Since \( l = 3 \), \( E(G[\{y_1, y_2, y_3\}]) = \emptyset \). Then \( G[x_1, x_2, x_3, y_1, y_2, y_3] = F_6 \), which is a contradiction.

**Case 4.2.** \( |N(V(C)) \cap R| = 2 \). Say \( |N(V(C)) \cap R| = \{y_i \mid x_i y_i \in E(G), i = 1, 2\} \). Since \( l = 3 \), \( y_1 y_2 \notin E(G) \). Suppose that there exists an induced path \( x_1 y_1 u_1 v_1 \). Since \( G \) is \( P_7 \)-free, \( N(y_2) \setminus V(C) = \emptyset \).

Suppose that there exists a vertex \( u \) such that \( u \in N(y_1) \setminus \{x_1, u_1\} \). If \( u_1 u \notin E(G) \), then \( G[u, x_1, x_2, y_1, u_1, v_1] = F_1 \), which is a contradiction. If \( u_1 u \in E(G) \), then \( \{u_1, x_2\} \) is a dominating set of \( G \), which is a contradiction. If \( N(y_1) \setminus \{x_1, u_1\} = \emptyset \), then \( \{u_1, x_2\} \) is a dominating set of \( G \), which is a contradiction. Hence, we can assume that every vertex in \( V(G) \setminus (V(C) \cup \{y_1, y_2\}) \) is adjacent to exactly one vertex in \( \{y_1, y_2\} \).
If \( N(y_i) \cap (V(G) \setminus (V(C) \cup \{y_1, y_2\})) = \emptyset \), then \( \{x_i, y_j\} \) is a dominating set of \( G \), where \( i, j \in \{1, 2\} \) and \( j \neq i \), which is a contradiction. Suppose that \( N(y_i) \cap (V(G) \setminus (V(C) \cup \{y_1, y_2\})) \neq \emptyset \) for \( i \in \{1, 2\} \). If \( x_3 \) is not adjacent to \( y_1 \) and \( y_2 \), then \( G[\{x_1, x_2, x_3, y_1, y_2, s_1, y_2, s_2\}] = F_{10} \), where \( s_i \in N(y_i) \), which is a contradiction. If \( x_3 \) is adjacent to \( y_1 \) or \( y_2 \), then \( \{y_1, y_2\} \) is a dominating set of \( G \), which is a contradiction.

**Case 4.3.** \( ||N(V(C)) \cap R|| = 1 \). Say \( y_1x_1 \in E(G) \). Since \( \{x_1, y_1\} \) is not a dominating set of \( G \), there is a vertex \( u \) at distance 2 from \( y_1 \) in \( G \). Without loss of generality, we can assume that \( y_1vu \) be a induced path. If there exists a vertex \( t \) such that \( y_1t \in E(G) \). If \( tv \notin E(G) \), then \( G[\{u, v, x_2, y_1, t\}] = F_1 \), which is a contradiction. Suppose that \( tv \in E(G) \). If \( N(t) \setminus \{y_1, v\} \neq \emptyset \), say \( s \in N(t) \setminus \{y_1, v\} \), then \( G[\{t, s, u, v, u, x_1, y_1\}] = F_6 \), which is a contradiction. If \( N(t) \setminus \{y_1, v\} = \emptyset \) or \( d_G(y_1) = 2 \), then \( \{v, x_1\} \) is a dominating set of \( G \), which is a contradiction.

### 3. Remark

Henning et al. also gave the following conjecture.

**Conjecture 2** [5]. The set \( \mathcal{F} \) in Conjecture 1 can be chosen such that \( \gamma(F) = 3 \) and \( \gamma_e(F) = 2 \) for every graph \( F \) in \( \mathcal{F} \).

It is obvious that the conjecture holds for subcubic graphs.

### References


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