

VIZING'S CONJECTURE AND THE ONE-HALF ARGUMENT

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Abstract

The domination number of a graph G is the smallest order, $\gamma(G)$, of a dominating set for G . A conjecture of V. G. Vizing [5] states that for every pair of graphs G and H , $\gamma(G \square H) \geq \gamma(G)\gamma(H)$, where $G \square H$ denotes the Cartesian product of G and H . We show that if the vertex set of G can be partitioned in a certain way then the above inequality holds for every graph H . The class of graphs G which have this type of partitioning includes those whose 2-packing number is no smaller than $\gamma(G) - 1$ as well as the collection of graphs considered by Barcalkin and German in [1]. A crucial part of the proof depends on the well-known fact that the domination number of any connected graph of order at least two is no more than half its order.

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1. INTRODUCTION AND TERMINOLOGY

We consider only finite, simple, undirected graphs. The vertex set of a graph G will be denoted by $V(G)$ and its edge set by $E(G)$. For a subset D of $V(G)$, $\langle D \rangle$ is the induced subgraph on the vertices of D . The *neighborhood* of $v \in V(G)$ is $N(v) = \{x \mid xv \in E(G)\}$ and $N(D) = \cup_{x \in D} N(x)$; $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v and $N[D] = \cup_{v \in D} N[v]$. The set D is called a *dominating set* for G if $N[D] = V(G)$. The cardinality of a smallest dominating set for G is the *domination number* of G and is denoted by $\gamma(G)$. We will refer to any dominating set of G having cardinality $\gamma(G)$ as a γ -*set* of G . $C \subseteq V(G)$ will be called a *clique* if $\langle C \rangle$ is a (not necessarily maximal) complete graph.

A set $I \subseteq V(G)$ is a *2-packing* of G if $N[x] \cap N[y] = \emptyset$ for every pair $x, y \in I$, $x \neq y$. $P_2(G)$, the *2-packing number* of G , is the cardinality of the largest 2-packing of G . Note that since a dominating set in G must contain at least one vertex from every closed neighborhood of G , it is immediate that $\gamma(G) \geq P_2(G)$.

The *Cartesian product*, $G \square H$, of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and where two vertices are adjacent when they are equal in one coordinate and adjacent in the other. That is, (u, v) and (x, y) are adjacent in $G \square H$ when either $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

Let h be any vertex of H . We let G_h represent the induced subgraph $\langle \{(x, h) \mid x \in V(G)\} \rangle$. Note that G_h is isomorphic to G . We refer to this subgraph as *level h* of $G \square H$. We shall say that level t is a *neighboring level* of level h if and only if t and h are neighbors in H . Similarly, H_g for $g \in V(G)$ will denote the subgraph of $G \square H$ induced by the set of vertices $\{(g, y) \mid y \in V(H)\}$. If $S \subset V(G)$, then $H_S = \langle S \times V(H) \rangle$. This subgraph of $G \square H$ is isomorphic to $\langle S \rangle \square H$.

We are interested in the conjecture first suggested by Vizing [5]: *for all graphs G and H , $\gamma(G \square H) \geq \gamma(G)\gamma(H)$* . We will say that *Vizing's conjecture is true for a graph G* if the above inequality is true for every graph H .

Suppose that D is any dominating set for $G \square H$ and (g, h) is any vertex not in D . Since D must intersect the neighborhood of (g, h) , D must contain either a vertex (g, t) where level t is a neighboring level of level h or a vertex (s, h) where vertices s and g are adjacent in G .

Our approach will be to partition the vertices of G into $\gamma(G)$ sets, say $S_1, S_2, \dots, S_{\gamma(G)}$, and then to show that each H_{S_i} contains at least $\gamma(H)$ vertices of any dominating set D of $G \square H$, or if some of these induced

subgraphs have fewer than $\gamma(H)$ members of D then there must be sufficient extras in those which do to compensate for the shortage.

The simplest case occurs when one is guaranteed that each H_{S_i} has at least $\gamma(H)$ elements of D . For instance, it has been noted by a number of authors (see [1] and [4]) that if $P_2(G) = \gamma(G)$, then $\gamma(G \square H) \geq \gamma(G)\gamma(H)$. This follows by observing that for each vertex v in a maximum 2-packing of G there must be sufficient vertices in D to dominate H_v , and these vertices can only be from $H_{N[v]}$.

For example, if G is the 6-cycle having consecutive vertices a, b, c, d, e, f , then to dominate H_b only vertices from $H_a \cup H_b \cup H_c$ can be used. Similarly, H_e is dominated only from $H_d \cup H_e \cup H_f$. Hence at least $2\gamma(H)$ will be required in any dominating set of $C_6 \square H$.

The following observation proves useful in working with Vizing's conjecture. It can be used to conclude that Vizing's conjecture is true for a given class of graphs after proving it for those graphs in the class which are edge-maximal with respect to the domination number.

Lemma 1.1. *If G is a spanning subgraph of G' such that $\gamma(G) = \gamma(G')$ and if Vizing's conjecture is true for G' , then it is also true for G .*

Proof. Let G and G' be as in the statement of the lemma and let H be any graph. Since the domination number of a spanning subgraph is always at least as large as that of the original graph, it follows that $\gamma(G \square H) \geq \gamma(G' \square H) \geq \gamma(G')\gamma(H) = \gamma(G)\gamma(H)$. ■

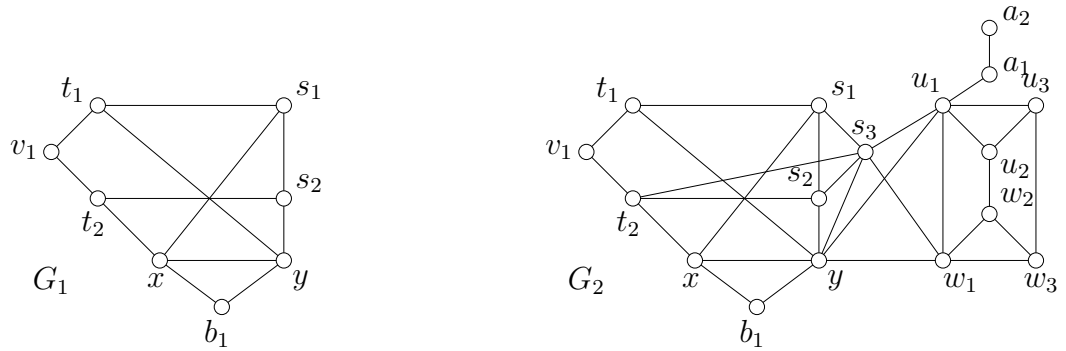
Another very useful result is that of Barcalkin and German [1] which is a more general partitioning condition than the 2-packing one mentioned above.

Theorem 1.2. *Suppose G is a spanning subgraph of a graph G' such that $\gamma(G) = \gamma(G')$ and such that $V(G')$ can be partitioned into $\gamma(G')$ subsets each of which induces a clique in G' . Then Vizing's conjecture is true for G .*

To see that the previous case, where G has a 2-packing $\{v_1, v_2, \dots, v_{\gamma(G)}\}$, is actually a special case of this theorem it suffices to observe that sufficient edges can be added to make each $N[v_i]$ a clique. Any vertices of G not included in $\cup_{i=1}^{\gamma(G)} N[v_i]$ can be assigned to any of the cliques. The resulting graph G' has domination number $\gamma(G)$. Thus in the C_6 example above, edges ac and df can be added to C_6 to form two triangles. However, this result also handles such diverse cases as $K_n \square K_n$ (as it partitions into n cliques) and C_7 with vertices (in order) a, b, c, d, e, f, g . Here one can

produce three cliques (one K_3 and two K_2 's) by adding the two edges from a to vertices d and e . In the first example note that the 2-packing number is much smaller than the domination number. Since in C_7 one cannot, for example, join vertices a and c without lowering the domination number, this example illustrates the care that must be taken to check if the Barcalkin and German result applies.

Our intent is to consider graphs G which cannot be partitioned into $\gamma(G)$ cliques (nor are they spanning subgraphs of such) and extend the partition concept to include some of them. For example, the four graphs in Figure 1 are potential candidates.



$$S_1 = \{v_1, t_1, t_2\}, SC = \{s_1, s_2, s_3\}, B_1 = \{b_1, x, y\}$$

$$C_1 = \{a_1, a_2\}, C_2 = \{u_1, u_2, u_3\}, C_3 = \{w_1, w_2, w_3\}$$

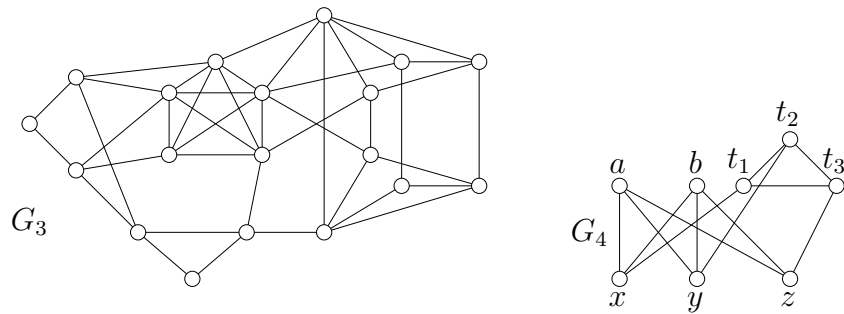


Figure 1

2. ILLUSTRATION OF PROOF TECHNIQUE

Before we consider graphs of a more general nature, we will illustrate the main new argument that we employ on a very small graph. In particular, consider $G = C_4$ with vertices (in order) a, b, c, d . Although this graph can be partitioned into $\gamma(G)$ cliques (for instance, $\{a, b\}$ and $\{c, d\}$), and so Theorem 1.2 states that Vizing's conjecture is true for G , we will show another argument that can be used.

For an arbitrary graph H let D be any dominating set of $G \square H$. If H_a is dominated by $D \cap H_a$, then $|D \cap H_a| \geq \gamma(H)$. If there are some vertices of H_a not dominated in this way, then we make a list of these "missing levels" for H_a . That is, vertex x of H is on the missing level list for H_a if $D \cap H_a$ contains no vertex of the form (a, y) for $y \in N[x]$. Since D dominates (a, x) it follows that D must contain a vertex from level x in either H_b or H_d . Similarly, any vertex of H_c not dominated by $D \cap H_c$ appears on the missing level list for H_c .

We will now show there are sufficient elements, say m , of D in $H_b \cup H_d$ so that some set of m vertices could be carefully chosen to dominate the missing levels in H_a and those in H_c . Consider the members of $D \cap H_d$ and project them onto H_b . That is, if $(d, i) \in D$, then this is projected to (b, i) . Consider the set A of vertices in H_b consisting of the original members of $D \cap H_b$ as well as the projected elements from $D \cap H_d$. In general the subgraph F of H_b induced by A will consist of components of order two or more as well as singleton vertices. The effect they had as far as H_a and H_c were concerned was to dominate (although a given one may not actually be required) the vertices in the corresponding levels. But consider any component C of F of order two or more. All of the vertices of C can be dominated by a γ -set of C , and a γ -set of C will contain no more than $\frac{1}{2}|C|$ vertices. Hence for any such component we could count half the elements of D in that component towards H_a and half towards H_c . They would be sufficient to dominate all missing levels (in H_a and H_c) that correspond to the vertices in C . Of course, there may be fewer missing levels than there are vertices in C , but the worst case is that they are all missing.

Now consider a component of F of order one consisting of the single vertex (b, i) . This means that no neighbor of (b, i) is in $H_b \cap D$ and no neighbor of (d, i) is in $H_d \cap D$. If i appears on the missing level list of only one of H_a or H_c , then we can count (b, i) there. On the other hand, if i appears on the missing level list at both H_a and H_c , then since D

contains no vertices from neighboring levels of level i in $H_b \cup H_d$, it follows that D contains both (b, i) and (d, i) since both of these vertices must be dominated by D . Hence we have a “duplicate” and can count one of these for H_a and the other for H_c . Note that since these arose from an isolated vertex component there is no danger of counting one of these vertices from D in two different ways.

We have shown that there are sufficient extra vertices of D in $H_b \cup H_d$ to be “earmarked” for H_a and H_c to dominate any missing levels. Hence D must contain at least $2\gamma(H)$ vertices and so Vizing’s conjecture is true for G . It should be noted that one cannot consider the missing levels one at a time and simply project the corresponding vertices from $D \cap H_b$ or $D \cap H_d$ to either H_a or H_c where needed, because in so doing it is possible to leave portions of the graphs H_b and H_d not yet considered no longer dominated.

As another illustration of this technique consider $G = G_4$ as shown in Figure 1. Let H be any graph and consider any dominating set D of $G \square H$. If $D \cap H_a$ does not dominate H_a , then make a list of the missing levels for H_a . Similarly, make a missing level list for any vertices of H_b that are not dominated by $D \cap H_b$. For any fixed vertex, say t_1 , of the triangle project all the elements of $D \cap (H_{t_2} \cup H_{t_3})$ onto H_{t_1} . Let W represent the set of vertices in H_{t_1} which were already in D together with those which are images of the projections. If there are any vertices in H_{t_1} which are not dominated by W , they form a missing level list for the triangle.

First note that if vertex h is on the missing level list of the triangle, then it must be the case that (x, h) , (y, h) and (z, h) are all in D in order to dominate the “triangle part” of level h . Hence we will be able to count one member of D for missing level h in the triangle and still have two members of D for H_a and H_b if required.

Now project the vertices in $D \cap (H_y \cup H_z)$ onto H_x . Let F denote the subgraph of H_x induced by the resulting set A of vertices from $H_x \cap D$ together with the projection images. As in the previous example, the effect on H_a and H_b of some vertex in $D \cap (H_x \cup H_y \cup H_z)$ is to dominate the corresponding vertices in the same level and not in any neighboring levels. Any component of F of order at least two can be dominated by a subset of cardinality at most half the order of the component. Thus we can compensate for any missing level in either H_a or H_b which corresponded to a vertex in such a component. If the triangle were also missing that level, then as noted above we have duplicates at that level.

Suppose $\{(x, h)\}$ is a component of F . If only one of H_a or H_b is

missing level h , then we can count the vertex (x, h) where it is missing. In the event that h is on the missing level list for both H_a and H_b , then either there are at least two of (x, h) , (y, h) and (z, h) in D (and hence we can count one for each of H_a and H_b), or only one of these is in D . In this case there must be at least two of (t_1, h) , (t_2, h) and (t_3, h) in D to dominate $\{(x, h), (y, h), (z, h)\}$. Hence again we have a duplicate in the triangle at that level along with the single member from (x, h) , (y, h) and (z, h) for H_a and H_b . This counting gives $|D| \geq 3\gamma(H) = \gamma(G_4)\gamma(H)$. Thus Vizing's conjecture is true for G_4 .

3. THE MAIN THEOREM

Consider a graph G with $\gamma(G) = n = k + t + m + 1$ and such that $V(G)$ can be partitioned into $S \cup SC \cup BC \cup C$, where $S = S_1 \cup S_2 \cup \dots \cup S_k$, $BC = B_1 \cup B_2 \cup \dots \cup B_t$, and $C = C_1 \cup C_2 \cup \dots \cup C_m$. Each of $SC, B_1, \dots, B_t, C_1, \dots, C_m$ induces a clique. Every vertex of SC (special clique) has at least one neighbor outside SC whereas each of B_1, \dots, B_t (the buffer cliques), say B_i , has at least one vertex, say b_i , which has no neighbors outside B_i . Each $S_i \in \{S_1, S_2, \dots, S_k\}$ is star-like in that it contains a star centered at a vertex v_i which is adjacent to each vertex in $T_i = S_i - \{v_i\}$. The vertex v_i has no neighbors besides those in T_i . Although other pairs of vertices in T_i may be adjacent (and hence S_i does not necessarily induce a star), S_i does *not* induce a clique nor can more edges be added in $\langle S_i \rangle$ without lowering the domination number of G . Furthermore, there are no edges between vertices in S and vertices in C . For ease of reference we will say such a graph G is of *Type \mathcal{X}* .

It should be noted that a graph of Type \mathcal{X} need not have a clique having the properties of SC , and any of t , m or k is allowed to be 0. However, if such an SC is not in G , then $\gamma(G) = n = k + t + m$. Also, if SC is not present and BC is empty, but S as well as C are not empty, then the graph is disconnected. SC can not be the only one of these which is nonempty since by definition its vertices must have neighbors outside SC . As illustrations of graphs of Type \mathcal{X} see G_1, G_2 and G_3 in Figure 1.

Theorem 3.1. *Let G be a graph of Type \mathcal{X} . Vizing's conjecture is true for G .*

Proof. Let H be an arbitrary graph and suppose D is any dominating set for $G \square H$. For each set, an S_i or B_i , in the partition of $S \cup BC$, there

are sufficient elements of D to entirely dominate a copy of H . In particular, consider $S_j \in S$ and the copy of H represented by H_{v_j} . Any vertex (v_j, h) of H_{v_j} not dominated by $D \cap H_{v_j}$ must be dominated by a neighboring vertex in level h and hence by some vertex from $D \cap H_{T_j}$. In either case, all of H_{v_j} is dominated by $D \cap H_{S_j}$. Similarly, for each buffer clique B_j , there must be sufficient vertices in $D \cap H_{B_j}$ to entirely dominate H_{b_j} since b_j has no neighbors in $V(G) - B_j$.

However, H_{SC} and H_{C_j} may not be entirely dominated by vertices of $D \cap H_{SC}$ and $D \cap H_{C_j}$, respectively. We must show that there are sufficient elements of D to yield at least $\gamma(H)$ for each of the sets in $\{SC, C_1, \dots, C_m\}$ as well as $\{S_1, \dots, S_k, B_1, \dots, B_t\}$.

To obtain a measure of the shortfall at each clique, say K , in $\{SC, C_1, \dots, C_m\}$, select one vertex $w \in K$ and project all the elements of $D \cap H_K$ onto H_w . That is, a vertex $(u, h) \in (D \cap H_K) - H_w$ is projected onto the vertex (w, h) . If all of H_w is dominated by the elements of D originally in H_w or those projected onto H_w , then there is no shortfall as we must already have at least $\gamma(H)$ elements of D in H_K . On the other hand, if certain vertices (levels) of H_w are not so dominated, form a missing level list for H_w consisting of these. Observe that if vertex h is in the missing level list at H_w this means that no vertex of H_K in level h is in D nor was any vertex from a neighboring level to level h in $D \cap H_K$. This implies that all vertices from level h in H_K must be dominated by neighbors in level h in D corresponding to other sets in the partition of $V(G)$.

For example, let $G = G_1$ in Figure 1 and $H = P_5$ with vertices labeled 1,2,3,4,5 in order. Suppose the only vertex in $H_{s_1} \cap D$ is $(s_1, 1)$, and $(s_2, 1)$ and $(s_2, 5)$ are the only members of $H_{s_2} \cap D$. When projected onto H_{s_1} vertex 3 would be a missing level. Hence either $(t_1, 3)$ or $(x, 3)$ (or both) must belong to D to dominate $(s_1, 3)$. Similarly, $(t_2, 3)$ or $(y, 3)$ must belong to D to dominate $(s_2, 3) \in H_{s_2}$.

Note that there may well be duplicates at certain levels when the projecting occurs as is the case with level 1 in H_{s_1} . We note the extra occurrences in a particular clique of a level h vertex in D (i.e., all but one) may be required for counting towards any shortages at that level in other cliques. For instance in Figure 1, if both $(x, 3)$ and $(y, 3)$ are in D , one of these is sufficient for H_{b_1} , and the other could be counted towards the missing level 3 in SC .

We must modify this argument for the sets T_1, T_2, \dots, T_k . First observe that because of the structure of G , $(u, h) \in D \cap H_{T_j}$ is only needed to

dominate vertex (v_j, h) in H_{v_j} (not any vertex in a neighboring level in H_{v_j}). The vertices of $D \cap H_{T_j}$ may help dominate a missing level of H_{SC} as well. Although they may dominate vertices in some H_{T_r} or H_{B_p} , both H_{S_r} and H_{B_p} have sufficient members of D to dominate a copy of H and do not require external assistance in the count. Now for each d , $1 \leq d \leq k$, choose a vertex $e \in T_d$ and project all members of $D \cap T_d$ onto H_e . The projected vertices will induce a subgraph of H_e consisting of isolated vertices or of components of order two or more.

Let F be one of these components of order at least two. For $(e, f) \in F$ the corresponding vertex $(v_d, f) \in H_{v_d}$ may need to be dominated. Thus it is possible that $\gamma(F)$ vertices from F must be available to dominate H_{v_d} . But $\gamma(F) \leq \frac{1}{2}|F|$ and hence we could count $\gamma(F)$ for H_{v_d} and $\gamma(F)$ for H_{SC} , even though these vertices may not be needed at H_{SC} . All components of order two or more could be treated similarly. The singleton components will be treated separately.

Let us now proceed to consider $G \square H$. For each clique Q in $\{SC, C_1, C_2, \dots, C_m\}$ fix a vertex $a \in Q$ and project $D \cap H_Q$ onto H_a . Create a missing level list for each such Q . As discussed before, in the case of each member of $\{S_1, \dots, S_k, B_1, \dots, B_t\}$ there are sufficient members of D to dominate a copy of H , and hence there are no missing levels.

Now consider each vertex of H which is a missing level in at least one clique. Suppose a total of r cliques other than SC are missing level h . Then every vertex in level h corresponding to a vertex from one of these r cliques must be dominated by a member of D (in level h) corresponding to a vertex from another clique, possibly in $SC \cup BC$. Suppose there are a total of s such cliques. That is, there are s cliques which have at least one element of D at level h which is adjacent to at least one vertex in the set of r cliques missing level h . Call the set of vertices at level h in D from these s cliques D_r . The claim is that $|D_r| \geq r + s$, since in G_h the set D_r dominates the s cliques it belongs to as well as the set of r cliques missing level h . For if there were fewer than $r + s$ vertices in D_r , then we can extend D_r to a dominating set of G_h by including $(v_1, h), (v_2, h), \dots, (v_k, h)$ and one per clique at level h for the cliques not in the $r + s$ already considered. This resulting dominating set has cardinality less than n which is a contradiction.

In case SC is missing level h but level h of SC is entirely dominated from neighboring cliques (as would be possible if $G = G_1$ in Figure 1), the same argument applies. If SC is missing level h but level h of SC is not entirely dominated by vertices of D in neighboring cliques, then we

must consider elements of D in $T_1 \cup T_2 \cup \dots \cup T_k$. (For example, this would necessarily be the case if $G = G_3$ in Figure 1 since two of the vertices of SC are not adjacent to vertices in any of the cliques.)

For each j , $1 \leq j \leq k$, choose $w_j \in T_j$ and project $D \cap H_{T_j}$ onto H_{w_j} . The projected vertices will, for each T_j , induce a subgraph F_j of H_{w_j} consisting of components of order two or more and singleton vertices. There are two possibilities:

Case 1. SC is missing level h and for every i , $1 \leq i \leq k$, T_i has either no elements of D or one element of D in level h . Furthermore, for each T_i with exactly one element of D at level h , no vertex from a neighboring level to level h in H_{T_i} belongs to D (hence, when projected, singletons would result). In addition, for each such T_i , in the corresponding H_{v_i} , neither (v_i, h) nor any vertex from a neighboring level to level h belongs to D . Let S' denote the collection of all S_i corresponding to such a T_i .

In this case suppose a total of r_1 cliques besides SC are missing level h . These r_1 cliques, as well as SC and the set of, say r_2 , members of S' must be dominated at level h by the r_2 members of D in S' as well as vertices of D which are neighbors in level h but in other cliques. Suppose there are a total of s such cliques. That is, there are s cliques which have at least one element of D in level h which is adjacent to at least one vertex in SC , in a member of S' , or in one of the r_1 cliques missing level h . But if there were fewer than $r_1 + r_2 + s + 1$ such elements of D , then all of G_h could be dominated by enlarging this set to include v_j for each $S_j \notin S'$ and one vertex from each of the cliques not counted above. This is a contradiction since the resulting dominating set for G_h would have fewer than n vertices.

Case 2. SC is missing level h and there is at least one j , $1 \leq j \leq k$, such that T_j has either

- (2a) two or more elements of D in level h (and hence there will be duplicates when projected), or
- (2b) a level h member of D as well as a neighboring level member of D (and hence, when projected, a component of order two or more will result), or
- (2c) a level h vertex in D , and $D \cap H_{v_j}$ has either a vertex in level h or a vertex from a neighboring level of level h .

First, if any F_j contains a level h vertex in a component L of order two or more, then L can be dominated in $\gamma(L) \leq \frac{1}{2}|L|$ elements, and so $\gamma(L)$ of the vertices in L could be counted towards dominating H_{v_j} and $\gamma(L)$

towards dominating those missing levels of SC . If no F_i contains the level h vertex in a component of order at least two, then either (2a) holds which means there is a duplicate level h vertex in D available for SC , or (2c) holds, in which case H_{v_j} does not require the level h vertex in $D \cap T_j$ and so it can be counted for dominating SC .

Hence any missing level h of SC can be handled by either Case 1 or Case 2. We have shown that $|D| \geq n\gamma(H)$ and so Vizing's conjecture is true for G . ■

The following more general result is an immediate corollary of Theorem 3.1 and Lemma 1.1.

Corollary 3.2. *If G is a graph of Type \mathcal{X} and F is a spanning subgraph of G such that $\gamma(F) = \gamma(G)$, then Vizing's conjecture is true for F .*

We also note that the result of Barcalkin and German, Theorem 1.2, is a special case of this theorem.

Corollary 3.3. *If G is a graph as in Theorem 3.1 except that the set S is empty, then Vizing's conjecture is true for G .*

As mentioned in Section 1, if $\gamma(G) = P_2(G)$ then Vizing's conjecture is true for G . The following corollary of Theorem 3.1 shows that this can now be extended.

Corollary 3.4. *If G is a graph and $\gamma(G) = P_2(G) + 1$, then Vizing's conjecture is true for G .*

Proof. Let $P_2(G) = k$ and $\gamma(G) = k + 1$. Suppose $\{v_1, v_2, \dots, v_k\}$ is a maximum 2-packing of G . For each i , $1 \leq i \leq k$, let S_i be the subgraph of G induced by $N[v_i]$ and let $W = V(G) - \cup_{1 \leq j \leq k} N[v_j]$. Add edges if necessary to make W into a clique SC , and if possible, add edges in each $N[v_i]$ as long as the domination number of the resulting graph is not smaller than $\gamma(G)$. It is clear that the resulting graph G' has domination number $k + 1$ and satisfies the hypothesis of Theorem 3.1. The fact that Vizing's conjecture is true for G now follows from Corollary 3.2. ■

Note that each of G_2 and G_3 in Figure 1 satisfies the hypothesis of Theorem 3.1 but is not covered by Theorem 1.2 or Corollary 3.4.

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