*Discussiones Mathematicae*
*Graph Theory 15 (1995) 185–193*

# GENERALIZED LIST COLOURINGS OF GRAPHS

Mieczysław Borowiecki
Ewa Drgas-Burchardt

*Institute of Mathematics*
*Technical University*
*Podgórna 50, 65-246 Zielona Góra, Poland*

and

Peter Mihók

*Department of Geometry and Algebra*
*P.J. Šafárik University*
*041 54 Košice, Slovakia*

**Abstract**

We prove: (1) that $\mathrm{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$ can be arbitrarily large, where $\mathrm{ch}_{\mathcal{P}}(G)$ and $\chi_{\mathcal{P}}(G)$ are $\mathcal{P}$-choice and $\mathcal{P}$-chromatic numbers, respectively, (2) the $(\mathcal{P}, L)$-colouring version of Brooks' and Gallai's theorems.

**Keywords:** hereditary property of graphs, list colouring, vertex partition number.

**1991 Mathematics Subject Classification:** 05C15, 05C70.

## 1. Introduction and Notation

We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$ and the edge set by $E(G)$. The notation $H \subseteq G$ means that $H$ is a subgraph of $G$. The vertex induced (we will say briefly: induced) subgraph $H$ of $G$ is denoted by $H \leq G$. We say that $G$ contains $H$ whenever $G$ contains a subgraph isomorphic to $H$. In general, we follow the notation and terminology of [5].

Let $\mathcal{I}$ denote the set of all mutually nonisomorphic graphs. If $\mathcal{P}$ is a nonempty subset of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property that a graph is

a member of the set $\mathcal{P}$. We shall use the terms *set of graphs* and *property of graphs* interchangeably.
A property $\mathcal{P}$ of graphs is said to be *induced hereditary* (shortly: *hereditary*) if whenever $G \in \mathcal{P}$ and $H$ is a vertex induced subgraph of $G$, then also $H \in \mathcal{P}$. For convenience, the empty set $\emptyset$ will be regarded as the set inducing the subgraph with any property $\mathcal{P}$.

A property $\mathcal{P}$ is *additive*, if for each graph $G$ all of whose components have the property $\mathcal{P}$ it follows that $G \in \mathcal{P}$. Let us denote by $\mathbf{M}$ and $\mathbf{M}^a$ the set of all hereditary and additive hereditary properties, respectively. Any hereditary property $\mathcal{P}$ of graphs is uniquely determined by the set $\mathcal{C}(\mathcal{P})$ of all *forbidden subgraphs* defined by

$$\mathcal{C}(\mathcal{P}) = \{H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H)\}.$$

It is easy to prove that a property $\mathcal{P} \in \mathbf{M}$ is additive if and only if all forbidden subgraphs $H \in \mathcal{C}(\mathcal{P})$ are connected.

A hereditary property $\mathcal{P} \in \mathbf{M}$ is said to be nontrivial if $\mathcal{P} \neq \mathcal{I}$. For a nontrivial property $\mathcal{P} \in \mathbf{M}$ there exists a number $c(\mathcal{P})$ called the *completness* of $\mathcal{P}$ defined as $\sup\{k : K_{k+1} \in \mathcal{P}\}$, and $c(\mathcal{P}) = \infty$ if for every $k$, $K_{k+2} \in \mathcal{P}$. Let $\delta(\mathcal{P}) = \min\{\delta(H) : H \in \mathcal{C}(\mathcal{P})\}$.

Let us denote by $\mathcal{O} = \{G : G \in \mathcal{I}, E(G) = \emptyset\}$. For this property we have $\mathcal{C}(\mathcal{O}) = \{K_2\}$ and $\delta(\mathcal{O}) = 1$.

A $\mathcal{P}$*-partition* (*colouring*) of a graph $G$ is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that the subgraph $< V_i >$ induced by the set $V_i$ has property $\mathcal{P}$ for each $i = 1, \ldots, n$. If $(V_1, \ldots, V_n)$ is a $\mathcal{P}$-partition of a graph $G$, then the corresponding vertex colouring $f$ is defined by $f(v) = i$ whenever $v \in V_i$, for $i = 1, \ldots, n$. The smallest integer $n$ for which $G$ has $\mathcal{P}$-partition is called the $\mathcal{P}$*-chromatic* (or $\mathcal{P}$*-vertex-partition*) *number* of $G$ and is denoted by $\chi_{\mathcal{P}}(G)$. The $\mathcal{O}$-chromatic number is the ordinary chromatic number. See [1] for a survey and more details.

Let $G$ be a graph and let $L(v)$ be a list of colours (as above, positive integers) prescribed for the vertex $v$, and $\mathcal{P} \in \mathbf{M}$. A $(\mathcal{P}, L)$*-colouring* is a graph $\mathcal{P}$-colouring $f(v)$ with the additional requirement that for all $v \in V(G), f(v) \in L(v)$. If $G$ admits a $(\mathcal{P}, L)$-colouring, then $G$ is said to be $(\mathcal{P}, L)$*-colourable.* The graph $G$ is $(k, \mathcal{P})$*-choosable* if it is $(\mathcal{P}, L)$-colourable for every list $L$ of $G$ satisfying $| L(v) |= k$ for every $v \in V(G)$. The $\mathcal{P}$*-choice number* $\mathrm{ch}_{\mathcal{P}}(G)$ is the smallest natural number $k$ such that $G$ is $(k, \mathcal{P})$-choosable.

Vizing [6] and Erdös, Rubin and Taylor [3] independly introduce the idea of considering $(\mathcal{O}, L)$-colouring and $(k, \mathcal{O})$-choosability.

The aim of this paper is to prove some extensions of two basic theorems in the colouring theory of graphs, namely the Brooks [2] and Gallai [4] theorems. If $L(v)$ is the same for all vertices of $G$, this results generalize also the corresponding results of [1]. Moreover, we prove that $\mathrm{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$ can be arbitrarily large.

## 2. Behaviour of the $\mathcal{P}$-Choice Number

To prove the main theorem of this section, use well-known Pigeonhole Principle in the following form.

**Pigeonhole Principle.** *Suppose that* $q_1, ..., q_t$ *are positive integers. If* $X = X_1 \cup ... \cup X_t$ *is a partition of the set* $X$ *and* $\mid X \mid \geq (\sum_{i=1}^{t} q_i) - t + 1$, *then* $\mid X_i \mid \geq q_i$ *for some* $i \in \{1, ..., t\}$.

**Theorem 1.** *Let* $\mathcal{P} \in \mathbf{M}^a$ *and* $1 \leq c(\mathcal{P}) < \infty$. *Then for any nonnegative integer* $s$ *there exists a graph* $G_s$ *such that* $\mathrm{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > s$.

***Proof.*** For a given $c(\mathcal{P})$ let the function $g(l)$ be defined by

$$g(l) = \left\lceil \frac{l(l-1) - 2 + c(\mathcal{P})(c(\mathcal{P})+3)}{2c(\mathcal{P})(c(\mathcal{P})+3)} \right\rceil - 1 - l$$

The function $g(l)$ tends to infinity with $l \to \infty$. Hence, for given $s$ and $c(\mathcal{P})$ there is an integer $l_0 \geq 3$ such that $g(l_0) \geq s$. For this integer $l_0$, let

$$(*) \qquad b = \left\lceil \frac{l_0(l_0-1) - 2 + c(\mathcal{P})(c(\mathcal{P})+3)}{2c(\mathcal{P})(c(\mathcal{P})+3)} \right\rceil - 1$$

Note that $b \geq s + l_0 \geq 3$.

Let the graph $G_s$ be given by the join $H_1 + \ldots + H_{l_0}$ of totally disconnected graphs $H_i$, $i = 1, \ldots, l_0$, all of order $t = \binom{2b-1}{b}$, i.e., $G_s$ is a complete $l_0$-partite graph with $t$ elements in each part.

Since $\mathcal{P} \in \mathbf{M}^a$ and $K_1 \in \mathcal{P}$, then $H_i \in \mathcal{P}$ for $i = 1, \ldots, l_0$. Hence, $\chi_{\mathcal{P}}(G_s) \leq l_0$. Note that it is sufficient to prove that $G_s$ is not $(b, \mathcal{P})$-choosable, i.e., $\mathrm{ch}_{\mathcal{P}}(G_s) > b$. By this and $(*)$ we have $\mathrm{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > b - l_0 \geq s$.

To prove this, let $C = \{1, \ldots, 2b-1\}$ be a set of colours and let $[C]^b = \{A : A \subseteq C, \mid A \mid = b\}$. Suppose that this set is indexed as follows $\{A_j : j = 1, \ldots, t\}$. Besides, let $V(H_i) = \{v_{ij} : j = 1, \ldots, t\}, i = 1, \ldots, l_0$ .
Define a list $L$ for the graph $G_s$ by $L(v_{ij}) = A_j$ for all $i$ and $j$. Let $f$ be a $(\mathcal{P}, L)$-colouring of $G_s$. Note that

(a) For any colouring of the graph $H_i$ from its lists we need at least $b$ different colours for any $i = 1, \ldots, l_0$.

(b) For $i_1, i_2 \in \{1, \ldots l_0\}, i_1 \neq i_2$, there is $j_1, j_2 \in \{1, \ldots, t\}$ and a colour $c \in C$ such that $f(v_{i_1j_1}) = f(v_{i_2j_2}) = c$. This follows by (a) and $\mid C \mid = 2b-1$.

The colour $c$ is said to be $\{i_1, i_2\}$-*colour*. Let $X = \{\{i_1, i_2\} : \{i_1, i_2\}\text{-colour}\}$ and $X_c = \{\{i_1, i_2\} : c \text{ is } \{i_1, i_2\}\text{-colour}\}, c \in C$.
Let be given a sequence of integers $q_1 = \ldots = q_{2b-1} = \binom{c(\mathcal{P})+2}{2}$.
Note that $X = X_1 \cup \ldots \cup X_{2b-1}$. By above and $(*)$ we have

$$\mid X \mid = \binom{l_0}{2} \geq \sum_{i=1}^{2b-1} q_i - (2b-1) + 1.$$

Hence, by Pigeonhole Principle it follows that there is $c_0 \in C$, such that $\mid X_{c_0} \mid \geq q_{c_0} = \binom{c(\mathcal{P})+2}{2}$. It implies that there are at least $c(\mathcal{P}) + 2$ pairwise different integers $i_1, \ldots, i_{c(\mathcal{P})+2} \in \{1, \ldots, l_0\}$ and for any $i_r$ there is an integer $j, 1 \leq j \leq t$, such that $f(v_{i_rj}) = c_0$. By above and the definition of $G_s$, the subgraph of $G_s$ induced by vertices with the assigned colour $c_0$ contains a complete graph of order $c(\mathcal{P}) + 2$, i.e., it is not $(b, \mathcal{P})$-choosable. Hence, $\mathrm{ch}_{\mathcal{P}}(G_s) > b$. ■

## 3. $(\mathcal{P}, L)$-Critical Graphs

For a nontrivial property $\mathcal{P} \in \mathbf{M}$ a graph $G$ is said to be $(\mathcal{P}, L)$-*critical* if $G$ has no $(\mathcal{P}, L)$-colouring but $G - v$ is $(\mathcal{P}, L)$-colourable for all $v \in V(G)$.

**Lemma 1.** *If $\mathcal{P} \in \mathbf{M}$ and $G$ is $(\mathcal{P}, L)$-critical, then $d_G(v) \geq \delta(\mathcal{P}) \mid L(v) \mid$ for any vertex $v$ of $G$.*

***Proof.*** Suppose that $d_G(u) < \delta(\mathcal{P}) \mid L(u) \mid$ for a vertex $u \in V(G)$. Then there is $i \in L(u)$ which is used in colouring of the vertices of $N_G(u)$ less than $\delta(\mathcal{P})$ times. Therefore, the vertex $u$ can be coloured by $i$, $< V_i >_G \in \mathcal{P}$ and $G$ is not $(\mathcal{P}, L)$-critical, a contradiction. ■

Let $\mathcal{P} \in \mathbf{M}$, $G$ be $(\mathcal{P}, L)$-critical and $x \in V(G)$. Define a new list assignment

$$L^x(v) = \begin{cases} \{l\}, v = x, \\ L(v), \text{ otherwise,} \end{cases}$$

where $l \notin L(v)$ for $v \in V(G)$.

Since, by the definition, $G - w$ is $(\mathcal{P}, L)$-colourable, then $G$ is $(\mathcal{P}, L^w)$-colourable. Thus, for a vertex $w$ of a $(\mathcal{P}, L)$-critical graph $G$ we shall say that $f$ *is a* $(\mathcal{P}, L^w)$-*colouring* of $G$, whenever $f(w) = l$ and $f(v) \in L(v)$ for $v \neq w$.
Note that the list $L^x$ is always created from the list $L$ by assignment of the colour $l$ to the vertex $x$ and preserving the remaining assignments.

Let us denote by $S(G) = \{v : v \in V(G),\ d_G(v) = \delta(\mathcal{P}) \mid L(v) \mid\}$.

From Lemma 1 we have immediatelly the following lemma.

**Lemma 2.** *Let $G$ be a $(\mathcal{P}, L)$-critical, $w \in S(G)$. Then for any $(\mathcal{P}, L^w)$-colouring $f$ of $G$, $\mid N_G(w) \cap V_i \mid = \delta(\mathcal{P})$ for any $i \in L(w)$, where $V_i = \{v : v \in V(G),\ f(v) = i\}$.* ■

**Lemma 3.** *Let $G$ be a $(\mathcal{P}, L)$-critical, $u, v \in S(G)$, $uv \in E(G)$, and let $f$ be a $(\mathcal{P}, L^v)$-colouring of $G$. Then there is a $(\mathcal{P}, L^u)$-colouring $f'$ of $G$ such hat $f'(v) = f(u)$ and $f'(w) = f(w)$ for all $w \in V(G) - \{u, v\}$.*

***Proof.*** Since $u, v \in S(G)$, by Lemma 2 we have $f(u) \in L(v)$. From this, the definition of the $(\mathcal{P}, L^x)$-colouring and again Lemma 2, the required $f'$ colouring follows. ■

**Lemma 4.** *Let $G$ be a $(\mathcal{P}, L)$-critical and $Q : v_0v_1...v_m$ be a walk in $G$ such that $V(Q) = \{v_0, v_1, ..., v_m\} \subseteq S(G)$. Let $f$ be a $(\mathcal{P}, L^{v_0})$-colouring of $G$. Then there is a $(\mathcal{P}, L^{v_m})$-colouring $f'$ of $G$ such that $f'(v_i) = f(v_{i+1})$ for $i = 1, ..., m-1$, $f'(v_m) = f(v_0) = l$ and $f'(w) = f(w)$ for all $w \in V(G) - V(Q)$.*

***Proof.*** By applying Lemma 3 to the consecutive adjacent vertices of $Q$ we obtain the required $f'$ colouring. ■

The procedure described by Lemma 4 will be called $(\mathcal{P}, L^x)$-*recolouring of* $Q$.

**Lemma 5.** *Let $G$ be a $(\mathcal{P}, L)$-critical and $C : v_0v_1...v_mv_0$ be a cycle in $G$ with $V(C) = \{v_0, v_1, ..., v_m\} \subseteq S(G)$ and let $f$ be a $(\mathcal{P}, L^{v_0})$-colouring of $G$. Then there is a $(\mathcal{P}, L^{v_0})$-colouring $f'$ of $G$ such that $f'(v_i) = f(v_{i+1})$ for $i = 0, 1, ..., m-1$, $f'(v_m) = f(v_1)$, $f'(w) = f(w)$ for all $w \in V(G) - V(C)$.*

***Proof.*** By $(\mathcal{P}, L^x)$ recolouring of $Q : v_0v_1 \dots v_mv_0$ it follows. ■

**Lemma 6.** *Let $C : v_0v_1...v_mv_0$ be an even cycle in a $(\mathcal{P}, L)$-critical graph $G$ with $V(C) \subseteq S(G)$. If there exists a vertex $v_j$ which is not incident to any diagonal of $C$, then in any $(\mathcal{P}, L^{v_j})$-colouring $f$ of $G$ all vertices of $C$ but $v_j$ have the same colour.*

***Proof.*** By the assumption, $N_C(v_j) = \{v_{j-1}, v_{j+1}\}$. Let $f$ be an arbitrary $(\mathcal{P}, L^{v_j})$-colouring of $G$. By Lemma 2 the vertex $v_j$ has $\delta(\mathcal{P})$ neighbours in each colour from its list. Lemma 4 implies that this property is preserved after $(\mathcal{P}, L^x)$-recolouring of a walk $W : v_{j+1} \dots v_mv_0 \dots v_{j-1}$. Since $C$ is an even cycle we have that all vertices of $C$ but $v_j$ have the same colour. ■

**Lemma 7.** (Dirac, see [4], p.170). *If each even cycle in the block $B$ of a graph $G$ has at least two diagonals in $G$, then the block $B$ is a complete subgraph of $G$.* ■

**Theorem 2.** *Let $\mathcal{P} \in \mathbf{M}$, $G$ be a $(\mathcal{P}, L)$-critical graph. Then any block of $< S(G) >_G$ is one of the following types:*

(i) *$B$ is a complete graph,*
(ii) *$B$ is a $\delta(\mathcal{P})$-regular graph belonging to $\mathcal{C}(\mathcal{P})$,*
(iii) *$B \in \mathcal{P}$ and $\Delta(B) \leq \delta(\mathcal{P})$,*
(iv) *$B$ is an odd cycle.*

***Proof.*** We have considered three cases.

*Case* 1. The block $B$ of $< S(G) >_G$ contains no even cycles.
Then either $B = K_2$ or $B$ contains an odd cycle $C_{2p+1}$. In the case when $B \neq C_{2p+1}$, either $C_{2p+1}$ has a diagonal in $< S(G) >_G$ or there exists a vertex $u$ of $B$ not belonging to $C_{2p+1}$. In both cases there is an even cycle in $B$, a contradiction.

*Case* 2. There is an even cycle $C_{2p} : v_0v_1 \dots v_{2p-1}v_0$ in $B$ which contains a vertex $v_j$ not incident to any diagonal of $C_{2p}$.
Let $f$ be a $(\mathcal{P}, L^{v_j})$-colouring of $G$. By Lemma 6 all vertices of $C_{2p}$ but $v_j$ have the same colour. Let us suppose that $f(v_i) = a, i \neq j$. We are going to prove that all vertices of $B$ but $v_j$ are coloured in $f$ by $a$. Suppose that there exists a vertex $z \neq v_j$ such that $f(z) \neq a$. Then since $B$ is 2-connected and $z \notin V(C_{2p})$ there exists a cycle $C' : v_jv_{j+1} \dots z \dots v_j$. By applying Lemma 5 to the cycle $C'$ we can obtain $(\mathcal{P}, L^{v_j})$-colouring of $G$ such that the vertices $v_{j+1}, \dots, v_{2p-1}, v_0, \dots, v_j$ of $C_{2p}$ are not coloured the same, a contradiction. Thus, by Lemma 2, we have $\Delta(B) \leq \delta(\mathcal{P})$. If $B$ is $\delta(\mathcal{P})$-regular, then

$B \in \mathcal{C}(\mathcal{P})$; otherwise $v_j$ could be recoloured by $a$, which contradicts that $G$ is $(\mathcal{P}, L)$-critical. If there is a vertex $u \in V(B)$ with $d_B(u) < \delta(\mathcal{P})$, then according to Lemma 4 by $(\mathcal{P}, L^x)$-recolouring of a walk $Q : v_j \ldots u$, we obtain $(\mathcal{P}, L^u)$-colouring of $G$ with all vertices of $B$ but $u$ coloured by $a$. Since $d_B(u) < \delta(\mathcal{P})$, we have $B \in \mathcal{P}$.

*Case* 3. Each vertex of any even cycle $C$ in $B$ is incident with at least one diagonal of $C$.

In this case, by Lemma 7, $B$ is a complete subgraph of $G$. ■

## 4. $(k, \mathcal{P})$-CHOICE CRITICAL GRAPHS. GENERALIZATIONS OF GALLAI'S AND BROOKS' THEOREMS

For a nontrivial property $\mathcal{P} \in \mathbf{M}$, a graph $G$ is said to be *(vertex) $(k, \mathcal{P})$-choice critical* if $\mathrm{ch}_{\mathcal{P}}(G) = k \geq 2$ but $\mathrm{ch}_{\mathcal{P}}(G - v) < k$ for all vertices $v$ of $G$. According to the previous definitions, it follows immediatelly that if $G$ is $(k+1, \mathcal{P})$-choice critical, then $G$ is $(\mathcal{P}, L)$-critical with some list $\mid L(v) \mid = k$ for all $v \in V(G)$.

Hence, by Theorem 2 we have the following generalization of Gallai's Theorem.

**Theorem 3.** *Let $\mathcal{P} \in \mathbf{M}$ and $G$ be a $(k, \mathcal{P})$-choice critical graph. Then any block of $< S(G) >_G$ is one of the following types:*

(i) *$B$ is a complete graph,*
(ii) *$B$ is a $\delta(\mathcal{P})$-regular graph belonging to $\mathcal{C}(\mathcal{P})$,*
(iii) *$B \in \mathcal{P}$ and $\Delta(B) \leq \delta(\mathcal{P})$,*
(iv) *$B$ is an odd cycle.*

Note that in Theorem 3, $S(G) = \{v : v \in V(G), d_G(v) = \delta(\mathcal{P})(k-1)\}$.

**Lemma 8.** *Let $\mathcal{P} \in \mathbf{M}^a$ and let $G$ be a connected graph with $\mid L(v) \mid \geq \frac{d_G(v)}{\delta(\mathcal{P})}$ for all $v \in V(G)$, and let there exist a vertex $v_0 \in V(G)$ such that $\mid L(v_0) \mid \geq \frac{d_G(v_0)+1}{\delta(\mathcal{P})}$. Then $G$ is $(\mathcal{P}, L)$-colourable.*

***Proof.*** The proof is by induction on the order of $G$. Let $\mid V(G) \mid = 1$. Then $G = K_1$, $\mid L(v) \mid \geq 1$ for $v \in V(G)$ and Lemma is true.

Assume the Lemma holds for all graphs of order $\leq n$. Let $\mid V(G) \mid = n + 1$ and let $L$ be a list satisfying the assumptions of the Lemma. Consider the graph $G - v_0$ and its list $\widetilde{L}(v) = L(v)$ for all $v \in V(G) - \{v_0\}$. Since $G$ is connected, then in each component $G'$ of $G - v_0$ there is a vertex $u$ such that $uv_0 \in E(G)$. Thus,

$$| \widetilde{L}(u) | \geq \tfrac{d_G(u)}{\delta(\mathcal{P})} = \tfrac{d_{G'}(u)+1}{\delta(\mathcal{P})} .$$

For the remaining vertices of each component

$$| \widetilde{L}(v) | \geq \tfrac{d_G(v)}{\delta(\mathcal{P})} \geq \tfrac{d_{G'}(v)}{\delta(\mathcal{P})} \text{ holds.}$$

Then, by the induction hypothesis, each component $G'$ of $G - v_0$ is $(\mathcal{P}, \widetilde{L})$-colourable. Since $\mathcal{P} \in \mathbf{M}^a$, we have $(\mathcal{P}, \widetilde{L})$-colourability $f'$ of $G - v_0$. We will prove that $f'$ can be extended to $(\mathcal{P}, L)$-colourability $f$ of $G$. Suppose, contrary to our claim. It implies that the vertex $v_0$ has at least $\delta(\mathcal{P})$ neighbours in each colour class $V_i$ for $i \in L(v_0)$. Thus,

$d_G(v_0) \geq \delta(\mathcal{P}) \mid L(v_0) \mid \geq \delta(\mathcal{P}) \tfrac{d_G(v_0)+1}{\delta(\mathcal{P})} = d_G(v_0) + 1$, a contradiction. ■

**Theorem 4.** *Let $\mathcal{P} \in \mathbf{M}^a$ and $G$ be a connected graph other than*

(i) *a complete graph of order $n\delta(\mathcal{P}) + 1, n \geq 0$,*
(ii) *a $\delta(\mathcal{P})$-regular graph belonging to $\mathcal{C}(\mathcal{P})$,*
(iii) *an odd cycle if $\mathcal{P} = \mathcal{O}$.*

*Then*

$$\mathrm{ch}_{\mathcal{P}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(\mathcal{P})} \right\rceil .$$

***Proof.*** By Lemma 8, the Theorem is true for all not regular graphs and for any graph $G$ with $\Delta(G) \not\equiv 0 \pmod{\delta(\mathcal{P})}$. So, let $G$ be a regular graph with $\Delta(G) = (k-1)\delta(\mathcal{P})$. Since $G \neq K_1$ we have $k \geq 2$. Now, suppose the assertion of the Theorem is false for a list $L$, with $| L(v) | = k - 1$ for all $v \in V(G)$. By Lemma 8 it follows that for each vertex $v \in V(G)$ and each component $G'$ of $G - v$ there exists $(\mathcal{P}, L)$-colouring. Since, $\mathcal{P} \in \mathbf{M}^a$ we have $\mathrm{ch}_{\mathcal{P}}(G - v) \leq k - 1$ for all $v \in V(G)$. Thus, $\mathrm{ch}_{\mathcal{P}}(G) = k$, i.e., $G$ is $(k, \mathcal{P})$-choice critical. Since $S(G) = V(G)$, then by Theorem 3, for $k$=2 it follows that $G$ is a $\delta(\mathcal{P})$-regular graph with $\mathrm{ch}_{\mathcal{P}}(G) > 1$. Hence, $G \in \mathcal{C}(\mathcal{P})$, a contradiction.

If $k \geq 3$, also by Theorem 3, we have that $G$ is a complete graph of order $(k-1)\delta(\mathcal{P}) + 1$ or G is an odd cycle (only in the case when $\mathcal{P} = \mathcal{O}$), a contradiction. ■

## References

[1] M. Borowiecki and P. Mihók, *Hereditary Properties of Graphs*, in: Advances in Graph Theory (Vishwa International Publications, 1991) 41–68.

[2] R.L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. **37** (1941) 194–197.

[3] P. Erdös, A.L. Rubin and H. Taylor, *Choosability in graphs*, in: Proc. West Coast Conf. on Combin., Graph Theory and Computing, Congressus Numerantium XXVI (1979) 125–157.

[4] T. Gallai, *Kritiche Graphen I,* Publ. Math. Inst. Hung. Acad. Sci. **8** (1963) 373–395.

[5] F. Harary, Graph Theory (Addison Wesley, Reading, Mass. 1969).

[6] V.G. Vizing, *Colouring the vertices of a graph in prescribed colours* (in Russian), Diskret. Analiz **29** (1976) 3–10.

Received 10 April 1995