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SPANNING CATERPILLARS WITH BOUNDED DIAMETER

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Abstract

A *caterpillar* is a tree with the property that the vertices of degree at least 2 induce a path. We show that for every graph G of order n, either G or $\bar{G}$ has a spanning caterpillar of diameter at most $2\log n$. Furthermore, we show that if G is a graph of diameter 2 (diameter 3), then G contains a spanning caterpillar of diameter at most $cn^{3/4}$ (at most n).

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1. Introduction

It is easy to show that for every graph G, either G or the complement $\bar{G}$ is connected. Consequently, if $\mathcal{T}_n$ denotes the family of all trees of order n, then for every graph G of order n, either G or $\bar{G}$ contains a member of $\mathcal{T}_n$ (as a spanning subgraph). Such a family is called complete, that is, a family $\mathcal{F}_n$ of graphs of order n is *complete* if for every graph G of order n, either G or $\bar{G}$ contains a member of $\mathcal{F}_n$. Thus, $\mathcal{T}_n$ is complete and it is easy to show that the subfamily $\mathcal{T}_n(4)$ of trees of order n and diameter at most 4 is also complete. In Section 2, we will discuss other complete families of trees and show, in particular, that $C_n(2\log n)$ is complete, where $C_n(2\log n)$ is the family of caterpillars of order n and diameter at most $2\log n$. In Section 3 we will investigate graphs of order n and diameter at most 3 and show that if G has diameter 2 (diameter 3), then G contains a spanning caterpillar of diameter at most $cn^{3/4}$ (at most n).

2. Complete Families of Trees

We begin this section by proving a theorem from graph theory folklore. For vertices x and y of a graph G, $d_G(x,y)$ will denote the distance between x and y in G, i.e., the number of edges in a shortest path from x to y. The *diameter* of G, denoted $diam(G)$, is the largest distance between pairs of vertices of G.

Theorem 1. *Let $\mathcal{T}_n(4)$ denote the family of trees of order n and diameter at most 4. Then $\mathcal{T}_n(4)$ is complete.*

Proof. Without loss of generality, we may assume $n \geq 5$. Let G be a graph of order n. If $diam(G) \leq 2$, then clearly G contains a spanning tree with diameter at most 4. Thus we may assume that either G is disconnected or G has diameter at least 3. In either case, G contains nonadjacent vertices u and v which have no common neighbors. Therefore, in $\bar{G}$, u and v are adjacent and every other vertex is adjacent to at least one of u and v. Thus, $\bar{G}$ contains a spanning tree of diameter at most 4. ■

Let G be the graph of order $5s$ obtained by replacing each vertex of a 5-cycle with a copy of the complete graph K_s and adding edges between two vertices in different copies of K_s if the corresponding vertices of the 5-cycle were adjacent. Then neither G nor $\bar{G}$ contains a spanning tree of diameter at most 3. Thus, with respect to diameter, Theorem 1 cannot be improved.

Recently, Bialostocki, Dierker and Voxman [1] investigated other complete families of trees. Moreover, they conjectured that the family $\mathcal{B}_n$ of brooms of order n is complete, where a broom (of order n) is a tree consisting of a star and a path, with one end of the path identified with the central vertex of the star. The brooms of order 6 are shown in Figure 1.

In [2], Burr settled their conjecture in the affirmative and suggested that, in fact, only about half of B_n is needed for a complete family. We note that any complete subfamily of B_n necessarily contains the broom of diameter $n-1$, i.e. the path of order n.

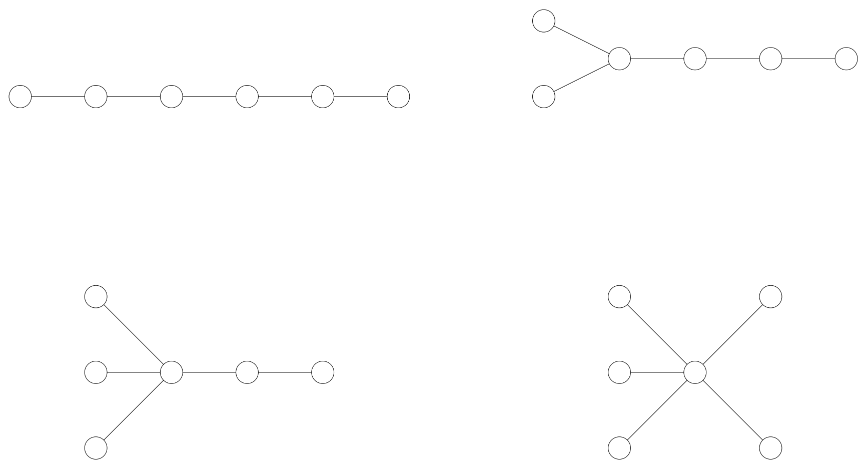

Figure 1

One property of brooms is that all non-endvertices lie along a single path. In the remainder of this paper we will focus primarily on complete families of trees with this property having small diameter.

A *caterpillar* is a tree with the property that the vertices of degree at least 2 induce a path. These vertices form the *spine* of the caterpillar. Note that if S is the spine of a caterpillar C of order at least 3, then $diam(C) = |S|+1$. In Theorem 2, we will show that $C_n(2\log n)$ is complete, where $C_n(2\log n)$ is the family of caterpillars of order n and diameter at most $2\log n$. (Here, $\log n$ is $\log_2 n$.) The following lemma will be useful.

Lemma 1. *Let G be a graph of order n and diameter* 2*. If G contains a caterpillar C of diameter d, then G contains a spanning caterpillar with* *diameter at most* $d + (|V(G)| - |V(C)|)$.

Proof. Let $v_1, v_2, \ldots, v_{d-1}$ be the vertices in the spine of C, where $v_i v_{i+1} \in E(C), 1 \leq i \leq d-2$. We first construct a caterpillar C' such that (i) $|V(C')| = |V(C)| + 1$ and (ii) $diam(C') \leq diam(C) + 1$.

Without loss of generality we may assume that if x is an endvertex of C and x is adjacent to v_i, then x is not adjacent to v_j for $j < i$. For convenience, we will say that the end vertices have been "shifted left". Furthermore, we may assume that no vertex in the spine is adjacent to a vertex of $V(G) - V(C)$ since in that case we immediately obtain C' with $diam(C') = diam(C)$. Let $y \in V(G) - V(C)$. Then, since $d_G(y, v_1) = 2$ it follows that there is a vertex x of C such that $xv_1 \in E(C)$ and $yx \in E(G)$. Thus we obtain C' with spine $\{x, v_1, v_2, \ldots, v_{d-1}\}$ and $diam(C') = diam(C) + 1$.

Clearly, by repeating this procedure we obtain the desired spanning caterpillar. ■

A set X of vertices in a graph G is a *dominating set* if every vertex of $V(G) - X$ is adjacent to at least one vertex of X. In [3] it was shown that for every graph G of order n, either G or $\bar{G}$ has a dominating set X with $|X| \leq \log n$. This result will be used in the proof of Theorem 2.

Theorem 2. *Let* $C_n(2 \log n)$ *denote the family of caterpillars of order* n *and diameter at most* $2 \log n$. *Then* $C_n(2 \log n)$ *is complete.*

Proof. It is straightforward to verify the result for $n \leq 4$. Thus we assume $n \geq 5$. If G or $\bar{G}$ is complete, then G or $\bar{G}$ contains a spanning caterpillar of diameter 2 (i.e., a spanning star), where $2 \leq 2 \log n$. Furthermore, if G or $\bar{G}$ is disconnected or has diameter at least 3 then, as in the proof of Theorem 1, either G or $\bar{G}$ contains a spanning caterpillar of diameter at most 3 and $3 \leq 2 \log n$. Thus we may assume that $diam(G) = diam(\bar{G}) = 2$.

Let $uv \in E(G)$ and let A denote those vertices adjacent to neither u nor v in G. Suppose $|A| \leq 2 \log n - 3$. Then, in $\bar{G} - A$, u and v are either in different components or at distance at least 3. Consequently, as in the proof of Theorem 1, $\bar{G} - A$ contains a spanning caterpillar of diameter at most 3. Thus $\bar{G}$ contains a caterpillar of diameter at most 3 and it follows from Lemma 1 that $\bar{G}$ contains a spanning caterpillar of diameter at most $3 + |A| \leq 2 \log n$. Thus we may assume that if $uv \in E(G)$ then u and v have at least $2 \log n - 3$ common neighbors in $\bar{G}$. Similarly, if $uv \notin E(G)$, then u and v have at least $2 \log n - 3$ common neighbors in G.

Let $X \subseteq V(G)$ with $|X| \leq \log n$ such that X is a dominating set in G or $\bar{G}$. (The existence of such a set is guaranteed by the aforementioned result in [3]). Assume, without loss of generality, that X dominates G and $X = \{v_1, v_2, \ldots, v_t\}$. We claim that there is a $v_1 - v_t$ path in G containing the vertices of X in the order $v_1, v_2, \ldots, v_t$ and such that between v_i and v_{i+1} there is at most one vertex. Suppose such a $v_1 - v_l$ path P has been constructed for $l < t$. If $v_l v_{l+1} \in E(G)$ then we may extend P to include v_{l+1}. If $v_l v_{l+1} \notin E(G)$ then v_l and v_{l+1} have at least $2 \log n - 3 \geq 2l - 1$ common neighbors in G. Consequently there is a common neighbor $w \in V(G) - V(P) - X$ and P can be extended to include v_{l+1}. Thus G contains a $v_1 - v_t$ path of order at most $2t - 1$ containing X and this path forms the spine of a spanning caterpillar of diameter at most $2 \log n$. ■

In [3] it was shown that for fixed $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for each $n \geq n_0$ there is a graph G of order n such that no set of at most $(1 - \varepsilon) \log n$ vertices dominates either G or $\bar{G}$. Thus the bound in Theorem 2 on the diameter of the spanning caterpillars is, in fact, the correct order of magnitude.

In the proof of Theorem 2, we began with either a caterpillar of diameter at most 3 or a dominating set of cardinality at most $\log n$ and built a spanning caterpillar of diameter at most $2 \log n$. The same proof technique can be used to establish Theorem 3.

Theorem 3. *If $\mathcal{D}_n$ denotes the family of trees of order n with diameter at most 6 and domination number at most $\log n$, then $\mathcal{D}_n$ is complete.*

3. Spanning Trees of Small Diameter Graphs

If G is the graph of Figure 2, then G has diameter 4 and no spanning caterpillar. In this section we will show that every graph of diameter at most 3 has a spanning caterpillar.

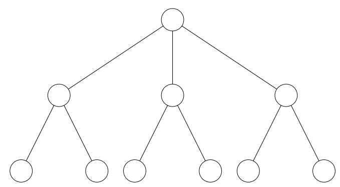

Figure 2

Theorem 4. *If G is a graph with diameter at most 3, then G contains a spanning caterpillar.*

Proof. If $diam(G) = 1$ then G is complete and contains a spanning star. If $diam(G) = 2$ then Lemma 1 guarantees the existence of a spanning caterpillar. Thus we need only show that if G is a graph of diameter 3 then G has a spanning caterpillar. Assume, to the contrary, that G is an edge-maximal counterexample. Thus, by edge maximality, G contains two vertex disjoint caterpillars that together span G. Among all such pairs C_1, C_2 of disjoint caterpillars that together span G select a pair such that $|V(C_1)|$ is as large as possible. Let $v_1, v_2, \ldots, v_l$ be the vertices (in order) of the spine of C_1 and $v_{l+1}, v_{l+2}, \ldots, v_m$ be the vertices of the spine of C_2. As in the proof of Lemma 1, assume that the endvertices of C_1 have been "shifted left". Let w be an endvertex of C_1 adjacent to v_l and let u be an endvertex of C_2 adjacent to v_{l+1}. If C_2 is trivial, let $u = v_{l+1}$. Clearly, $d_G(u, w) \neq 1$ since, by assumption, G has no spanning caterpillar. Thus, $2 \leq d_G(u, w) \leq 3$. Furthermore, by the choice of C_1 and C_2 we know that:

(1) w is adjacent to no vertex of C_2,
(2) w is adjacent to no v_i, $i < l$,
(3) v_l is adjacent to no vertex of C_2,
(4) u is adjacent to no v_i, $i \leq l$, and
(5) there is no $u - w$ path whose interior vertices are all endvertices of C_1 and C_2.

By (1) and (2), every adjacency of w other than v_l in G is an endvertex of C_1. Thus, by (4) and (5) there is no $u - w$ path of length 2. Therefore, $d_G(u, w) = 3$. Let u, x_1, x_2, w be a $u - w$ path of length 3. Then by (1) and (2), either $x_2 = v_l$ or x_2 is an endvertex of C_1. If $x_2 = v_l$ then by (3) and (4) it follows that x_1 is an endvertex of C_1. Subsequently C_1 can be extended by including x_1 in the spine and u as an endvertex, contradicting the maximality of C_1. Therefore x_2 is an endvertex of C_1. However, then by (4) and (5), x_1 must be a spine vertex of C_2 and again the maximality of C_1 is contradicted, and the proof is complete. ■

For even n, let G be the graph of order n obtained from the graph $K_{n/2} \cup \bar{K}_{n/2}$ by adding a matching between the set of $n/2$ isolated vertices and the remaining $n/2$ vertices. Then every spanning caterpillar has diameter $n/2 + 1$. Thus the (implied) bound in Theorem 4 of $n - 1$ on the smallest diameter of a spanning caterpillar is the correct order of magnitude for graphs of diameter 3. For graphs of diameter 2, some improvement can be made. The following notation will be useful. Let G be a graph, u a vertex of G, and H a subgraph of G. Then

$$N_H[u] = \{w \in V(H) | uw \in E(G)\} \cup \{u\}.$$

Theorem 5. *There is a constant c such that if G is a graph with $diam(G) = 2$, then G contains a spanning caterpillar of diameter at most $cn^{3/4}$.*

Proof. We first show that G contains a dominating set with at most $2n^{3/4}$ vertices. Let u_1 be a vertex of G with $deg_G u_1 \geq n^{1/4}$ and set $\mathcal{U}_1 = N_G[u_1]$. Let $u_2 \in V(G)$ with $deg_{G-\mathcal{U}_1} u_2 \geq n^{1/4}$ and set $\mathcal{U}_2 = N_{G-\mathcal{U}_1}[u_2]$. Continue in this fashion to obtain a maximal length sequence of vertices $u_1, u_2, \ldots, u_t, t \geq 1$, where $deg_{G-\mathcal{U}_1-\mathcal{U}_2-\ldots-\mathcal{U}_{l-1}} u_l \geq n^{1/4}$ and $\mathcal{U}_l = N_{G-\mathcal{U}_1-\mathcal{U}_2-\ldots-\mathcal{U}_{l-1}}[u_l]$ for $l = 1, 2, \ldots, t$, and let $A = V(G) - \mathcal{U}_1 - \mathcal{U}_2 - \ldots - \mathcal{U}_t$. Then $t \leq n^{3/4}$ and $\Delta(< A >) < n^{1/4}$. If $|A| \leq n^{3/4}$, then $A \cup \{u_1, u_2, \ldots, u_t\}$ is the desired dominating set. We show that this must be the case. Assume, to the contrary, that $|A| = kn^{3/4}$, where $k > 1$. Each of the $\binom{|A|}{2}$ pairs of vertices of A are at distance 1 or 2 in G. Since $\Delta(< A >) < n^{1/4}$, $< A >$ has fewer than $(|A| \cdot n^{1/4})/2$ edges. Furthermore, the number of pairs of vertices of A with a common neighbor in A is less than $|A| \cdot \binom{n^{1/4}}{2}$. Thus, more than

$$\binom{kn^{3/4}}{2} - \frac{kn}{2} - kn^{3/4} \cdot \binom{n^{1/4}}{2}$$

pairs of vertices of A have a common neighbor in $V(G) - A$, implying that more than

$$\frac{k^2 n^{3/2}}{2} - \frac{kn}{2} - \frac{kn^{5/4}}{2}$$

pairs of vertices in A have a common neighbor in $V(G) - A$. However, each vertex in $V(G) - A$ is adjacent to fewer than $n^{1/4}$ vertices of A. Therefore the number of pairs of vertices in A with a common neighbor in $V(G) - A$ is less than

$$n \cdot \binom{n^{1/4}}{2}.$$

We conclude that

$$\frac{k^2 n^{3/2}}{2} - \frac{kn}{2} - \frac{kn^{5/4}}{2} < \frac{n^{3/2}}{2} - \frac{n^{5/4}}{2},$$

which is a contradiction for $k > 1$ and n sufficiently large. Thus G has a dominating set X with $t \leq 2n^{3/4}$ vertices.

We complete the proof by showing that the vertices of X are contained in the spine S of a caterpillar of G in which

(1) consecutive vertices of X in $< S >$ are at distance at most 3 in $< S >$ and

(2) $< S >$ begins and ends with a vertex of X.

Suppose $l < t$ vertices of X are contained in such a caterpillar C with spine S'. We assume that no vertex of X is an endvertex of C and that the endvertices of C have been "shifted left." Furthermore, we assume that if $u \in V(G) - X - S'$ and u is adjacent to a vertex in S', then u is an endvertex of C. Let $x_1 \in X$ be the rightmost spine vertex of C and let $x_2 \in X - V(C)$. Furthermore, let w be an endvertex of C adjacent to x_1. If no such w exists, then we may replace x_1 in X by its predecessor on the spine S' and continue. Then $d_G(w, x_2) \leq 2$. If $wx_2 \in E(G)$ we can easily extend C to include w and x_2 as spine vertices. If $d_G(w, x_2) = 2$, then, as in the proofs of previous results, w and x_2 must have a common neighbor y that is not on the spine of C (where y may or may not be in X.) In either case, we can extend the spine of C to include w, y, x_2, and the proof is complete. ∎

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