\textbf{P-BIPARTITIONS OF MINOR HEREDITARY PROPERTIES}

\textbf{PIOTR BOROWIECKI}

\textit{Institute of Mathematics}
\textit{Technical University}
\textit{Podgórna 50, 65–246 Zielona Góra, Poland}
\textit{e-mail:} p.borowiecki@im.pz.zgora.pl

\textbf{AND}

\textbf{JAROSLAV IVANČO}

\textit{Department of Geometry and Algebra}
\textit{P.J. Šafárik University}
\textit{Jesenná 5, 041 54 Košice, Slovakaia}
\textit{e-mail:} ivanco@duro.upjs.sk

\textbf{Abstract}

We prove that for any two minor hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{P}_2$ covers $\mathcal{P}_1$, and for any graph $G \in \mathcal{P}_2$ there is a $\mathcal{P}_1$-bipartition of $G$. Some remarks on minimal reducible bounds are also included.

\textbf{Keywords:} minor hereditary property of graphs, generalized colouring, bipartitions of graphs.

\textbf{1991 Mathematics Subject Classification:} 05C70, 05C15.

1. Introduction and Notation

According to [3] we denote by $\mathcal{I}$ the class of all finite simple graphs. A \textit{graph property} is a nonempty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$. For properties $\mathcal{P}_1, \mathcal{P}_2$ of graphs a vertex \textit{$(\mathcal{P}_1, \mathcal{P}_2)$-partition} of a graph $G$ is a partition $(V_1, V_2)$ of $V(G)$ such that the subgraph $G[V_i]$ induced by the set $V_i$ has the property $\mathcal{P}_i$ for each $i = 1, 2$. The class of all vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable graphs is denoted by $\mathcal{P}_1 \circ \mathcal{P}_2$. If $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$, then a $(\mathcal{P}_1, \mathcal{P}_2)$-partition (as in [4]) we call a $\mathcal{P}$-\textit{bipartition}. 
Let be given a graph $G \in \mathcal{I}$. A contraction of the graph $G$ is a graph obtained from $G$ by repeated contractions of edges, where contraction of an edge $(v_1, v_2)$ of the graph $G$ is obtained by deleting $v_1$ and $v_2$ and all incident edges from $G$ and adding a new vertex $u$ and all the edges required to satisfy the following condition $N(u) = N(v_1) \cup N(v_2) \setminus \{v_1, v_2\}$.

A graph $H$ obtained from $G$ by deletions of vertices or edges, or contractions of edges is called a minor of $G$. So, the graph $H$ is a minor of the graph $G$ if $H$ is a subgraph of $G$ or can be obtained from a subgraph of $G$ by contractions of edges. We express this relation between the graphs $H$ and $G$ by $H < G$.

A property $\mathcal{P}$ of graphs is called minor hereditary (hereditary) if it is closed under minors (subgraphs), i.e., if whenever $G \in \mathcal{P}$ and $H$ is a minor (subgraph) of $G$, then also $H \in \mathcal{P}$.

Any minor hereditary property $\mathcal{P}$ can be uniquely determined by the set of forbidden minors which can be defined in the following way:

$$\mathcal{F}_M(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each minor } H \text{ of } G, H \neq G, \text{ belongs to } \mathcal{P}\}.$$ 

A property $\mathcal{P}$ is called additive if it is closed under disjoint union of graphs, i.e., if for each graph $G$ all of whose connected components have a property $\mathcal{P}$ it follows that $G$ has a property $\mathcal{P}$, too. It is easy to see that a minor hereditary property $\mathcal{P}$ is additive if and only if all minors $H \in \mathcal{F}_M(\mathcal{P})$ are connected.

Many well-known properties of graphs are both minor hereditary and additive. According to [2], [3] we list some of them to introduce the necessary notions which will be used in the paper. It is convenient to work with an arbitrary nonnegative integer $k$.

$$\mathcal{O} = \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},$$

$$\mathcal{D}_1 = \{G \in \mathcal{I} : G \text{ is 1-degenerate, i.e., the minimum degree } \delta(H) \leq 1 \text{ for each } H \subseteq G\},$$

$$\mathcal{T}_k = \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or } K_{\left\lfloor \frac{k+3}{2} \right\rfloor, \left\lceil \frac{k+3}{2} \right\rceil}, k \leq 3\},$$

$$\mathcal{SP} = \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_4\}.$$

We have $\mathcal{D}_1 = \mathcal{T}_1$ to be the class of all forests, $\mathcal{T}_2$ and $\mathcal{T}_3$ the class of all outerplanar and all planar graphs, respectively and $\mathcal{SP}$ the class of all series-parallel graphs.

For the properties given above we have:

$$\mathcal{F}_M(\mathcal{O}) = \{K_2\},$$

$$\mathcal{F}_M(\mathcal{D}_1) = \{K_3\},$$
Let us define the next properties.

\[ F_M(LF) = \{K_3, K_{1,3}\}, \]
\[ F_M(S) = \{K_4, K_{1,3} + K_1\}. \]

All additive minor hereditary (hereditary) properties of graphs, partially ordered by a set-inclusion, form a lattice \( T^a \), \((L^a)\) with \( \cap \) as a meet operation and \( \mathcal{O} \) as the smallest element (see [2]).

All the above listed properties form in \( T^a \) the following chain:

\[ \mathcal{O} \subset LF \subset D \subset T_2 \subset S \subset SP \subset T_3. \]

2. \( \mathcal{P} \)-Bipartition Theorem

**Definition.** Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) be two additive minor hereditary properties. We say that \( \mathcal{P}_2 \) covers \( \mathcal{P}_1 \) whenever for every graph \( G_1 \in F_M(\mathcal{P}_1) \) there exists a graph \( G_2 \in F_M(\mathcal{P}_2) \) such that \( G_2 - v \) is a minor of \( G_1 \) for some vertex \( v \in V(G_2) \).

**Theorem 1.** If \( \mathcal{P}_2 \) covers \( \mathcal{P}_1 \), then the vertex set of a graph \( G \in \mathcal{P}_2 \) can be partitioned into two subsets such that each of them induces a subgraph of \( G \) belonging to \( \mathcal{P}_1 \).

**Proof.** Let us consider a given graph \( G \in \mathcal{P}_2 \) with an arbitrarily chosen vertex \( v \). It is sufficient to consider a case when \( G \) is connected. We define the subsets \( U_k = \{u \in V(G) : d(v, u) = k\} \), where \( d(u, v) \) is the length of the shortest path between \( v \) and \( u \). Put \( e = \max\{k : U_k \neq \emptyset\} \). Then \( U_0, U_1, \ldots, U_e \) is a partition of \( V(G) \) into \( e + 1 \) pairwise disjoint subsets. Moreover, a subgraph induced by \( U_0 = \{v\} \) belongs to \( \mathcal{P}_1 \). Now, let us assume to the contrary, that one of the subsets \( U_k, k = 1, \ldots, e \), induces a subgraph of \( G \), which is not in \( \mathcal{P}_1 \). Thus there is a minor \( H \) of \( G[U_k] \) belonging to \( F_M(\mathcal{P}_1) \). Since the subgraph of \( G \) induced by \( U' = \bigcup_{i=0}^{k-1} U_i \) is connected and every vertex of \( U_k \) is adjacent to a vertex of \( U_{k-1} \subset U' \), then the graph \( H + K_1 \) is a minor of \( G \). Since \( \mathcal{P}_2 \) covers \( \mathcal{P}_1 \), then \( F_M(\mathcal{P}_2) \) contains a graph \( H' \) such that \( H' - u \) is a minor of \( H \), for some \( u \in V(H') \). Obviously, \( H' \) is a minor of \( H + K_1 \). Hence, since \( H + K_1 \) is a minor of \( G \), then \( H' \) is a minor of \( G \), contrary to \( G \in \mathcal{P}_2 \). Therefore, each of the subsets \( U_i, i = 0, 1, \ldots, e \) induces a subgraph of \( G \) belonging to \( \mathcal{P}_1 \). Since vertices...
$u \in U_i$ and $w \in U_j$, for $|i - j| > 1$ are non-adjacent in $G$, then both of the sets $V_1 = \bigcup_{i=1}^{\lfloor e/2 \rfloor} U_{2i-1}$ and $V_2 = \bigcup_{i=0}^{\lfloor e/2 \rfloor} U_{2i}$ induce subgraphs of $G$ belonging to $P_1$, i.e., the partition $(V_1, V_2)$ is the required $P_1$-bipartition of $V(G)$. ■

From the theorem given above, a series of well-known results follows:

(a) $D_1 \subset O^2$,
(b) $T_2 \subset LF^2$
proven by Mihók [10], Broere and Mynhardt [5], Wang [13], and Goddard [8],
(c) $SP \subset D_1^2$
which is the result of Dirac [7],
(d) $T_3 \subset T_2^2$
proven by Broere and Mynhardt [5], Wang [13] and Poh [12].

The new conclusions can be drawn, too. For the class $S$ defined by $F_M(S) = \{K_4, K_{1,3} + K_1\}$ we have:

(e) $S \subset LF^2$.

3. Minimal Reducible Bounds

An additive hereditary property $R$ is called reducible in $L^a$, if there exist additive hereditary properties $P_1, P_2$ such that $P = P_1 \circ P_2$, and it is called irreducible, otherwise.

For a given property $P$, a reducible property $R$ is called minimal reducible bound for $P$ if $P \subseteq R$ and there is no reducible property $R' \subset R$ satisfying $P \subseteq R'$. The set of all minimal reducible bounds for $P$ will be denoted by $B(P)$. The notion of minimal reducible bounds have been introduced in [11]. In this paper Mihók proved that the class $T_2$ of outerplanar graphs has exactly two minimal reducible bounds, i.e., $B(T_2) = \{LF^2, O \circ D_1\}$. A similar results for $SP$ and $D_2$ can be found in [1], namely, $B(SP) = B(D_2) = \{O \circ D_1\}$.

By the transitivity and Mihók’s proof (see [11]) we have the following minimal reducible bounds for the property $S \supset T_2$.

**Theorem 2.** $B(S) = \{LF^2, O \circ D_1\}$.

References


Received 25 February 1997