

A CONSTRUCTIVE EXTENSION OF THE
CHARACTERIZATION ON POTENTIALLY
 $K_{s,t}$ -BIGRAPHIC PAIRS¹

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Abstract

Let $K_{s,t}$ be the complete bipartite graph with partite sets of size s and t . Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \dots \geq a_m$ and $c_1 \geq c_2 \geq \dots \geq c_n$. We say that $L = (L_1; L_2)$ is potentially $K_{s,t}$ (resp. $A_{s,t}$)-bigraphic if there is a simple bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$ and G contains $K_{s,t}$ as a subgraph (resp. the induced subgraph of $\{x_1, \dots, x_s, y_1, \dots, y_t\}$ in G is a $K_{s,t}$). In this paper, we give a characterization of L that is potentially $A_{s,t}$ -bigraphic. As a corollary, we also obtain a characterization of L that is potentially $K_{s,t}$ -bigraphic if $b_1 \geq b_2 \geq \dots \geq b_m$ and $d_1 \geq d_2 \geq \dots \geq d_n$. This is a constructive extension of the characterization on potentially $K_{s,t}$ -bigraphic pairs due to Yin and Huang (Discrete Math. **312** (2012) 1241–1243).

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1. INTRODUCTION

Let $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ be two nonincreasing sequences of nonnegative integers. The pair $S = (A; B)$ is said to be *bigraphic* if there exists a

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simple bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $d_G(x_i) = a_i$ for $1 \leq i \leq m$ and $d_G(y_i) = b_i$ for $1 \leq i \leq n$. In this case, G is referred to as a *realization* of S . The following well-known theorem due to Gale [2] and Ryser [4] independently gave a characterization of S that is bigraphic.

Theorem 1 [2, 4]. $S = (A; B)$ is bigraphic if and only if $\sum_{i=1}^m a_i = \sum_{i=1}^n b_i$ and

$$\sum_{i=1}^k a_i \leq \sum_{i=1}^n \min\{k, b_i\} \text{ for all } k \text{ with } 1 \leq k \leq m.$$

The pair $S = (A; B)$ is said to be *potentially $K_{s,t}$ -bigraphic* if there is a realization of S containing $K_{s,t}$ as a subgraph. Yin and Huang [6] presented a characterization of S that is potentially $K_{s,t}$ -bigraphic.

Theorem 2 [6]. $S = (A; B)$ is potentially $K_{s,t}$ -bigraphic if and only if $a_s \geq t$, $b_t \geq s$, $\sum_{i=1}^m a_i = \sum_{i=1}^n b_i$ and

$$\sum_{i=1}^p a_i + \sum_{i=s+1}^{s+q} a_i \leq \sum_{i=1}^t \min\{p+q, b_i - s + p\} + \sum_{i=t+1}^n \min\{p+q, b_i\}$$

for all p and q with $0 \leq p \leq s$ and $0 \leq q \leq m - s$.

Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \dots \geq a_m$ and $c_1 \geq c_2 \geq \dots \geq c_n$. We say that $L = (L_1; L_2)$ is *bigraphic* if there exists a simple bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$ and $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$. In this case, G is referred to as a *realization* of L . Garg *et al.* [3] obtained a characterization of L that is bigraphic.

Theorem 3 [3]. $L = (L_1; L_2)$ is bigraphic if and only if

$$\sum_{i=1}^k a_i \leq \sum_{j=1}^n \min\{k, d_j\} \text{ for all } k \text{ with } 1 \leq k \leq m$$

and

$$\sum_{i=1}^k c_i \leq \sum_{j=1}^m \min\{k, b_j\} \text{ for all } k \text{ with } 1 \leq k \leq n.$$

Theorem 3 reduces to Theorem 1 when $a_i = b_i$ for $1 \leq i \leq m$ and $c_i = d_i$ for $1 \leq i \leq n$. We say that $L = (L_1; L_2)$ is *potentially $K_{s,t}$ (resp. $A_{s,t}$)-bigraphic* if there is a simple bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$

for $1 \leq i \leq n$ and G contains $K_{s,t}$ as a subgraph (resp. the induced subgraph of $\{x_1, \dots, x_s, y_1, \dots, y_t\}$ in G is a $K_{s,t}$).

The purpose of this paper is to investigate a characterization of L that is potentially $K_{s,t}$ -bigraphic. We first give a characterization of L that is potentially $A_{s,t}$ -bigraphic as follows.

Theorem 4. *Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \dots \geq a_m$ and $c_1 \geq c_2 \geq \dots \geq c_n$. If $a_s \geq t$ and $c_t \geq s$, then $L = (L_1; L_2)$ is potentially $A_{s,t}$ -bigraphic if and only if*

$$(1) \quad \sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i \leq \sum_{i=1}^t \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^n \min\{p_1 + q_1, d_i\}$$

for all p_1 and q_1 with $0 \leq p_1 \leq s$ and $0 \leq q_1 \leq m - s$ and

$$(2) \quad \sum_{i=1}^{p_2} c_i + \sum_{i=t+1}^{t+q_2} c_i \leq \sum_{i=1}^s \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^m \min\{p_2 + q_2, b_i\}$$

for all p_2 and q_2 with $0 \leq p_2 \leq t$ and $0 \leq q_2 \leq n - t$.

If $s = t = 0$, then $p_1 = p_2 = 0$ and Theorem 4 reduces to Theorem 3. If we further assume that $b_1 \geq b_2 \geq \dots \geq b_m$ and $d_1 \geq d_2 \geq \dots \geq d_n$, then we can prove the following theorem.

Theorem 5. *Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \dots \geq a_m$ and $c_1 \geq c_2 \geq \dots \geq c_n$. If $b_1 \geq b_2 \geq \dots \geq b_m$ and $d_1 \geq d_2 \geq \dots \geq d_n$, then $L = (L_1; L_2)$ is potentially $K_{s,t}$ -bigraphic if and only if it is potentially $A_{s,t}$ -bigraphic.*

Combining Theorem 4 with Theorem 5, we have the following corollary.

Corollary 6. *Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \dots \geq a_m$ and $c_1 \geq c_2 \geq \dots \geq c_n$. If $a_s \geq t$, $c_t \geq s$, $b_1 \geq b_2 \geq \dots \geq b_m$ and $d_1 \geq d_2 \geq \dots \geq d_n$, then $L = (L_1; L_2)$ is potentially $K_{s,t}$ -bigraphic if and only if (1) holds for all p_1 and q_1 with $0 \leq p_1 \leq s$ and $0 \leq q_1 \leq m - s$ and (2) holds for all p_2 and q_2 with $0 \leq p_2 \leq t$ and $0 \leq q_2 \leq n - t$.*

Corollary 6 reduces to Theorem 2 when $a_i = b_i$ for $1 \leq i \leq m$ and $c_i = d_i$ for $1 \leq i \leq n$.

2. PROOFS OF THEOREMS 4 AND 5

The proof technique of Theorem 4 was developed earlier by Tripathi, Venugopalan and West [5].

Proof of Theorem 4. For the necessity, we suppose that G is a realization of $L = (L_1; L_2)$ with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$ and the induced subgraph of $\{x_1, \dots, x_s, y_1, \dots, y_t\}$ in G is a $K_{s,t}$. For p_1 and q_1 with $0 \leq p_1 \leq s$ and $0 \leq q_1 \leq m - s$, it is easy to see that $\sum_{i=1}^t \min\{p_1 + q_1, d_G(y_i) - s + p_1\} + \sum_{i=t+1}^n \min\{p_1 + q_1, d_G(y_i)\}$ is the maximum contribution to $\sum_{i=1}^{p_1} d_G(x_i) + \sum_{i=s+1}^{s+q_1} d_G(x_i)$ from edges incident to y_1, \dots, y_n . Thus,

$$\begin{aligned} \sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i &\leq \sum_{i=1}^{p_1} d_G(x_i) + \sum_{i=s+1}^{s+q_1} d_G(x_i) \\ &\leq \sum_{i=1}^t \min\{p_1 + q_1, d_G(y_i) - s + p_1\} + \sum_{i=t+1}^n \min\{p_1 + q_1, d_G(y_i)\} \\ &\leq \sum_{i=1}^t \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^n \min\{p_1 + q_1, d_i\}, \end{aligned}$$

that is, (1) holds for p_1 and q_1 . Similarly, we can prove that (2) holds for p_2 and q_2 with $0 \leq p_2 \leq t$ and $0 \leq q_2 \leq n - t$.

For the sufficiency, we assume that (1) holds for p_1 and q_1 with $0 \leq p_1 \leq s$ and $0 \leq q_1 \leq m - s$ and (2) holds for p_2 and q_2 with $0 \leq p_2 \leq t$ and $0 \leq q_2 \leq n - t$. A *subrealization* of $L = (L_1; L_2)$ is a bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $d_G(x_i) \leq b_i$ for $1 \leq i \leq m$ and $d_G(y_i) \leq d_i$ for $1 \leq i \leq n$. If $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$ and $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$, then G is a realization of L . We will construct a realization of L through successive subrealizations. The initial subrealization is $K_{s,t} \cup \bar{K}_{m-s} \cup \bar{K}_{n-t}$, where \bar{K}_r is the complement of K_r , $K_{s,t}$ has partite sets $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$, $V(\bar{K}_{m-s}) = \{x_{s+1}, \dots, x_m\}$ and $V(\bar{K}_{n-t}) = \{y_{t+1}, \dots, y_n\}$.

In each successive subrealization, let p_1 be the largest index such that $d(x_i) = a_i$ for $1 \leq i < p_1$ and $d(x_{p_1}) < a_{p_1}$ and q_1 be the largest index such that $d(x_i) = a_i$ for $s + 1 \leq i < s + q_1$ and $d(x_{s+q_1}) < a_{s+q_1}$. While $p_1 \leq s$ or $q_1 \leq m - s$, we can obtain a new subrealization containing the initial subrealization and having smaller deficiency $(a_{p_1} - d(x_{p_1})) + (a_{s+q_1} - d(x_{s+q_1}))$ at x_{p_1} and x_{s+q_1} while not changing the degree of any vertex x_i with $i \in \{1, \dots, p_1 - 1, s + 1, \dots, s + q_1 - 1\}$.

Let $X_1 = \{x_{p_1+1}, \dots, x_s\}$ and $X_2 = \{x_{s+q_1+1}, \dots, x_m\}$. We maintain the condition that $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, there is no edge between $\{y_1, \dots, y_t\}$ and X_2 and there is no edge between $\{y_{t+1}, \dots, y_n\}$ and $X_1 \cup X_2$,

which certainly hold initially. For convenience, we write $v_i \leftrightarrow v_j$ for “ v_i is adjacent to v_j ” and $v_i \not\leftrightarrow v_j$ for “ v_i is not adjacent to v_j ”.

Case 0. Suppose $x_{p_1} \not\leftrightarrow y_k$ for some $k > t$ such that $d(y_k) < d_k$. Add $x_{p_1}y_k$.

Case 1. Suppose $x_{s+q_1} \not\leftrightarrow y_k$ for some k such that $d(y_k) < d_k$. Add $x_{s+q_1}y_k$.

Case 2. Suppose $d(y_k) \neq \min\{p_1 + q_1, d_k\}$ for some k with $k \geq t + 1$. In a subrealization, $d(y_k) \leq d_k$. Since there is no edge between $\{y_{t+1}, \dots, y_n\}$ and $X_1 \cup X_2$, $d(y_k) \leq p_1 + q_1$. Hence, $d(y_k) < \min\{p_1 + q_1, d_k\}$. Case 0 and Case 1 apply, unless $x_{p_1} \leftrightarrow y_k$ and $x_{s+q_1} \leftrightarrow y_k$. Since $d(y_k) < p_1 + q_1$, there exists $i \in \{1, \dots, p_1 - 1, s + 1, \dots, s + q_1 - 1\}$ such that $x_i \not\leftrightarrow y_k$. If $i \in \{1, \dots, p_1 - 1\}$, by $p_1 \leq s$ and $d(x_i) = a_i \geq a_{p_1} > d(x_{p_1})$, there exists $u \in N(x_i) \setminus N(x_{p_1})$, then replace ux_i by $\{x_iy_k, ux_{p_1}\}$. If $i \in \{s + 1, \dots, s + q_1 - 1\}$, by $d(x_i) > d_{s+q_1}$, there exists $u \in N(x_i) \setminus N(x_{s+q_1})$, then replace ux_i by $\{x_iy_k, ux_{s+q_1}\}$.

Case 3. Suppose $d(y_k) - s + p_1 \neq \min\{p_1 + q_1, d_k - s + p_1\}$ for some k with $k \leq t$. In a subrealization, $d(y_k) - s + p_1 \leq d_k - s + p_1$. Since there is no edge between $\{y_1, \dots, y_t\}$ and X_2 , $d(y_k) - s + p_1 \leq p_1 + q_1$. Hence $d(y_k) - s + p_1 < \min\{p_1 + q_1, d_k - s + p_1\}$. Case 1 applies unless $x_{s+q_1} \leftrightarrow y_k$. Since $d(y_k) - s + p_1 < p_1 + q_1$ and $x_i \leftrightarrow y_k$ for $1 \leq i \leq p_1$, there exists $i \in \{s + 1, \dots, s + q_1 - 1\}$ such that $x_i \not\leftrightarrow y_k$. By $d(x_i) > d(x_{s+q_1})$, there exists $u \in N(x_i) \setminus N(x_{s+q_1})$, then replace ux_i by $\{x_iy_k, ux_{s+q_1}\}$.

If none of Cases 0–3 applies, then $d(y_k) = \min\{p_1 + q_1, d_k\}$ for $k \geq t + 1$ and $d(y_k) - s + p_1 = \min\{p_1 + q_1, d_k - s + p_1\}$ for $k \leq t$. Since $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, there is no edge between $\{y_1, \dots, y_t\}$ and X_2 and there is no edge between $\{y_{t+1}, \dots, y_n\}$ and $X_1 \cup X_2$, we have that

$$\sum_{i=1}^{p_1} d(x_i) + \sum_{i=1}^{q_1} d(x_{s+i}) = \sum_{i=1}^t \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^n \min\{p_1 + q_1, d_i\}.$$

By (1) and the observation that $d(x_i) = a_i$ for $1 \leq i \leq p_1 - 1$ and $d(x_{s+i}) = a_{s+i}$ for $1 \leq i \leq q_1 - 1$, we get that $\sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i = \sum_{i=1}^{p_1} d(x_i) + \sum_{i=1}^{q_1} d(x_{s+i})$, which implies that $d(x_{p_1}) = a_{p_1}$ and $d(x_{s+q_1}) = a_{s+q_1}$. Now we have shown that while $p_1 \leq s$ or $q_1 \leq m - s$, we obtain a new subrealization containing the initial subrealization and having $d(x_{p_1}) = a_{p_1}$ and $d(x_{s+q_1}) = a_{s+q_1}$ while not changing the degree of any vertex x_i with $i \in \{1, \dots, p_1 - 1, s + 1, \dots, s + q_1 - 1\}$. Increase p_1 by 1 and q_1 by 1, and repeat the process from Case 0 to Case 3. Thus when $p_1 = s$ and $q_1 = m - s$, a subrealization G' containing the initial subrealization can be obtained so that $d(x_i) = a_i$ for $1 \leq i \leq m$ and $d(y_i) \leq d_i$ for $1 \leq i \leq n$.

We now regard G' as a new initial subrealization. In the following, for each successive subrealization, we define p_2 to be the largest index such that $d(y_i) \geq c_i$ for $1 \leq i < p_2$ and $d(y_{p_2}) < c_{p_2}$, and q_2 to be the largest index such that $d(y_i) \geq c_i$ for $t+1 \leq i < t+q_2$ and $d(y_{t+q_2}) < c_{t+q_2}$. While $p_2 \leq t$ or $q_2 \leq n - t$, we can obtain

a new subrealization having smaller deficiency $(c_{p_2} - d(y_{p_2})) + (c_{t+q_2} - d(y_{t+q_2}))$ at y_{p_2} and y_{t+q_2} while maintaining the conditions that $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, $d(y_i) \geq c_i$ for $i \in \{1, \dots, p_2 - 1, t + 1, \dots, t + q_2 - 1\}$ and $a_i \leq d(x_i) \leq b_i$ for $1 \leq i \leq m$. The process can only stop when the subrealization is a realization of L .

Case 4. Suppose, for some $j > s$, $x_j \leftrightarrow y_k$ for some $p_2 + 1 \leq k \leq t$ and $x_j \not\leftrightarrow y_\ell$ for some $\ell \leq p_2$. If $\ell = p_2$, then replace $y_k x_j$ by $y_{p_2} x_j$. If $\ell < p_2$, then replace $\{y_k x_j, y_\ell v\}$ by $\{y_\ell x_j, y_{p_2} v\}$, where $v \in N(y_\ell) \setminus N(y_{p_2})$.

Case 5. Suppose, for some $j \in \{1, \dots, m\}$, $x_j \leftrightarrow y_k$ for some $k > t + q_2$ and $x_j \not\leftrightarrow y_\ell$ for some $1 + t \leq \ell \leq t + q_2$. If $\ell = t + q_2$, then replace $x_j y_k$ by $x_j y_{t+q_2}$. If $t + 1 \leq \ell < t + q_2$, then replace $\{x_j y_k, y_\ell v\}$ by $\{v y_{t+q_2}, y_\ell x_j\}$, where $v \in N(y_\ell) \setminus N(y_{t+q_2})$.

Case 6. Suppose $d(x_j) < b_j$ for some $j > s$ and $x_j \not\leftrightarrow y_\ell$ for some $\ell \leq p_2$. If $\ell = p_2$, then add $x_j y_{p_2}$. If $\ell < p_2$, then replace $v y_\ell$ by $\{v y_{p_2}, y_\ell x_j\}$, where $v \in N(y_\ell) \setminus N(y_{p_2})$.

Case 7. Suppose $d(x_j) < b_j$ for some $j \in \{1, \dots, m\}$ and $x_j \not\leftrightarrow y_\ell$ for some $t + 1 \leq \ell \leq t + q_2$. If $\ell = t + q_2$, then add $x_j y_{t+q_2}$. If $t + 1 \leq \ell < t + q_2$, then replace $v y_\ell$ by $\{v y_{t+q_2}, y_\ell x_j\}$, where $v \in N(y_\ell) \setminus N(y_{t+q_2})$.

Case 8. Suppose, for some $j > s$, $x_j \leftrightarrow y_k$ for some $p_2 + 1 \leq k \leq t$ and $x_j \not\leftrightarrow y_\ell$ for some $t + 1 \leq \ell \leq t + q_2$. If $\ell = t + q_2$, then replace $x_j y_k$ by $x_j y_{t+q_2}$. If $t + 1 \leq \ell < t + q_2$, then replace $\{x_j y_k, y_\ell v\}$ by $\{v y_{t+q_2}, y_\ell x_j\}$, where $v \in N(y_\ell) \setminus N(y_{t+q_2})$.

Case 9. Suppose, for some $j > s$, $x_j \leftrightarrow y_k$ for some $k > t + q_2$ and $x_j \not\leftrightarrow y_\ell$ for some $\ell \leq p_2$. If $\ell = p_2$, then replace $x_j y_k$ by $x_j y_{p_2}$. If $\ell < p_2$, then replace $\{x_j y_k, v y_\ell\}$ by $\{v y_{p_2}, x_j y_\ell\}$, where $v \in N(y_\ell) \setminus N(y_{p_2})$.

Case 10. Suppose $d(y_i) > c_i$ for some $i \in \{1, \dots, p_2 - 1, t + 1, \dots, t + q_2 - 1\}$. If $i \in \{1, \dots, p_2 - 1\}$, then replace $v y_i$ by $v y_{p_2}$, where $v \in N(y_i) \setminus N(y_{p_2})$. If $i \in \{t + 1, \dots, t + q_2 - 1\}$, then replace $v y_i$ by $v y_{t+q_2}$, where $v \in N(y_i) \setminus N(y_{t+q_2})$.

If none of Cases 4–9 applies, we can prove the following claim.

Claim. Assume that none of Cases 4–9 applies. Then

(i) For each $x_j \in \{x_1, \dots, x_s\}$, $\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\}$ and $\min\{p_2 + q_2, b_j - t + p_2\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j .

(ii) For each $x_j \in \{x_{s+1}, \dots, x_m\}$, $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\}$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j .

Proof. If $x_j \in \{x_1, \dots, x_s\}$, we consider the following two cases depending on whether x_j is adjacent to all the vertices in $\{y_{t+1}, \dots, y_{t+q_2}\}$ or not.

Suppose $x_j \leftrightarrow y_k$ for all $t+1 \leq k \leq t+q_2$. Since $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, x_j is adjacent to every vertex in $\{y_1, \dots, y_t\}$. Thus, $p_2 + q_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j . By $b_j - t + p_2 \geq d(x_j) - t + p_2 \geq p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\} = p_2 + q_2$ and $\min\{p_2 + q_2, b_j - t + p_2\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j .

Suppose $x_j \not\leftrightarrow y_k$ for some $t+1 \leq k \leq t+q_2$. Since Case 5 and Case 7 cannot apply, we have that $x_j \not\leftrightarrow y_\ell$ for all $\ell > t+q_2$ and $d(x_j) = b_j$. This implies that $b_j - t + p_2 = d(x_j) - t + p_2 < p_2 + q_2$, $\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\} = b_j - t + p_2$ and $b_j - t + p_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j .

If $x_j \in \{x_{s+1}, \dots, x_m\}$, we consider the following two cases depending on whether x_j is adjacent to all the vertices in $\{y_1, \dots, y_{p_2}\}$ or not.

Suppose $x_j \leftrightarrow y_k$ for all $k \leq p_2$. If $x_j \leftrightarrow y_\ell$ for all $t+1 \leq \ell \leq t+q_2$, then $p_2 + q_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j . By $b_j \geq d(x_j) \geq p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = p_2 + q_2$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j . Assume that $x_j \not\leftrightarrow y_\ell$ for some $t+1 \leq \ell \leq t+q_2$. Since Case 5, Case 7 and Case 8 cannot apply, we have that $x_j \not\leftrightarrow y_k$ for all $k > t+q_2$, $d(x_j) = b_j$ and $x_j \not\leftrightarrow y_k$ for all $p_2+1 \leq k \leq t$. Thus, b_j is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j . By $b_j = d(x_j) < p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = b_j$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j .

Suppose $x_j \not\leftrightarrow y_k$ for some $k \leq p_2$. Since Case 4, Case 6 and Case 9 cannot apply, we have that $x_j \not\leftrightarrow y_\ell$ for all $p_2+1 \leq \ell \leq t$, $x_j \not\leftrightarrow y_\ell$ for all $\ell > t+q_2$ and $d(x_j) = b_j$. By $b_j = d(x_j) < p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = b_j$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j . The claim is proved. \square

We now continue to proceed with the proof of theorem. By the previous claim, we have that $\sum_{i=1}^s \min\{p_2 + q_2, d(x_i) - t + p_2\} + \sum_{i=s+1}^m \min\{p_2 + q_2, d(x_i)\} = \sum_{i=1}^s \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^m \min\{p_2 + q_2, b_i\}$, and $\sum_{i=1}^s \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^m \min\{p_2 + q_2, b_i\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_1, \dots, x_m . If Case 10 cannot apply, then $d(y_i) = c_i$ for all $i \in \{1, \dots, p_2 - 1, t + 1, \dots, t + q_2 - 1\}$. Thus, we obtain that

$$\begin{aligned} & \sum_{i=1}^{p_2-1} c_i + \sum_{i=1}^{q_2-1} c_{t+i} + d(y_{p_2}) + d(y_{t+q_2}) = \sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i}) \\ & = \sum_{j=1}^s \min\{p_2 + q_2, b_j - t + p_2\} + \sum_{j=s+1}^m \min\{p_2 + q_2, b_j\}. \end{aligned}$$

By (2), we further have that $c_{p_2} + c_{t+q_2} \leq d(y_{p_2}) + d(y_{t+q_2})$, which implies that $d(y_{p_2}) = c_{p_2}$ and $d(y_{t+q_2}) = c_{t+q_2}$. Now we have shown that while $p_2 \leq t$ or $q_2 \leq n-t$, we obtain a new subrealization having $d(y_{p_2}) = c_{p_2}$ and $d(y_{t+q_2}) = c_{t+q_2}$ while maintaining the conditions that $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, $d(y_i) \geq c_i$ for $i \in \{1, \dots, p_2 - 1, t + 1, \dots, t + q_2 - 1\}$ and $a_i \leq d(x_i) \leq b_i$ for $1 \leq i \leq m$. Increase p_2 by 1 and q_2 by 1, and repeat the process from Case 4 to Case 10. When $p_2 = t$ and $q_2 = n - t$, we finally get a realization of L containing $K_{s,t}$ and satisfying $a_i \leq d(x_i) \leq b_i$ for $1 \leq i \leq m$ and $c_i \leq d(y_i) \leq d_i$ for $1 \leq i \leq n$, where $V(K_{s,t}) = \{x_1, \dots, x_s, y_1, \dots, y_t\}$. In other words, L is potentially $A_{s,t}$ -bigraphic. The proof of Theorem 4 is completed. ■

This constructive proof can be implemented as an algorithm to construct a realization of L containing $K_{s,t}$. The following lemma due to Ferrara *et al.* [1] will be useful as we proceed with the proof of Theorem 5.

Lemma 7 [1]. *Let S be a bigraphic pair with realization $G = (X \cup Y, E)$ having partite sets X and Y . Let $H = (X' \cup Y', E')$ be a subgraph of G such that X' and Y' are contained in X and Y , respectively. Then there exists a realization $G_1 = (X \cup Y, E_1)$ of S containing H as a subgraph such that X' and Y' lie on the vertices of highest degree in X and Y , respectively.*

Proof of Theorem 5. We only need to show that if $L = (L_1; L_2)$ is potentially $K_{s,t}$ -bigraphic, then it is potentially $A_{s,t}$ -bigraphic. Let G be a simple bipartite graph with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$ and G contains $K_{s,t} = (X' \cup Y', E')$ as a subgraph. Denote $d_{1i} = d_G(x_i)$ for $1 \leq i \leq m$ and $d_{2i} = d_G(y_i)$ for $1 \leq i \leq n$. Let $A = (d_{11}, \dots, d_{1m})$ and $B = (d_{21}, \dots, d_{2n})$. By Lemma 7, $(A; B)$ has a realization $G_1 = (X \cup Y, E_1)$ satisfying $d_{G_1}(x_i) = d_{1i}$ for $1 \leq i \leq m$, $d_{G_1}(y_i) = d_{2i}$ for $1 \leq i \leq n$ and G_1 contains $K_{s,t}$ so that X' and Y' lie on the vertices of highest degree in X and Y , respectively. Let $D = \{x_1, \dots, x_s\} \setminus X'$, $D' = \{x_{s+1}, \dots, x_m\} \cap X'$, $C = \{y_1, \dots, y_t\} \setminus Y'$ and $C' = \{y_{t+1}, \dots, y_n\} \cap Y'$. Then, it is easy to see that

$$\begin{aligned} \max\{a_i|x_i \in D\} &\leq \max\{d_{1i}|x_i \in D\} \leq d_{1j} \leq \min\{b_i|x_i \in D\} \text{ for each } x_j \in D', \\ \max\{a_i|x_i \in D'\} &\leq d_{1j} \leq \min\{d_{1i}|x_i \in D'\} \leq \min\{b_i|x_i \in D'\} \text{ for each } x_j \in D, \\ \max\{c_i|y_i \in C\} &\leq \max\{d_{2i}|y_i \in C\} \leq d_{2j} \leq \min\{d_i|y_i \in C\} \text{ for each } y_j \in C', \\ \max\{c_i|y_i \in C'\} &\leq d_{2j} \leq \min\{d_{2i}|y_i \in C'\} \leq \min\{d_i|y_i \in C'\} \text{ for each } y_j \in C. \end{aligned}$$

Thus, we can see that $(L_1; L_2)$ is potentially $A_{s,t}$ -bigraphic by exchanging D with D' and C with C' . ■

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