

THE DICHROMATIC NUMBER OF INFINITE FAMILIES OF CIRCULANT TOURNAMENTS

NAHID JAVIER AND BERNARDO LLANO

*Departamento de Matemáticas
Universidad Autónoma Metropolitana Iztapalapa
San Rafael Atlixco 186, Colonia Vicentina
09340, México, D.F., Mexico*

e-mail: {nahid,llano}@xanum.uam.mx

Abstract

The *dichromatic number* $dc(D)$ of a digraph D is defined to be the minimum number of colors such that the vertices of D can be colored in such a way that every chromatic class induces an acyclic subdigraph in D . The *cyclic* circulant tournament is denoted by $T = \vec{C}_{2n+1}(1, 2, \dots, n)$, where $V(T) = \mathbb{Z}_{2n+1}$ and for every jump $j \in \{1, 2, \dots, n\}$ there exist the arcs $(a, a+j)$ for every $a \in \mathbb{Z}_{2n+1}$. Consider the circulant tournament $\vec{C}_{2n+1}(k)$ obtained from the cyclic tournament by reversing one of its jumps, that is, $\vec{C}_{2n+1}(k)$ has the same arc set as $\vec{C}_{2n+1}(1, 2, \dots, n)$ except for $j = k$ in which case, the arcs are $(a, a - k)$ for every $a \in \mathbb{Z}_{2n+1}$. In this paper, we prove that $dc(\vec{C}_{2n+1}(k)) \in \{2, 3, 4\}$ for every $k \in \{1, 2, \dots, n\}$. Moreover, we classify which circulant tournaments $\vec{C}_{2n+1}(k)$ are vertex-critical r -dichromatic for every $k \in \{1, 2, \dots, n\}$ and $r \in \{2, 3, 4\}$. Some previous results by Neumann-Lara are generalized.

Keywords: tournament, dichromatic number, vertex-critical r -dichromatic tournament.

2010 Mathematics Subject Classification: 05C20, 05C38.

1. INTRODUCTION

A *tournament* T is an orientation of the complete graph. If T contains no directed cycles, T is called *transitive* and denoted by TT_k , where $k \in \mathbb{N}$ is its order.

The definition of the dichromatic number of a digraph and the first important results were introduced by Neumann-Lara in 1982 [9]. Independently, Jacob and Meyniel defined the same notion in 1983, see [6]. In 1977, Erdős visited Mexico

and began to work on the dichromatic number of a graph (see the definition below) with Neumann-Lara. The results of this collaboration were summarized in a survey by Erdős in 1979 (see [3] for details).

Other results concerning this topic can be found in a paper by Erdős, Gimbel and Kratsch [4]. According to this paper, the *dichromatic number of a digraph* D , denoted by $dc(D)$, is “the minimum number of parts the vertex set of D must be partitioned into, so that each part induces an acyclic digraph.” Equivalently, the dichromatic number of D is the minimum number of colors such that the vertices of D can be colored in such a way that every chromatic class induces an acyclic subdigraph in D . The main result of paper [4] for tournaments is the following: every tournament with n vertices can be colored with $O(n/\log n)$ and there exists tournaments (for example, random tournaments) having dichromatic number $\Omega(n/\log n)$ (see Theorem 5 of the aforementioned paper). There are more interesting asymptotic results in [5] by Harutyunyan. In particular, Theorem 2.3.8 states that if T is a tournament of order n , then $dc(T) \leq \frac{n}{\log n}(1 + o(1))$.

The *dichromatic number of a graph* G was defined by Erdős and Neumann-Lara in [3] as

$$dc(G) = \max \left\{ dc(\vec{G}) : \vec{G} \text{ is an orientation of } G \right\}.$$

Determining the dichromatic number of a general (di)graph is a very hard problem. Exact values of this parameter are only known for some special classes of digraphs, particularly in a few cases of circulant tournaments (see [1, 7, 9, 12, 10, 11] and [13]). In this paper, we prove that

$$(i) \quad dc(\vec{C}_{2n+1}\langle 1 \rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } 4 \leq n \leq 7, \\ 4 & \text{if } n \geq 8, \end{cases}$$

(Section 3, Corollary to Theorem 11),

$$(ii) \quad dc(\vec{C}_{2n+1}\langle 2 \rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8, \end{cases}$$

(Section 3, Corollary to Theorem 17),

(iii) if $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$ for $n \geq 7$ (Section 4, Theorem 22),

(iv) if $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$ (Section 5, Theorem 26), and

(v) $dc(\vec{C}_{2n+1}\langle k \rangle) = 3$ for $k = \lfloor \frac{2}{3}n \rfloor + 1, \dots, n$, where $n \geq 3$ (Section 5, Theorem 29).

Our results generalize some theorems obtained by Neumann-Lara. At the end of Section 5, we characterize the vertex-critical r -dichromatic circulant tournaments $\vec{C}_{2n+1}\langle k \rangle$ for every $k \in \{1, 2, \dots, n\}$ and $r \in \{2, 3, 4\}$, see Theorem 32, the main theorem of this paper.

2. PRELIMINARIES

Let \mathbb{Z}_m be the cyclic group of integers modulo m , where $m \in \mathbb{N}$ and J is a nonempty subset of $\mathbb{Z}_m \setminus \{0\}$ such that $w \in J$ if and only if $-w \notin J$ for every $w \in \mathbb{Z}_m$. The circulant digraph $\vec{C}_m(J)$ is defined by $V(\vec{C}_m(J)) = \mathbb{Z}_m$ and

$$A(\vec{C}_m(J)) = \{(i, j) : i, j \in \mathbb{Z}_m \text{ and } j - i \in J\}.$$

Notice that $\vec{C}_{2n+1}(J)$ is a circulant (or rotational) tournament if and only if $|J| = n$. We recall that circulant tournaments are regular and their automorphism group is vertex-transitive. We define

$$\begin{aligned} \vec{C}_{2n+1}(1, 2, \dots, n) &:= \vec{C}_{2n+1}\langle \emptyset \rangle \quad \text{and} \\ \vec{C}_{2n+1}(1, \dots, k-1, -k, k+1, \dots, n) &= \\ \vec{C}_{2n+1}(1, \dots, k-1, k+1, \dots, n, 2n+1-k) &:= \vec{C}_{2n+1}\langle k \rangle. \end{aligned}$$

Observe that the circulant $\vec{C}_m(1) = \vec{C}_m$ is the directed cycle of length m . If $V(\vec{C}_m) = \{a_1, a_2, \dots, a_m\}$, we denote $\vec{C}_m = (a_1, a_2, \dots, a_m, a_1)$. The tournament $\vec{C}_{2n+1}\langle \emptyset \rangle$ is called the *cyclic tournament*. It is straightforward to check that there is only one cyclic tournament on n vertices up to isomorphism for every $n \in \mathbb{N}$. The isomorphism between digraphs G and H is denoted by $G \cong H$. A digraph D is called r -dichromatic if $dc(D) = r$. It is *vertex-critical* r -dichromatic if $dc(D) = r$ and $dc(D - v) < r$ for every $v \in V(D)$. For general terminology, see [2]. In what follows, we will need the following results of [11] and [13].

Theorem 1 ([13], Theorem 1). *If T_{2n+1} is a regular tournament on $2n + 1$ vertices, then $dc(T_{2n+1}) = 2$ if and only if $T_{2n+1} \cong \vec{C}_{2n+1}\langle \emptyset \rangle$.*

Theorem 2 ([13], Theorem 2). *$\vec{C}_{2n+1}\langle n \rangle$ is a vertex-critical 3-dichromatic circulant tournament for $n \geq 3$.*

Theorem 3 ([11]). *$\vec{C}_{6m+1}\langle 2m \rangle$ is a vertex-critical 4-dichromatic circulant tournament for $m \geq 2$.*

Let T be a tournament and $k, l \in \mathbb{N}$. We recall that a transitive subtournament TT_k of T is *maximal* if there does not exist a transitive subtournament TT_l of T ($k < l$) such that TT_k is a subtournament of TT_l .

Remark 4 ([8]). Up to isomorphism

(i) there exists a unique circulant tournament of order 5, that is, $\vec{C}_5(1, 2) = \vec{C}_5\langle\emptyset\rangle$,

(ii) there exist two circulant tournaments of order 7 which are

$$\begin{aligned} \vec{C}_7(1, 2, 3) &= \vec{C}_7\langle\emptyset\rangle \cong \vec{C}_7\langle 1\rangle \cong \vec{C}_7\langle 2\rangle \quad \text{and} \\ \vec{C}_7(1, 2, 4) &= \vec{C}_7\langle 3\rangle, \end{aligned}$$

(iii) there exist three circulant tournaments of order 9 which are

$$\begin{aligned} \vec{C}_9(1, 2, 3, 4) &= \vec{C}_9\langle\emptyset\rangle, \\ \vec{C}_9(1, 2, 3, 5) &= \vec{C}_9\langle 4\rangle \cong \vec{C}_9\langle 1\rangle \cong \vec{C}_9\langle 3\rangle \quad \text{and} \\ \vec{C}_9(1, 3, 4, 7) &= \vec{C}_9\langle 2\rangle, \end{aligned}$$

(where $\vec{C}_9\langle 2\rangle \cong \vec{C}_3[\vec{C}_3]$ is the composition of \vec{C}_3 and \vec{C}_3).

In the following three sections we determine the exact value of $dc(\vec{C}_{2n+1}\langle k\rangle)$ for every $n, k \in \mathbb{N}$. We subdivide the calculations into five cases (the cases are illustrated in Figure 1):

- (i) $k = 1$,
- (ii) $k = 2$,
- (iii) $3 \leq k \leq \lceil \frac{n}{2} \rceil$,
- (iv) $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$ and
- (v) $\lfloor \frac{2}{3}n \rfloor + 1 \leq k \leq n$.

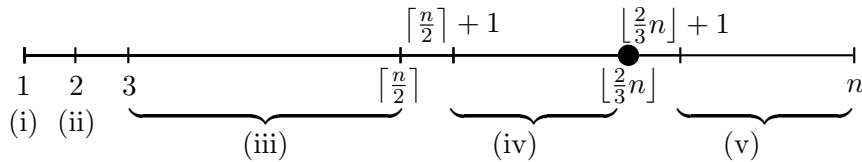


Figure 1

3. THE DICHROMATIC NUMBER OF $\vec{C}_{2n+1}\langle 1\rangle$ AND $\vec{C}_{2n+1}\langle 2\rangle$

To begin with, let us observe the following facts.

Remark 5.

- (i) For every $j \in \mathbb{Z}_{2n+1}$, the set of vertices $\{j - 2, j - 1, j\}$ induces a \vec{C}_3 in $\vec{C}_{2n+1}\langle 1\rangle$ and $\vec{C}_{2n+1}\langle 2\rangle$, respectively,
- (ii) $\vec{C}_{11}\langle 1\rangle \cong \vec{C}_{11}\langle 4\rangle$,
- (iii) $\vec{C}_{13}\langle 1\rangle \cong \vec{C}_{13}\langle 3\rangle$.

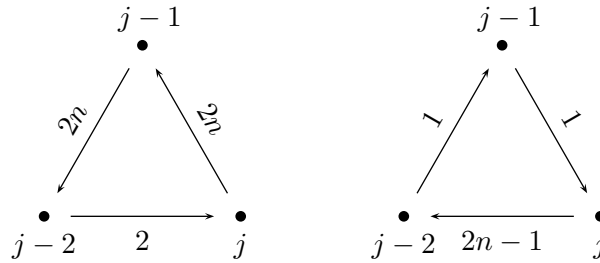


Figure 2. \vec{C}_3 in $\vec{C}_{2n+1}\langle 1 \rangle$ and $\vec{C}_{2n+1}\langle 2 \rangle$, respectively.

Proposition 6. $dc(\vec{C}_{11}\langle 1 \rangle) = dc(\vec{C}_{13}\langle 1 \rangle) = dc(\vec{C}_{15}\langle 1 \rangle) = 3$.

Proof. Consider $\vec{C}_{11}\langle 1 \rangle$. Since $\vec{C}_{11}\langle 1 \rangle \not\cong \vec{C}_{11}\langle \emptyset \rangle$, by Theorem 1, $dc(\vec{C}_{11}\langle 1 \rangle) \geq 3$. The partition of the vertices of $\vec{C}_{11}\langle 1 \rangle$ given by $P_1 = \{0, 3, 2, 5\}$, $P_2 = \{7, 10, 9, 1\}$ and $P_3 = \{4, 6, 8\}$ implies that $dc(\vec{C}_{11}\langle 1 \rangle) = 3$. Observe that P_1 and P_2 induce a TT_4 , and P_3 induces a TT_3 in $\vec{C}_{11}\langle 1 \rangle$, respectively. By Remark 5(ii), we have that $dc(\vec{C}_{11}\langle 4 \rangle) = 3$. Notice that the transitive subtournaments induced by P_1, P_2 and P_3 are maximal. If $\langle P_1 \rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is the vertex 10, but $(2, 5, 10, 2) \cong \vec{C}_3$. Then P_1 induces a maximal transitive subtournament. The arguments are similar for P_2 and P_3 . Analogously we can prove the others cases $n = 6$ and $n = 7$. ■

The following lemmas will be useful tools in order to prove Theorem 11. Let a and b nonnegative integers such that $0 \leq a < b \leq n$. We define the interval $[a, b] = \{a, a + 1, \dots, b\}$, $X_0 = [0, n]$, $X_3 = [0, 2n]$ and

$$Y_0 = \{j \in X_0 : j \equiv 1 \pmod 3\}.$$

Lemma 7. *The tournament $\vec{C}_{2n+1}\langle 1 \rangle$ contains a maximal transitive subtournament of order $n + 1 - \lfloor \frac{n}{3} \rfloor$ if $n \equiv 0 \pmod 3$ and of order $n + 2 - \lfloor \frac{n}{3} \rfloor$ vertices if $n \equiv 1 \pmod 3$.*

Proof. Consider $\vec{C}_{2n+1}\langle 1 \rangle = \vec{C}_{2n+1}(2, 3, 4, 5, 6, \dots, n, 2n)$. Recall that circulant tournaments are vertex-transitive, so it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that

$$N^+(0) = \{2, 3, 4, 5, 6, \dots, n, 2n\}.$$

We have two cases.

Case 1. $n \equiv 0 \pmod 3$. Let $n = 3k$ where $k \in \mathbb{N}$. Notice that the subset of vertices $j \equiv 0, 2 \pmod 3$ belonging to the set X_0 induces a transitive subtournament

$$H_0 = \langle X_0 \setminus Y_0 \rangle,$$

by Remark 5(i). Observe that $|Y_0| = \lfloor \frac{n}{3} \rfloor$ and $|H_0| = n + 1 - \lfloor \frac{n}{3} \rfloor = 2k + 1$. It remains to prove that H_0 is maximal. If H_0 was not maximal, then the only vertex we can add is $2n$ by Remark 5(i). Observe that in this case the set $\{n, 2n, n - 4\}$ induces a \vec{C}_3 in $\vec{C}_{2n+1}\langle 1 \rangle$, which implies that $H_0 \cup \{2n\}$ cannot induce a maximal transitive subtournament.

Case 2. $n \equiv 1 \pmod 3$. This case is analogous to Case 1. The maximal transitive subtournament is

$$H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0 \rangle. \quad \blacksquare$$

Lemma 8. *Let $\vec{C}_{2n+1}\langle 1 \rangle$ be such that $n \equiv 0, 1 \pmod 3$. Then the subtournaments induced by*

$$\begin{aligned} X_3 \setminus (X_0 \setminus Y_0) & \text{ if } n \equiv 0 \pmod 3 \text{ and} \\ X_3 \setminus ((X_0 \cup \{2n\}) \setminus Y_0) & \text{ if } n \equiv 1 \pmod 3 \end{aligned}$$

contain a maximal transitive subtournament of $n - \lfloor \frac{n}{3} \rfloor$ and $n - \lfloor \frac{n}{3} \rfloor - 1$ vertices, respectively.

Proof. Suppose that $n \equiv 0 \pmod 3$. By Lemma 7, $\vec{C}_{2n+1}\langle 1 \rangle$ contains the transitive subtournament H_0 . Consider $X_1 = [n + 1, 2n]$ and define

$$Y_1 = \{j \in X_1 : j \equiv 2 \pmod 3\} \text{ and } J_0 = \langle X_1 \setminus Y_1 \rangle.$$

By Remark 5(i), J_0 is transitive. Notice that J_0 has order $n - \lfloor \frac{n}{3} \rfloor$. We prove that J_0 is maximal in the same way as in the proof of Lemma 7. For a contradiction, if J_0 was not maximal, then the only vertex we can add is vertex 1 by Remark 5(i). Observe that in this case, the set $\{1, n + 1, n + 3\}$ induces a \vec{C}_3 in $\vec{C}_{2n+1}\langle 1 \rangle$, which implies that $J_0 \cup \{1\}$ cannot induce a maximal transitive subtournament.

When $n \equiv 1 \pmod 3$, the arguments are similar. The maximal transitive subtournament J_1 is given by

$$J_1 = \langle X_1 \setminus (Y_2 \cup \{2n\}) \rangle,$$

where $Y_2 = \{j \in X_1 : j \equiv 0 \pmod 3\}$. ■

Lemma 9. *A maximal transitive subtournament contained in $\vec{C}_{2n+1}\langle 1 \rangle$ has $n + 1 - \lfloor \frac{n}{3} \rfloor$ vertices if $n \equiv 2 \pmod 3$.*

Proof. It is similar to the proof of Lemma 7. The maximal transitive subtournament is

$$H_2 = \langle X_0 \setminus Y_0 \rangle. \quad \blacksquare$$

Lemma 10. *Let $\vec{C}_{2n+1}\langle 1 \rangle$ be such that $n \equiv 2 \pmod 3$. Then the subtournament induced by*

$$X_3 \setminus (X_0 \setminus Y_0)$$

contains a maximal transitive subtournament of order $n - \lfloor \frac{n}{3} \rfloor$.

Proof. It is similar to the proof of Lemma 8. In this case, every vertex $j \equiv 0, 1 \pmod 3$ in X_1 induces a transitive subtournament

$$J_2 = \langle X_1 \setminus Y_3 \rangle,$$

where $Y_3 = \{j \in X_1 : j \equiv 2 \pmod 3\}$. ■

Theorem 11. *Let $n \in \mathbb{N}$. Then $dc(\vec{C}_{2n+1}\langle 1 \rangle) = 4$ for every $n \geq 8$.*

Proof. By Theorem 1, we have that $dc(\vec{C}_{2n+1}\langle 1 \rangle) \geq 3$. In the first place, we prove that $dc(\vec{C}_{2n+1}\langle 1 \rangle) \geq 4$. For a contradiction, suppose that $dc(\vec{C}_{2n+1}\langle 1 \rangle) = 3$. Thus, $\vec{C}_{2n+1}\langle 1 \rangle$ has a partition of its vertices inducing three transitive subtournaments. Suppose that $n \equiv 0 \pmod 3$ (it is similar when $n \equiv 1, 2 \pmod 3$). By Lemmas 7 and 8, two maximal disjoint transitive subtournaments in $\vec{C}_{2n+1}\langle 1 \rangle$ are H_0 and J_0 . Hence, the remaining vertex set $X_3 \setminus \{V(H_0) \cup V(J_0)\}$,

$$\{1, 4, 7, \dots, n + 2, n + 5, \dots\},$$

induces the third transitive subtournament. Observe that the vertex set $\{1, 7, n + 2\}$ induces a \vec{C}_3 , this is a contradiction. Therefore, $dc(\vec{C}_{2n+1}\langle 1 \rangle) \geq 4$. We show that $dc(\vec{C}_{2n+1}\langle 1 \rangle) = 4$. By Lemmas 7 and 8, we have that the two maximal transitive subtournaments H_0 and J_0 have cardinality $n + 1 - \lfloor \frac{n}{3} \rfloor$ and $n - \lfloor \frac{n}{3} \rfloor$, respectively. Define a third subtournament

$$K_0 = \langle \{1\} \cup Y_1 \rangle.$$

Notice that $|K_0| = \lfloor \frac{n}{3} \rfloor + 1$ and K_0 is transitive by the definition of Y_1 . We will prove that K_0 is a maximal transitive subtournament in $\vec{C}_{2n+1}\langle 1 \rangle \setminus \{H_0 \cup J_0\}$. If K_0 was not a maximal transitive subtournament, then we can add at least one vertex of $Y_0 \setminus \{1\}$. Notice that if $i \in Y_0 \setminus \{1\}$, we have that $(i, i + n - 2, i + n + 1, i) \cong \vec{C}_3$. Therefore, K_0 is a maximal transitive subtournament. Finally, the subtournament $L_0 = \langle Y_0 \setminus \{1\} \rangle$ is transitive by the definition of Y_0 and maximal. Thus, $dc(\vec{C}_{2n+1}\langle 1 \rangle) = 4$. The proof is completely analogous for the cases when $n \equiv 1, 2 \pmod 3$. The partition into transitive subtournaments is

$$H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0 \rangle, J_1 = \langle X_1 \setminus Y_2 \cup \{2n\} \rangle, K_1 = \langle Y_2 \cup \{1\} \rangle, L_1 = \langle Y_0 \setminus \{1\} \rangle,$$

for $n \equiv 1 \pmod 3$. For $n \equiv 2 \pmod 3$, we have that

$$H_2 = \langle X_0 \setminus Y_0 \rangle, J_2 = \langle X_1 \setminus Y_3 \rangle, K_2 = \langle (Y_3 \cup \{1\}) \rangle, L_2 = \langle Y_0 \setminus \{1\} \rangle. \quad \blacksquare$$

From Proposition 6, Theorems 1, 2 and 11 and Remark 4(iii), we obtain the following consequence.

Corollary 12.

$$dc(\vec{C}_{2n+1}\langle 1 \rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } 4 \leq n \leq 7, \\ 4 & \text{if } n \geq 8. \end{cases}$$

Theorem 13. *Let $r \in \{2, 3, 4\}$. Then $\vec{C}_{2n+1}\langle 1 \rangle$ is a vertex-critical r -dichromatic circulant tournament if and only if $n \in \{1, 4\}$.*

Proof. If $r = 2$, clearly $\vec{C}_3\langle 1 \rangle$ is a vertex-critical 2-dichromatic.

If $r = 3$, we need to check for which values of $4 \leq n \leq 7$, the circulant tournament $\vec{C}_{2n+1}\langle 1 \rangle$ is a vertex-critical 3-dichromatic. Notice that $\vec{C}_9\langle 1 \rangle \cong \vec{C}_9\langle 4 \rangle$ by Remark 4(iii). It is a vertex-critical 3-dichromatic by Theorem 2. For $n = 5$, the circulant tournament $\vec{C}_{11}\langle 1 \rangle$ is not a vertex-critical 3-dichromatic by Proposition 6. Using analogous arguments, $\vec{C}_{13}\langle 1 \rangle$ and $\vec{C}_{15}\langle 1 \rangle$ are not vertex-critical.

If $r = 4$, the circulant tournament $\vec{C}_{2n+1}\langle 1 \rangle$ is 4-dichromatic for every $n \geq 8$ by Theorem 11. It was partitioned into four maximal transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0, 1, 2$. Notice that $\vec{C}_{2n+1}\langle 1 \rangle$ is a vertex-critical 4-dichromatic if the cardinality of L_i is equal to one for $i = 0, 1, 2$. Since $|L_i| = |Y_0| - 1 = \lfloor \frac{n}{3} \rfloor - 1$, we have that $|L_i| = 1$ if and only if $\lfloor \frac{n}{3} \rfloor = 2$. It occurs when $n = 6, 7$ or 8 . By Theorem 11, it is only possible for $n \geq 8$. Observe that $|L_2| \geq 2$ for $T = \vec{C}_{2n+1}\langle 1 \rangle$ if $n \geq 8$. Since this partition is maximal, T is not vertex-critical.

Therefore, $\vec{C}_{2n+1}\langle 1 \rangle$ is a vertex-critical r -dichromatic circulant tournament if and only if n is 1 or 4. ■

Let us recall that

Remark 14. $\vec{C}_9\langle 2 \rangle = \vec{C}_3[\vec{C}_3]$ is 3-dichromatic, a particular case of Theorem 8 from [9]. Notice that it is not vertex-critical.

Remark 15 ([10], Theorem 2.6). $\vec{C}_{11}\langle 2 \rangle$ is vertex-critical 4-dichromatic.

Proposition 16. *If $n = 6$ and 7 , then $dc(\vec{C}_{2n+1}\langle 2 \rangle) = 3$.*

Proof. Observe that $\vec{C}_{15}\langle 2 \rangle \not\cong \vec{C}_{15}\langle \emptyset \rangle$. Then by Theorem 1, $dc(\vec{C}_{15}\langle 2 \rangle) \geq 3$. Consider the following partition of $V(\vec{C}_{15}\langle 2 \rangle)$:

$$P_1 = \{0, 1, 3, 4, 6, 7\}, P_2 = \{5, 8, 9, 11, 12\} \text{ and } P_3 = \{2, 10, 13, 14\}.$$

We have that $\langle P_1 \rangle \cong TT_6$, $\langle P_2 \rangle \cong TT_5$ and $\langle P_3 \rangle \cong TT_4$. Therefore, $dc(\vec{C}_{15}\langle 2 \rangle) = 3$. Note that the transitive subtournaments induced by P_1 , P_2 and P_3 are maximal. If $\langle P_1 \rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is vertex 13. We cannot add the vertex 5 by Remark 5(i). But $(4, 7, 13, 4) \cong \vec{C}_3$. Then P_1 induces a maximal transitive subtournament. The same conclusion is valid for P_2 and P_3 . Observe that $\vec{C}_{15}\langle 2 \rangle$ is not vertex-critical. The proof is analogous for $n = 6$. ■

Theorem 17. *Let $n \in \mathbb{N}$. Then $dc(\vec{C}_{2n+1}\langle 2 \rangle) = 4$ for every $n \geq 8$.*

Proof. It is analogous to the proof of Theorem 11. Therefore, Remark 5(i) is applied for $\vec{C}_{2n+1}\langle 2 \rangle$. The corresponding partitions are following.

(i) $n \equiv 0 \pmod 3$, we define $Y_4 = \{j \in X_0 : j \equiv 2 \pmod 3\}$,

$$H_0 = \langle X_0 \setminus Y_4 \rangle, J_0 = \langle X_1 \setminus Y_2 \rangle, K_0 = \langle Y_2 \cup \{2\} \rangle, L_0 = \langle Y_4 \setminus \{2\} \rangle.$$

(ii) $n \equiv 1 \pmod 3$, we define $Y_5 = \{j \in X_1 : j \equiv 1 \pmod 3\}$,

$$H_1 = \langle X_0 \setminus Y_4 \rangle, J_1 = \langle X_1 \setminus Y_5 \rangle, K_1 = \langle Y_5 \cup \{2\} \rangle, L_1 = \langle Y_4 \setminus \{2\} \rangle.$$

(iii) $n \equiv 2 \pmod 3$, we define $Y_6 = \{j \in X_1 : j \equiv 1 \pmod 3\}$,

$$H_2 = \langle X_0 \setminus Y_4 \rangle, J_2 = \langle (X_1 \cup \{n\}) \setminus Y_6 \rangle, K_2 = \langle Y_6 \rangle, L_2 = \langle Y_4 \setminus \{n\} \rangle. \quad \blacksquare$$

The next corollary is an immediate consequence of Remarks 4(ii)–(iii), 14, 15, Proposition 16 and Theorems 1, 2 and 17.

Corollary 18.

$$dc(\vec{C}_{2n+1}\langle 2 \rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8. \end{cases}$$

Theorem 19. *Let $r \in \{2, 3, 4\}$. Then $\vec{C}_{2n+1}\langle 2 \rangle$ is a vertex-critical r -dichromatic circulant tournament if and only if $n = 5$.*

Proof. If $r = 2$, by Theorem 1, $\vec{C}_7\langle 2 \rangle \cong \vec{C}_7\langle \emptyset \rangle$ is 2-dichromatic, but it is not vertex-critical.

Let $r = 3$. For $n = 4$, we have that $\vec{C}_9\langle 2 \rangle$ is not vertex-critical by Remark 14. For $n = 6$ and 7 by Proposition 16, $\vec{C}_{13}\langle 2 \rangle$ and $\vec{C}_{15}\langle 2 \rangle$ are not vertex-critical.

If $r = 4$, then by Remark 15, $\vec{C}_{11}\langle 2 \rangle$ is vertex-critical. By Theorem 17, $\vec{C}_{2n+1}\langle 2 \rangle$ is 4-dichromatic for every $n \geq 8$. It was partitioned into four maximal

transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0, 1, 2$. Notice that $\vec{C}_{2n+1}\langle 2 \rangle$ is vertex-critical 4-dichromatic if the cardinality of L_i is equal to one for $i = 0, 1, 2$. Since $|L_i| = |Y_4| - 1 = \lfloor \frac{n}{3} \rfloor - 1$, we have that $|L_i| = 1$ if and only if $\lfloor \frac{n}{3} \rfloor = 2$. It occurs when $n = 6, 7$ or 8 . By Theorem 17 it is only possible for $n \geq 8$. Observe that $|L_2| \geq 2$ for $T = \vec{C}_{2n+1}\langle 2 \rangle$ if $n \geq 8$. Since this partition is maximal, T is not vertex-critical. Therefore, $\vec{C}_{2n+1}\langle 2 \rangle$ is vertex-critical if and only if $n = 5$. ■

4. THE DICHROMATIC NUMBER OF $\vec{C}_{2n+1}\langle k \rangle$ FOR $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$

We prove that $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$, for $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 7$.

Lemma 20. *If $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$, then $\vec{C}_{2n+1}\langle k \rangle$ contains a maximal transitive subtournament H .*

Proof. Let n and k be nonnegative integers and consider the interval $[0, n]$. Applying the Euclidean division algorithm to $n + 1$ and $2k - 1$, there exist unique $\alpha, r \in \mathbb{N}$ such that

$$n + 1 = \alpha(2k - 1) + r \quad \text{where } 0 \leq r < 2k - 1.$$

Consider the partition of the interval $[0, 2k - 2] = [0, k - 1] \cup [k, 2k - 2]$ and define

$$n + 1 = \begin{cases} \alpha(2k - 1) + s_1 & \text{if } s_1 \in [0, k - 1], \\ \alpha(2k - 1) + s_2 & \text{if } s_2 \in [k, 2k - 2]. \end{cases}$$

Observe that since $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we have that $s_1 \in [1, k - 1]$.

Let

$$W = \bigcup_{i=0}^{\alpha-1} [i(2k - 1), i(2k - 1) + (k - 1)].$$

We define the subtournament H of $\vec{C}_{2n+1}\langle k \rangle$ in the following way.

- (i) If $s_1 \in [1, k - 1]$, then $H = \langle W \cup [\alpha(2k - 1), n] \rangle$. Moreover, if $k = \frac{n+1}{2}$ and n is odd, then $H = \langle W \cup \{n, 2n + 1 - k\} \rangle$.
- (ii) $H = \langle W \cup [\alpha(2k - 1), \alpha(2k - 1) + k - 1] \rangle$ for every $s_2 \in [k, 2k - 2]$.

Note that H is a transitive subtournament by construction, since its vertex set does not contain induced \vec{C}_3 's. We prove that H is maximal by contradiction. Since $\vec{C}_{2n+1}\langle k \rangle$ is vertex-transitive, without loss of generality, choose the vertex 0. Observe that $N^+(0) = \{1, 2, \dots, k - 1, k + 1, \dots, n, 2n + 1 - k\}$ and

$$N^+(0) \setminus V(H) = (X_0 \cup \{2n + 1 - k\}) \setminus (V(H) \cup \{k\}).$$

For every vertex $x \in N^+(0) \setminus V(H)$ there exist $h_1, h_2 \in V(H)$ such that the vertex set $\{h_1, h_2, x\}$ induces a \vec{C}_3 (for instance, $x = k + 1, h_1 = 1$ and $h_2 = k - 1$), a contradiction. Therefore, H is maximal. ■

Lemma 21. *If $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $X_3 \setminus V(H)$ contains a maximal transitive subtournament J , where H is the subtournament defined in Lemma 20.*

Proof. The construction of J is similarly obtained as in the proof of Lemma 20 for H , but we have two ways of defining J .

Case 1. $\alpha = 1$.

- (i) If $s_1 \in [1, k - 1]$, then $J = [k, 2k - 2] \cup [3k - 2, 3k + s_1 - 2]$. Notice that if $k = \frac{n+1}{2}$ with n is odd if and only if $s_1 = 1$. Then $J = [k, 2k - 2] \cup \{3k - 2\}$ by the construction of H .
- (ii) If $s_2 \in [k, 2k - 2]$, then $J = [k, 2k - 2] \cup [3k - 1, 4k - 2]$.

Case 2. $\alpha > 1$. Let

$$U = \bigcup_{i=0}^{\alpha-1} [(n + 1) + i(2k - 1), (n + 1) + i(2k - 1) + (k - 1)].$$

- (i) If $s_1 \in [1, k - 1]$, then $J = \langle U \cup [(n + 1) + \alpha(2k - 1), 2n] \rangle$.
- (ii) $J = \langle U \cup [(n + 1) + \alpha(2k - 1), (n + 1) + \alpha(2k - 1) + k - 1] \rangle$ for every $s_2 \in [k, 2k - 2]$.

Notice that H is a maximal transitive subtournament in $\vec{C}_{2n+1}\langle k \rangle$ by Lemma 20. We claim that J is maximal in $V(\vec{C}_{2n+1}\langle k \rangle) \setminus V(H)$. If J was not maximal, we could add at least one vertex of $V(\vec{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J))$.

For Case 1, consider

$$\{n + k + 1, 3k - 3\} \subseteq V(\vec{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J)).$$

We have that $(k, n + k, n + k + 1, k) \cong \vec{C}_3$ or $(2k - 2, 3k - 3, 3k - 2, 2k - 2) \cong \vec{C}_3$. Therefore, J is maximal.

For Case 2, consider

$$k \in V(\vec{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J)).$$

We have that $(k, n + k, n + 2k + 1, k) \cong \vec{C}_3$. Hence, J is maximal. ■

Theorem 22. *If $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$ for $n \geq 7$.*

Proof. By Theorem 1, $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 3$. We prove that $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 4$. For a contradiction, suppose that $dc(\vec{C}_{2n+1}\langle k \rangle) = 3$. Thus, $\vec{C}_{2n+1}\langle k \rangle$ has a partition of its vertices consisting of three transitive subtournaments. By Lemmas 20 and 21, two maximal transitive disjoint subtournaments in $\vec{C}_{2n+1}\langle k \rangle$ are H and J . Hence, the remaining vertex set $X_3 \setminus (V(H) \cup V(J))$ induces the third transitive subtournament.

We consider three cases.

Case 1. $J = \langle [k, 2k - 2] \cup [3k - 2, 3k + s_1 - 2] \rangle$ obtained by Case 1(i) of Lemma 21. Therefore, $K = \langle [3k + s_1 - 1, 2n] \rangle$. Moreover, $|J| = k + s_1$ and $|H| = n - k + 2$. Since $k \leq \lceil \frac{n}{2} \rceil$, we have that $|K| = 2n + 1 - (|H| + |J|) = 2k - 3 > k$. In this case, K is induced by at least $k + 1$ consecutive vertices. Therefore, K cannot be a transitive subtournament by the definition of $\vec{C}_{2n+1}\langle k \rangle$. Hence, $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 4$. The following cases are necessary because the structure of K and L changes with different values of s_1 .

(i) If $s_1 = 1, 2$ or 3 , then

$$K = \langle [n + 1, 3k - 3] \cup [4k - 3, 2n] \rangle \text{ and } L = \langle [3k + s_1 - 1, 4k - 4] \rangle.$$

(ii) If $s_1 \in [4, k - 5]$, then

$$K = \langle [n + 1, 3k - 3] \cup [4k - 3, 5k - 4] \rangle \text{ and } L = \langle [3k + s_1 - 1, 4k - 4] \cup [5k - 3, 2n] \rangle.$$

(iii) If $s_1 = k - 4$, then

$$K = \langle [n + 1 = 3k - 4, 3k - 3] \cup [4k - 3, 5k - 4] \rangle \text{ and } L = \langle [4k - 5, 4k - 4] \cup [5k - 3, 2n] \rangle.$$

(iv) If $s_1 = k - 3$, then

$$K = \langle [n + 1 = 3k - 5, 3k - 3] \cup [4k - 3, 5k - 4] \rangle \text{ and } L = \langle \{4k - 4\} \cup [5k - 3, 2n] \rangle.$$

(v) If $s_1 = k - 1$ or $k - 2$, then

$$K = \langle [n + k + 1, n + 2k] \rangle \text{ and } L = \langle [n + 2k + 1, 2n] \rangle.$$

By construction and the definition of $\vec{C}_{2n+1}\langle k \rangle$, the subtournaments K and L are transitive. Observe that if $s_1 \in [1, k - 3]$, $4k - 4 \notin V(K)$ and $(4k - 4, 4k - 3, 3k - 3, 4k - 4) \cong \vec{C}_3$. Therefore, K is maximal in $\vec{C}_{2n+1}\langle k \rangle \setminus (H \cup J)$. If $s_1 = k - 1$ or $k - 2$, then $n + 2k + 1 \notin V(K)$ and $(n + k + 1, n + 2k, n + 2k + 1, n + k + 1) \cong \vec{C}_3$. Thus, K is maximal in $\vec{C}_{2n+1}\langle k \rangle \setminus (H \cup J)$. Hence, $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$.

Case 2. $J = [k, 2k - 2] \cup [3k - 1, 4k - 2]$ obtained by Case 1(ii) of Lemma 21. We have that $X_3 \setminus (V(H) \cup V(J)) = [4k - 1, 2n]$, but $2n - 4k + 2 > k$ implies that the subtournament induced by $X_3 \setminus (V(H) \cup V(J))$ has at least $k + 1$ consecutive vertices and a \vec{C}_3 is induced by $X_3 \setminus (V(H) \cup V(J))$. Therefore, $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 4$. The following cases show the partition of $\vec{C}_{2n+1}\langle k \rangle$ into transitive subtournaments.

(i) If $s_2 = k$, then

$$K = \langle [4k - 1, 5k - 2] \rangle \text{ and } L = \langle [5k - 1, 2n] \rangle.$$

(ii) If $s_2 = k + 1$, then

$$K = \langle [4k - 1, 5k - 2] \cup \{6k - 2 = 2n\} \rangle \text{ and } L = \langle [5k - 1, 6k - 3] \rangle.$$

(iii) If $s_2 = k + 2$, then

$$K = \langle [4k - 1, 5k - 2] \cup [6k - 2, 6k] \rangle \text{ and } L = \langle [5k - 1, 6k - 3] \rangle.$$

(iv) If $s_2 \in [k + 3, 2k - 2]$, then

(a) if $2n \leq 7k - 3$, we have that

$$K = \langle [4k - 1, 5k - 2] \cup [6k - 2, 2n] \rangle \text{ and } L = \langle [5k - 1, 6k - 3] \rangle,$$

(b) if $2n > 7k - 3$, then

$$K = \langle [4k - 1, 5k - 2] \cup [6k - 2, 7k - 3] \rangle \text{ and } L = \langle [5k - 1, 6k - 3] \cup [7k - 2, 2n] \rangle.$$

By construction and the definition of $\vec{C}_{2n+1}\langle k \rangle$, the subtournaments K and L are transitive. Observe that $5k - 1 \notin V(K)$ and $(4k - 1, 5k - 2, 5k - 1, 4k - 1) \cong \vec{C}_3$. Therefore, K is maximal in $\vec{C}_{2n+1}\langle k \rangle \setminus (H \cup J)$. Hence, $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$.

Case 3. J is obtained by Case 2(i) and (ii) of Lemma 21. If $dc(\vec{C}_{2n+1}\langle k \rangle) = 3$, then $X_3 \setminus (V(H) \cup V(J))$ induces a transitive subtournament, but the vertex set $\{k, 3k - 1, n + k + 1\} \subseteq X_3 \setminus (V(H) \cup V(J))$ induces a \vec{C}_3 . Hence, $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 4$. The partition of $\vec{C}_{2n+1}\langle k \rangle$ into transitive subtournaments is $H, J, K = \langle X_1 \cup \{k\} \setminus V(J) \rangle$ and

$$L = \langle X_3 \setminus (V(H) \cup V(J) \cup V(K)) \rangle.$$

By construction and the definition of $\vec{C}_{2n+1}\langle k \rangle$, the subtournaments K and L are maximal transitive. Observe that $k + 1 \notin V(K)$. Then the vertex set $\{k, k + 1, n + k + 1\}$ induces a \vec{C}_3 . Therefore, K is maximal in $\vec{C}_{2n+1}\langle k \rangle \setminus (H \cup J)$.

This proves that $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$. ■

The following example illustrates Theorem 22, Case 2(ii). The tournament $\vec{C}_{29}\langle 5 \rangle$ has the following partition into four transitive subtournaments

$$H = \langle [0, 4] \cup [9, 13] \rangle, K = \langle [5, 8] \cup [14, 18] \rangle, J = \langle [19, 23] \cup \{28\} \rangle \text{ and } L = \langle [24, 27] \rangle.$$

Observe that $H \cong TT_{10}$, $J \cong TT_9$, $K \cong TT_6$ and $L \cong TT_4$.

Theorem 23. *If $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $\vec{C}_{2n+1}\langle k \rangle$ is a vertex-critical 4-dichromatic circulant tournament if and only if*

- (i) $n = 7$ and $k \in \{3, 4\}$,
- (ii) $n = 9$ and $k = 4$,
- (iii) $n = 10$ and $k = 5$,
- (iv) $n = 13$ and $k = 6$.

Proof. By Theorem 22, $\vec{C}_{2n+1}\langle k \rangle$ is 4-dichromatic, where H, J, K and L are maximal transitive subtournaments. Note that by the partition of the vertices of $\vec{C}_{2n+1}\langle k \rangle$, the cases that need to be considered are when $\alpha = 1$, because it is when the order of L can be one. In this case, $\vec{C}_{2n+1}\langle k \rangle$ is a vertex-critical 4-dichromatic. We have two cases when $\alpha = 1$.

Case 1. $s_1 \in [1, k - 1]$.

- (i) If $s_1 \in \{1, 2, 3\}$, then by Theorem 22 Case 1(i), we have that $|L| = k - 3, k - 4, k - 5$, respectively. The tournament is vertex-critical if and only if $|L| = 1$ if and only if $k = 4$ and $n = 7$, $k = 5$ and $n = 10$, $k = 6$ and $n = 13$, respectively
- (ii) If $s_1 = k - 3$, then by the proof of Theorem 22 Case 1(iv), it is vertex-critical if and only if $|L| = 1$ if and only if $2n = 5k - 4$ and $n = 3k - 5$ if and only if $k = 6$ and $n = 13$.
- (iii) If $s_1 = k - 2$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = k - 2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
- (iv) If $s_1 = k - 1$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = n - 2k$. It is vertex-critical if and only if $|L| = 1$ if and only if $n = 2k + 1$ and $n = 3k - 3$ if and only if $k = 4$ and $n = 9$.

Case 2. $s_2 \in [k, 2k - 2]$.

- (i) If $s_2 = k$, then by the proof of Theorem 22 Case 2(i), we have that $|L| = k - 2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
- (ii) If $s_2 \in [k + 1, 2k - 2]$, then by the proof of Theorem 22 Case 2(ii)–(iv)(a), we have that $|L| = k - 2$, but it is not necessarily vertex-critical if $|L| = 1$, because the last vertices remain in K . When L is obtained by the proof of Theorem 22 Case 2(iv)(b), $|L|$ never is one. In any case, $\vec{C}_{2n+1}\langle k \rangle$ is not a vertex-critical 4-dichromatic circulant tournament. ■

5. THE DICHROMATIC NUMBER OF $\vec{C}_{2n+1}\langle k \rangle$ FOR $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n$.

In this part we prove that the tournaments $\vec{C}_{2n+1}\langle k \rangle$ are 4-dichromatic if $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$ for $n \geq 8$.

Lemma 24. *If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $\vec{C}_{2n+1}\langle k \rangle$ contains a maximal transitive subtournament of order k .*

Proof. Since $\vec{C}_{2n+1}\langle k \rangle$ is vertex-transitive, it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that $N^+(0) = \{1, 2, \dots, k-1, k+1, \dots, n, 2n+1-k\}$. We define $H = \langle [0, k-1] \rangle$. It is transitive by the definition of $\vec{C}_{2n+1}\langle k \rangle$. If H was not maximal, then we could add one vertex of $N^+(0) \setminus [0, k-1]$. Let $j \in [k+1, n]$. Without loss of generality, choose $j = k+1$. Thus, the set of vertices $\{1, t, k+1\}$ with $t \in [2, k-1]$ induces a \vec{C}_3 . The same occurs for the vertex $2n+1-k$. Observe that $(3, k-1, 2n+1-k, 3) \cong \vec{C}_3$, a contradiction. Therefore, H is maximal. ■

Lemma 25. *If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $\vec{C}_{2n+1}\langle k \rangle$ contains three maximal transitive subtournaments of k vertices.*

Proof. By Lemma 24, $\vec{C}_{2n+1}\langle k \rangle$ contains a maximal transitive subtournament H . Notice that $|N^+(0)| - |H| < k$. Consider the following subtournaments

$$J = \langle [k, 2k-1] \rangle \quad \text{and} \quad K = \langle [2k, 3k-1] \rangle.$$

Observe that J and K are isomorphic to H . Let $\varphi_1 : H \rightarrow J$ such that $\varphi_1(j) = j+k$ with $0 \leq j \leq k-1$, (φ_1 is bijective and it is clear that H is isomorphic to J). Analogously, $\varphi_2 : H \rightarrow K$ is an isomorphism between H and K . As in Lemma 24, we can prove that J and K are maximal transitive subtournaments. Then $\vec{C}_{2n+1}\langle k \rangle$ contains three maximal transitive subtournaments on k vertices. ■

Theorem 26. *If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$.*

Proof. First we prove that $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 4$. By Lemma 25, we have that $\vec{C}_{2n+1}\langle k \rangle$ contains three maximal transitive subtournaments of k vertices. Then $|\vec{C}_{2n+1}\langle k \rangle| - 3k > 0$. Thus, $V(\vec{C}_{2n+1}\langle k \rangle)$ cannot be partitioned into three transitive subtournaments. Then $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 4$. We verify that $dc(\vec{C}_{2n+1}\langle k \rangle) = 4$. By Lemma 25, we have that H, J and K are maximal transitive subtournaments of order k . The fourth transitive subtournament is $L = \langle [3k, 2n] \rangle$. Therefore, $\vec{C}_{2n+1}\langle k \rangle$ is 4-dichromatic. ■

Theorem 27. *If $\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $\vec{C}_{2n+1}\langle k \rangle$ is a vertex-critical 4-dichromatic circulant tournament if and only if $n \equiv 0 \pmod 3$.*

Proof. By Theorem 26, $\vec{C}_{2n+1}\langle k \rangle$ is 4-dichromatic. Observe that the order of H , J and K is k and $|L| = 2n - 3k + 1$. Notice that $\vec{C}_{2n+1}\langle k \rangle$ is vertex critical 4-dichromatic if the cardinality of L is equal to one, and it occurs if and only if $k = \frac{2}{3}n$ when $n \equiv 0 \pmod 3$. By Theorem 3, $\vec{C}_{2n+1}\langle \frac{2}{3}n \rangle$ with $n \equiv 0 \pmod 3$ is a vertex-critical circulant tournament 4-dichromatic. ■

Corollary 28 ([11]). *$\vec{C}_{6m+1}\langle 2m \rangle$ is a vertex-critical 4-dichromatic circulant tournament for $m \geq 2$.*

Theorem 29. *Let $n \geq 3$. Then $dc(\vec{C}_{2n+1}\langle k \rangle) = 3$ for $k = \lfloor \frac{2}{3}n \rfloor + 1, \dots, n$.*

Proof. Let $n \geq 3$. By Theorem 1, $dc(\vec{C}_{2n+1}\langle k \rangle) \geq 3$. Take the following partition of the vertices of $\vec{C}_{2n+1}\langle k \rangle$:

$$H = [0, k - 1], J = [k, 2k - 1] \text{ and } K = [2k, 2n].$$

Observe that H induces a TT_k because $N^+(i) = \{i + 1, i + 2, \dots, k + 1\}$ for $k \leq i \leq 2k - 1$, also J and K induce a TT_k and a $TT_{2n-2k+1}$, respectively. Then $dc(\vec{C}_{2n+1}\langle k \rangle) = 3$. ■

Theorem 30. *If $k = \lfloor \frac{2}{3}n \rfloor + 1, \dots, n$, $n \geq 3$. Then $\vec{C}_{2n+1}\langle k \rangle$ is a vertex-critical 3-dichromatic circulant tournament if and only if $n = k$.*

Proof. By Theorem 29, $\vec{C}_{2n+1}\langle k \rangle$ is 3-dichromatic and its partition into three maximal transitive subtournaments was

$$|H| = |J| = k \text{ and } |K| = 2n - 2k + 1.$$

Since $k = \lfloor \frac{2}{3}n \rfloor + 1, \dots, n$, we have that $k \geq 2n - 2k + 1$. Hence, $\vec{C}_{2n+1}\langle k \rangle$ is vertex-critical if and only if $2n - 2k + 1 = 1$, if and only if $n = k$. ■

Corollary 31 ([13], Theorem 2). *$\vec{C}_{2n+1}\langle n \rangle$ is a vertex-critical 3-dichromatic circulant tournament for $n \geq 3$.*

By Theorems 13, 19, 23, 27 and 30, we have the following.

Theorem 32. *Let $r \in \{2, 3, 4\}$, $\vec{C}_{2n+1}\langle k \rangle$ is vertex-critical r -dichromatic if and only if*

- (i) $r = 2$, $n = 1$ and $k = 1$;
- (ii) $r = 3$,
- (a) $n = 4$ and $k = 1$,

- (b) $n \geq 3$ and $k = n$;
- (iii) $r = 4$,
 - (a) $n = 5$ and $k = 2$,
 - (b) $n = 7$ and $k \in \{3, 4\}$,
 - (c) $n = 9$ and $k = 4$,
 - (d) $n = 10$ and $k = 5$,
 - (e) $n = 13$ and $k = 6$,
 - (f) $n = 3m$ and $k = 2m$ ($m \geq 2$).

REFERENCES

- [1] G. Araujo-Pardo and M. Olsen, *A conjecture of Neumann-Lara on infinite families of r -dichromatic circulant tournaments*, Discrete Math. **310** (2010) 489–492.
doi:10.1016/j.disc.2009.03.028
- [2] J. Bang-Jensen and G. Gutin, *Digraphs. Theory, Algorithms and Applications*, Second Edition (Springer Monographs in Mathematics, Springer-Verlag London, London, 2009).
- [3] P. Erdős, *Problems and results in number theory and graph theory*, Proceedings of the Ninth Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba, Winnipeg, Man., 1979), Congr. Numer. **XXVII** (1979) 3–21.
- [4] P. Erdős, J. Gimbel and D. Kratsch, *Some extremal results in cochromatic and dichromatic theory*, J. Graph Theory **15** (1991) 579–585.
doi:10.1002/jgt.3190150604
- [5] A. Harutyunyan, *Brooks-type results for coloring of digraphs*, PhD thesis supervised by B. Mohar (Simon Fraser University, 2011).
<http://www.math.univ-toulouse.fr/~aharutyu/thes-short.pdf>
- [6] H. Jacob and H. Meyniel, *Extensions of Turan's and Brooks theorem and new notions of stability and colouring in digraphs*, Ann. Discrete Math. **17** (1983) 365–370.
- [7] B. Llano and M. Olsen, *On a conjecture of Víctor Neumann-Lara*, Electron. Notes Discrete Math. **30** (2008) 207–212.
doi:10.1016/j.endm.2008.01.036
- [8] B. McKay, *Combinatorial Data*, published online.
<http://cs.anu.edu.au/~bdm/data>
- [9] V. Neumann-Lara, *The dichromatic number of a digraph*, J. Combin. Theory, Ser. B **33** (1982) 265–270.
doi:10.1016/0095-8956(82)90046-6
- [10] V. Neumann-Lara, *The 3 and 4-dichromatic tournaments of minimum order*, Discrete Math. **135** (1994) 233–243.
doi:10.1016/0012-365X(93)E0113-I

- [11] V. Neumann-Lara, *Vertex critical 4-dichromatic circulant tournaments*, Discrete Math. **170** (1997) 289–291.
doi:10.1016/S0012-365X(96)00128-8
- [12] V. Neumann-Lara, *Dichromatic number, circulant tournaments and Zykov sums of digraphs*, Discuss. Math. Graph Theory **20** (2000) 197–207.
doi:10.7151/dmgt.1119
- [13] V. Neumann-Lara and J. Urrutia, *Vertex critical r -dichromatic tournaments*, Discrete Math. **49** (1984) 83–87.
doi:10.1016/0012-365X(84)90154-7

Received 30 June 2016

Revised 7 April 2016

Accepted 7 April 2016