

ON q -POWER CYCLES IN CUBIC GRAPHS

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Abstract

In the context of a conjecture of Erdős and Gyárfás, we consider, for any $q \geq 2$, the existence of q -power cycles (i.e., with length a power of q) in cubic graphs. We exhibit constructions showing that, for every $q \geq 3$, there exist arbitrarily large cubic graphs with no q -power cycles. Concerning the remaining case $q = 2$ (which corresponds to the conjecture of Erdős and Gyárfás), we show that there exist arbitrarily large cubic graphs whose all 2-power cycles have length 4 only, or 8 only.

Keywords: cubic graphs, q -power cycles, Erdős-Gyárfás conjecture.

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1. INTRODUCTION

Throughout this note, given some $q \geq 2$, by a q -power cycle of some undirected simple graph G , we refer to a cycle of G whose length is a power of q . Our work is related to the following conjecture attributed to Erdős and Gyárfás (see [1]).

Conjecture 1 (Erdős and Gyárfás). *Every graph with minimum degree at least 3 has a 2-power cycle.*

Although Erdős and Gyárfás themselves suspected that Conjecture 1 should admit counterexamples, the question is still open in general. Conjecture 1 being seemingly complicated, only a few works have been dedicated to it. Markström, through a computer search, proved that any cubic counterexample to Conjecture 1 should have at least 30 vertices [4]. He also exhibited small cubic graphs whose all 2-power cycles have length 16. Conjecture 1 was nevertheless verified in some situations, such as for planar claw-free graphs (by Daniel and Shauger [2]), and

for $K_{1,m}$ -free graphs with minimum degree at least $m + 1$ or maximum degree at least $2m - 1$ (by Shauger [6]). More recently, the conjecture was also verified for 3-connected planar cubic graphs by Heckman and Krakovski [3]. More details on this subject may be found in [5].

Caro, as reported by West [7], raised, in the context of Conjecture 1, a more general question about whether every graph with minimum degree at least 3 has a cycle whose length is a non-trivial power of some natural number.

Conjecture 2 (Caro). *For every graph G with minimum degree at least 3, there is a natural number $q \geq 2$ such that G has q^p -cycles for some $p > 1$.*

Conjecture 1, if true, would basically answer positively to Conjecture 2 (with setting $q = 2$). We here consider a generalization of Conjecture 1, which is obtained by fixing q in Conjecture 2, and whose some particular cases will be answered in this note.

Question 3. *For any $q \geq 2$, does every large enough graph with minimum degree at least 3 have a q -power cycle?*

Although quite natural in the context of Conjecture 1 (the case $q = 2$ of Question 3 is exactly Conjecture 1), we did not find any progress on Question 3 in literature.

We here exhibit constructions yielding a negative answer to Question 3 for every $q \geq 3$. These constructions all provide graphs being cubic and planar. As our construction tools and techniques do not apply for the case $q = 2$ of Question 3, this confirms that this remaining case, Conjecture 1, is by far the hardest case.

2. TERMINOLOGY AND CONSTRUCTIONS

Most of our constructions are based on the following scheme. We start from an *internally cubic tree*, namely a tree whose all non-leaf nodes have degree exactly 3. Clearly an internally cubic tree may have arbitrarily large order. In order to get a cubic graph with no q -power cycles for some given q , the main idea now is to start from some internally cubic tree T , and make T cubic by attaching some gadgets to its leaves. So that we control the lengths of the cycles we create in this way, one way to proceed is to attach gadgets to the leaves of T in such a way that all leaves of T remain articulation vertices in the obtained cubic graph G . In doing so, the only cycles in G will be the cycles in the gadgets used to raise the degrees — so we just need to find such gadgets with no q -power cycles.

We will mainly use two kinds of degree-raising gadgets — see Figure 1 for an illustration of the described operations. The first kind of gadgets are called *edge-gadgets*. The main property of an edge-gadget G is that it is a subcubic graph with only two vertices t_1 and t_2 , called the *ends* of G , being of degree less

than 3. Furthermore, t_1 and t_2 must be of degree exactly 1 in G . Basically, an edge-gadget G with ends t_1 and t_2 can be used to replace an edge in some graph H : assuming uv is an edge of H , one may just delete uv , add a copy of G to H , identify u and t_1 , and identify v and t_2 in H . In doing so, note that the degrees of u and v are not affected by the operation. Edge-gadgets can also be used to raise by 2 the degree of any vertex w of H . To that end, one may just add a loop at w , and replace that loop by an edge-gadget, in the same fashion as described above.

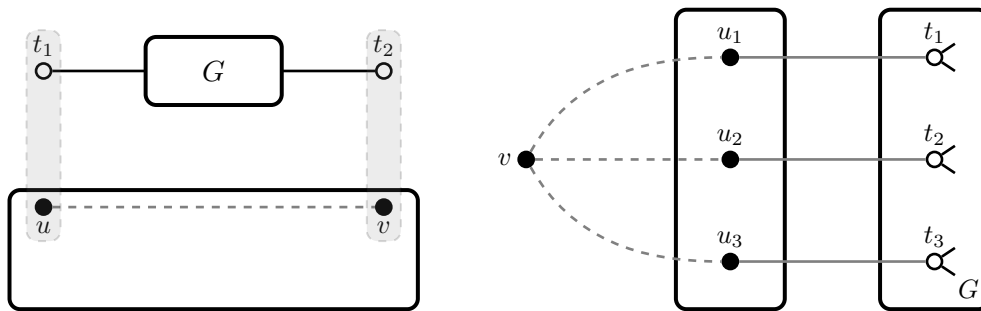


Figure 1. Replacing an edge by an edge-gadget (left), and a degree-3 vertex by a vertex-gadget (right).

The second kind of gadgets we will use are *vertex-gadgets*. A vertex-gadget G is a subcubic graph with only three vertices t_1, t_2 and t_3 , or *ends*, being of degree less than 3, with t_1, t_2 and t_3 being of degree exactly 2 in G . This time, a vertex-gadget G with ends t_1, t_2 and t_3 can be used to replace a vertex v with degree 3 in a graph H : assuming u_1, u_2 and u_3 are the three neighbours of v in H , one may just delete v from H , add a copy of G to H , and match the vertices among $\{t_1, t_2, t_3\}$ with those among $\{u_1, u_2, u_3\}$. Note that if H is cubic, then of course the obtained graph remains cubic after the modification.

3. CUBIC GRAPHS WITH NO q -POWER CYCLES FOR $q \geq 3$

We now exhibit constructions yielding, for any fixed $q \geq 3$, arbitrarily large cubic planar graphs with no q -power cycles. We start by considering the case $q \geq 6$, before considering every remaining case separately.

3.1. Case $q \geq 6$

The main gadget we use in this section is the edge-gadget G depicted in Figure 2, whose two ends are t_1 and t_2 , and whose number $k \geq 1$ of internal columns is parametrized. We note that, for every k , the different cycle lengths of G , as well as the path lengths from t_1 to t_2 , are easily deduced.

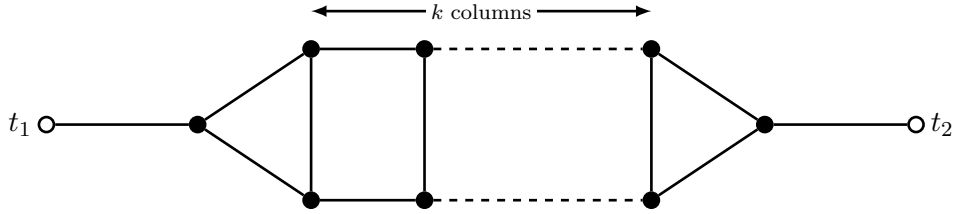


Figure 2. The edge-gadget G used for the case $q \geq 6$.

Observation 4. Assume $k \geq 1$ is fixed. All cycles of G have length among $\{3, 4, \dots, 2k + 2\}$.

Observation 5. Assume $k \geq 1$ is fixed. All $\{t_1, t_2\}$ -paths of G have length among $\{k + 3, k + 4, \dots, 2k + 3\}$.

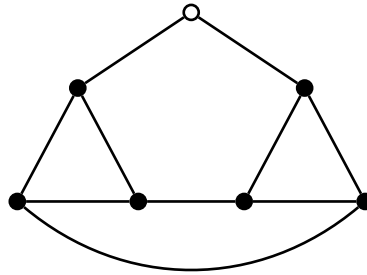


Figure 3. The gadget G' used for the case $q \geq 6$.

We will also make use of the gadget G' depicted in Figure 3, where we call the white vertex of G' its *root*. We note the following.

Observation 6. All cycles of G' have length among $\{3, 4, 5, 6, 7\}$.

We are now ready to construct a graph H with no q -power cycles for any fixed $q \geq 6$. Start from H being an internally cubic tree. Now consider every leaf v of T , add a new copy of G' to H , and identify v and the root of G' . Note that, so far, all cycles of H are the cycles in the copies of G' . To finish the construction, consider every gadget G' we attached to H , and replace every of its edges by a copy of G . Now each cycle of H is either a cycle of G or goes through some copies of G following some cycle of G' .

Observation 7. All cycles of H have length among

$$\{3, 4, \dots, 2k + 2\} \cup \{3k + 9, 3k + 10, \dots, 14k + 21\}.$$

Proof. As said above, a cycle C of H uses either (1) edges from a single copy of G , or (2) edges of several copies of G following some cycle of G' . In situation (1), from Observation 4 we know that the length of C is among $\{3, 4, \dots, 2k + 2\}$. Now, in situation (2), C goes through several G 's following some cycle in G' . Since the cycles in G' have length among $\{3, 4, 5, 6, 7\}$ according to Observation 6, and all $\{t_1, t_2\}$ -paths of G have length among $\{k + 3, k + 4, \dots, 2k + 3\}$ according to Observation 5, the smallest possible length for C is $3(k + 3) = 3k + 9$ while the biggest possible length for C is $7(2k + 3) = 14k + 21$. ■

Corollary 8. *If $q \in \{2k + 4, 2k + 5\}$, then H has no q -power cycle.*

Proof. According to Observation 7, all cycles of H have length among

$$L = \{3, 4, \dots, 2k + 2\} \cup \{3k + 9, 3k + 10, \dots, 14k + 21\}.$$

We remark first that $q \notin L$, so H has no q -cycle. Now note that $q^2 \in \{4k^2 + 16k + 16, 4k^2 + 20k + 25\}$, which is strictly greater than $14k + 21$ as long as $k \geq 1$. So, as long as $k \geq 1$, which is always verified, H has no q^p -cycle for any $p \geq 2$. So H has no q -power cycle. ■

The conclusion is then the following.

Theorem 9. *For every $q \geq 6$, there exist arbitrarily large planar cubic graphs having no q -power cycles.*

Proof. Consider the construction of H above. The original internally cubic tree can be taken arbitrarily large — so H has arbitrarily large order. To now make sure that H has no q -power cycle, just perform the construction with choosing k so that $q \in \{2k + 4, 2k + 5\}$, which is possible since $q \geq 6$. The conclusion follows from Corollary 8. ■

3.2. Case $q = 5$

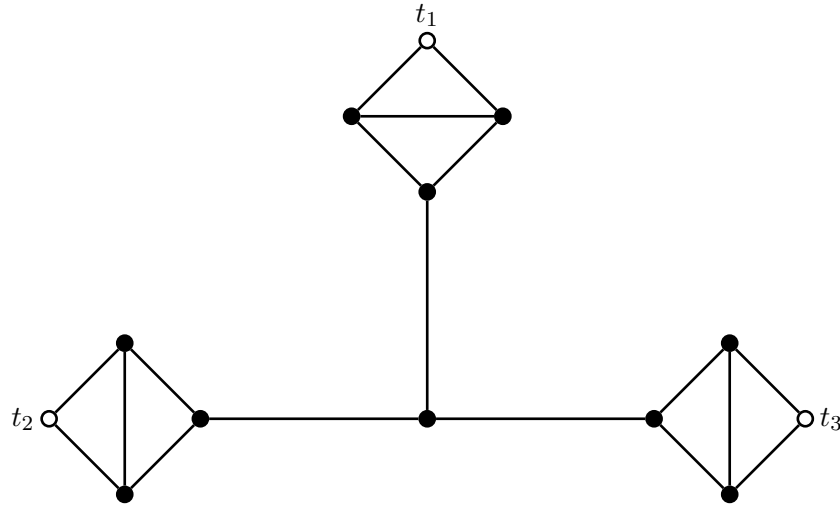
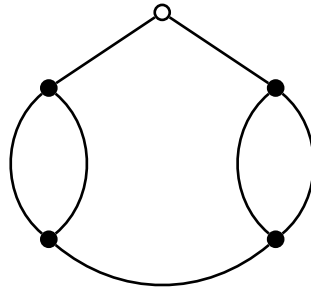
The main gadget to be used in this section is the vertex-gadget G depicted in Figure 4, whose ends are t_1 , t_2 and t_3 . Its cycle and path lengths properties of interest here are the following.

Observation 10. *All cycles of G have length among $\{3, 4\}$.*

Observation 11. *For every two distinct ends t_i and t_j of G , all $\{t_i, t_j\}$ -paths of G have length among $\{6, 7, 8\}$.*

Now consider the multigraph G' depicted in Figure 5, whose white vertex is the *root* of G' . Clearly, we have the following.

Observation 12. *All cycles of G' have length among $\{2, 5\}$.*

Figure 4. The vertex-gadget G used for the case $q = 5$.Figure 5. The gadget G' used for the case $q = 5$.

We now describe how to obtain a cubic graph H with no 5-power cycles. Start from H being an internally cubic tree. For every leaf v of H , identify v and the root vertex of a new copy of G' . Finally consider every vertex v of H (hence of degree 3) belonging to some copy of G' , and replace v by a copy of H (as described in Section 2). Due to the symmetric structure of G , note that we do not have to care about how the replacement is done (i.e., how the vertices are joined). Furthermore, the only cycles in H are cycles in some copies of G , or cycles going through copies of G following some cycle in G' . In particular, every such cycle cannot go twice through a single copy of G .

Observation 13. *All cycles of H have length among*

$$\{3, 4\} \cup \{14, 15, \dots, 18\} \cup \{35, 36, \dots, 45\}.$$

Proof. Let C be a cycle of H . If C is completely included in one of the copies of G , then C has length among $\{3, 4\}$ according to Observation 10. Now, if C goes through several copies of G , then, according to the arguments above, it goes only through 2 or 5 copies of G according to Observation 12. Recall that, when going through a copy of G , cycle C uses 6, 7 or 8 of its edges (Observation 11). In case C goes through exactly two copies of G , the length of C is among $\{14, 15, \dots, 18\}$. Now, if C goes through exactly five copies of G , then the length of C is among $\{35, 36, \dots, 45\}$. This concludes the proof. ■

Observation 13 now gives our conclusion, since the original internally cubic tree from which H is constructed can be arbitrarily large.

Theorem 14. *There exist arbitrarily large planar cubic graphs having no 5-power cycles.*

3.3. Case $q = 4$

We herein need the edge-gadget G depicted in Figure 6, whose ends are t_1 and t_2 . Its properties of interest are the following.

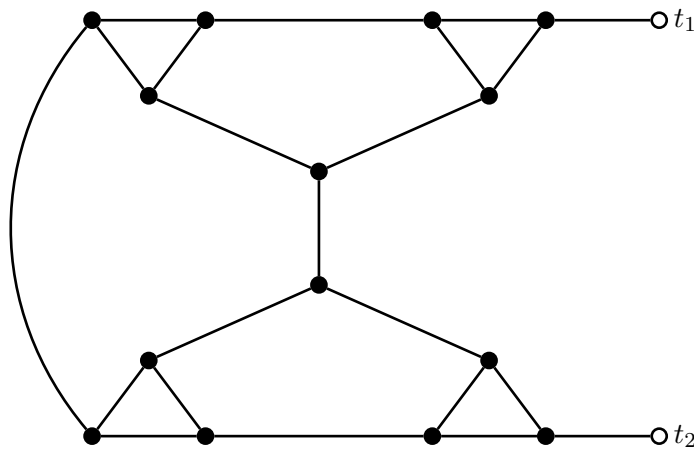


Figure 6. The edge-gadget G used for the case $q = 4$.

Observation 15. *All cycles of G have length among $\{3\} \cup \{5, 6, \dots, 14\}$.*

Observation 16. *All $\{t_1, t_2\}$ -paths of G have length among $\{7, 8, \dots, 15\}$.*

We now construct a cubic graph H with no 4-power cycles. Start from H being an internally cubic tree. Now consider every leaf v of H , and add a loop at v . Note that H is now cubic. Finally replace every such loop by a new copy of G . It should be clear that H remains cubic, and that the only cycles of H are located at its “leaves”. We make it more formal below.

Observation 17. *All cycles of H have length among*

$$\{3\} \cup \{5, 6, \dots, 15\}.$$

Proof. Consider a cycle C in H . It should be clear that C cannot include two vertices of H belonging to the original internally cubic tree from which H was constructed (because of its 1-connectivity). So C is either completely included in some copy of G , in which case C has length among $\{3\} \cup \{5, 6, \dots, 14\}$ (Observation 15), or is actually a path from t_1 to t_2 in G . In the latter, C has length among $\{7, 8, \dots, 15\}$ according to Observation 16. This concludes the proof. ■

Once again, since the original internally cubic tree (from which H is constructed) can be arbitrarily large, from Observation 17 we get the following.

Theorem 18. *There exist arbitrarily large planar cubic graphs having no 4-power cycles.*

3.4. Case $q = 3$

We will here use the edge-gadget G with ends t_1 and t_2 depicted in Figure 7. Its properties are the following.

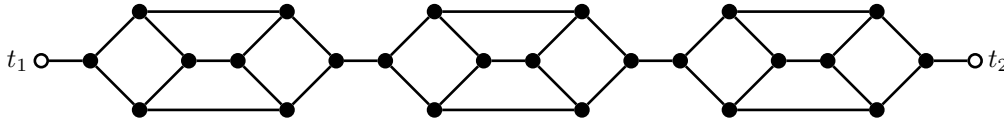


Figure 7. The edge-gadget G used for the case $q = 3$.

Observation 19. *All cycles of G have length among $\{4, 6, 8\}$.*

Observation 20. *All $\{t_1, t_2\}$ -paths of G have length among $\{13, 14, \dots, 25\}$.*

To obtain a cubic graph H with no 3-power cycles, just repeat the construction from Section 3.3 but with using the gadget G introduced in this section. We then get the following.

Observation 21. *All cycles of H have length among*

$$\{4, 6, 8\} \cup \{13, 14, \dots, 25\}.$$

Proof. The proof of Observation 17 can just be mimicked in the current context. The statement then follows from Observations 19 and 20. ■

For the same reasons as in Section 3.3, we can now state the following.

Theorem 22. *There exist arbitrarily large planar cubic graphs having no 3-power cycles.*

4. DISCUSSION

In this paper, we have considered a generalization of Conjecture 1, namely Question 3, and obtained some results about it. We have exhibited several constructions confirming that Conjecture 1, not surprisingly, is the hardest particular case of Question 3.

A few side results can be deduced from our results. Since the case $q = 4$ of Question 3 is wrong, we get that, in the context of Conjecture 1, there exists a family of arbitrarily large planar cubic graphs whose only 2-power cycles have length an odd power of 2. Similar statements may of course be formulated for any power of 2 greater than 4.

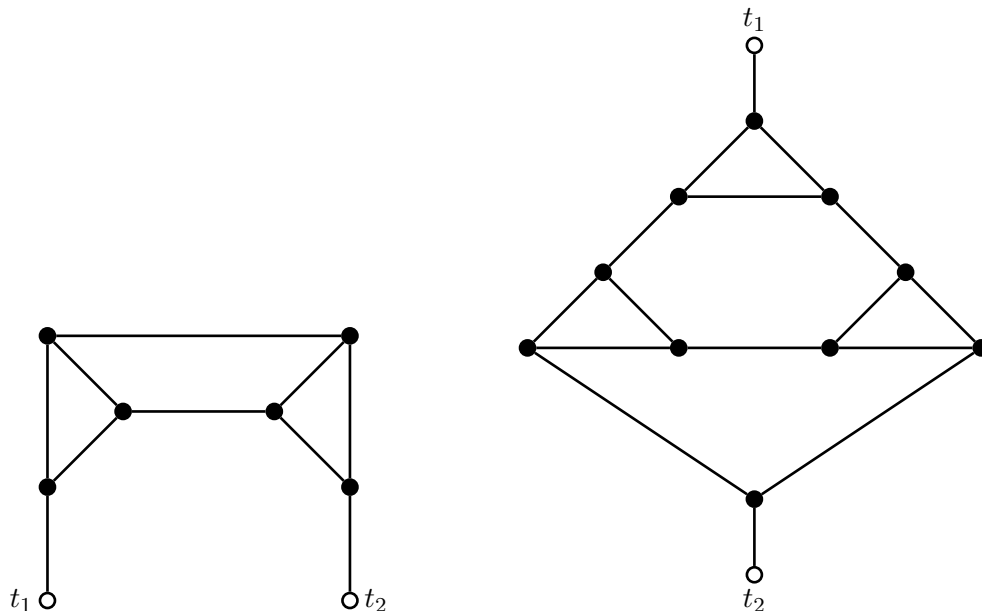


Figure 8. The edge-gadgets mentioned in the discussion section.

We also note that our constructions can be easily modified to get arbitrarily large planar cubic graphs whose 2-power cycles are all of the same length. To obtain such graphs whose all 2-power cycles are 4-cycles, we can just apply the construction in Section 3.3 but with using, as G , the first (left) edge-gadget depicted in Figure 8. Similarly, to obtain such graphs whose all 2-power cycles are 8-cycles, we can just repeat the same construction but with using, as G , the second (right) edge-gadget depicted in Figure 8. Towards Conjecture 1, it could be interesting to investigate whether, for any larger power of 2, say $2^k > 8$, there exist infinite families of cubic graphs whose all 2-power cycles have length 2^k .

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