

## DECOMPOSITION OF CERTAIN COMPLETE BIPARTITE GRAPHS INTO PRISMS

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### Abstract

Häggkvist [6] proved that every 3-regular bipartite graph of order  $2n$  with no component isomorphic to the Heawood graph decomposes the complete bipartite graph  $K_{6n,6n}$ . In [1] Cichacz and Froncek established a necessary and sufficient condition for the existence of a factorization of the complete bipartite graph  $K_{n,n}$  into generalized prisms of order  $2n$ . In [2] and [3] Cichacz, Froncek, and Kovar showed decompositions of  $K_{3n/2,3n/2}$  into generalized prisms of order  $2n$ . In this paper we prove that  $K_{6n/5,6n/5}$  is decomposable into prisms of order  $2n$  when  $n \equiv 0 \pmod{50}$ .

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### 1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. We use standard terminology and notation of graph theory.

Graph decompositions have been widely studied in many different settings. We say that a graph  $B$  has a  $G$ -decomposition if there are subgraphs  $G_1, G_2, \dots, G_s$  of  $B$ , all isomorphic to  $G$ , such that each edge of  $B$  belongs to exactly one  $G_i$ . If each  $G_i$  for  $i \in \{1, 2, \dots, s\}$  contains all vertices of  $B$  and none of them is isolated, then we say that  $B$  has a  $G$ -factorization.

Häggkvist [6] proved that every 3-regular bipartite graph of order  $2n$  with no component isomorphic to the Heawood graph decomposes the complete bipartite graph  $K_{6n,6n}$ .

Recall that a *prism*  $PR_n$  with  $2n$  vertices is the graph with two vertex disjoint cycles  $R_n^i = v_0^i, v_1^i, \dots, v_{n-1}^i, v_0^i$  for  $i \in \{1, 2\}$  of length  $n$  called *rims* and edges

$v_i^1 v_i^2$  for  $i \in \{0, 1, \dots, n-1\}$  called *spokes*. In [1], the notion was generalized and an  $(0, j)$ -prism (pronounced “oh-jay prism”) of order  $2n$  for  $n$  and  $j$  both even was defined as the graph with two vertex disjoint cycles  $R_n^i = v_0^i, v_1^i, \dots, v_{n-1}^i, v_0^i$  for  $i \in \{1, 2\}$  of length  $n$  called *rims* and edges  $v_1^1 v_1^2, v_3^1 v_3^2, v_5^1 v_5^2, \dots, v_{n-1}^1 v_{n-1}^2$  and  $v_0^1 v_j^2, v_2^1 v_{2+j}^2, v_4^1 v_{4+j}^2, \dots, v_{n-2}^1 v_{j-2}^2$  called *spokes of type 0* and *type j*, respectively (see Figure 1). It is easy to observe that an  $(0, j)$ -prism is a 3-regular graph and is isomorphic to an  $(0, -j)$ -prism,  $(j, 0)$ -prism and  $(-j, 0)$ -prism. We can therefore always assume that  $j \leq n/2$ . In our terminology the usual prism is an  $(0, 0)$ -prism.

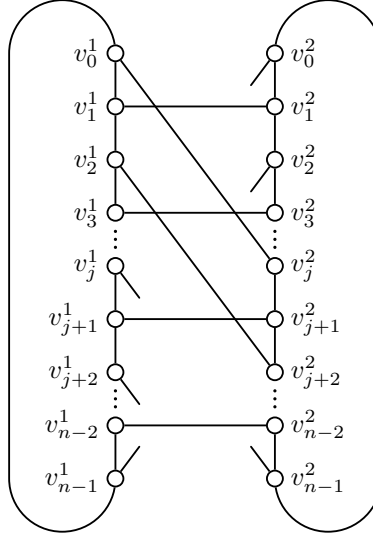


Figure 1.  $(0, j)$ -prism.

The problem of factorization of  $K_{n,n}$  into  $(0, j)$ -prisms was completely solved by Cichacz and Froncek in [1]. They proved that a factorization of  $K_{n,n}$  exists if and only if  $n \equiv 0 \pmod{6}$ .

Cichacz, Froncek, and Kovar in [2] found decompositions of  $K_{3n/2, 3n/2}$  into  $(0, j)$ -prisms with  $2n$  vertices for  $n \equiv 0 \pmod{8}$  and  $n/\gcd(n, j) \equiv 0 \pmod{2}$ .

The same authors in [3] found such decompositions for  $n \equiv j \equiv 0 \pmod{4}$ . One can observe that when  $n \equiv 0 \pmod{8}$  and  $n/\gcd(n, j) \equiv 1 \pmod{2}$ , then indeed  $j \equiv 0 \pmod{8}$ . It follows that  $n \equiv j \equiv 0 \pmod{4}$  and a decomposition of  $K_{3n/2, 3n/2}$  into  $(0, j)$ -prisms also exists.

Because the results were stated in [2] and [3] without the case where  $n \equiv 0 \pmod{8}$  and  $n/\gcd(n, j) \equiv 1 \pmod{2}$ , we state the complete result here for further reference. Notice that the necessary condition for  $K_{3n/2, 3n/2}$  to be decomposable into  $(0, j)$ -prisms with  $2n$  vertices is  $n \equiv 0 \pmod{4}$  and  $j \equiv 0 \pmod{2}$ .

For  $j \equiv 1 \pmod{2}$  no  $(0, j)$ -prism can exist, and when  $n \equiv 2 \pmod{4}$ , the number of edges of  $K_{3n/2, 3n/2}$  is odd while the  $(0, j)$ -prism has always an even number of edges.

**Theorem 1.** *Let  $n \equiv 0 \pmod{8}$  and  $j \equiv 0 \pmod{2}$  or  $n \equiv j \equiv 0 \pmod{4}$ . Then  $K_{3n/2, 3n/2}$  is decomposable into  $(0, j)$ -prisms with  $2n$  vertices.*

These results are in certain sense stronger than Häggkvist's theorem, because the order of the host graph is smaller. On the other hand, we notice that the obvious necessary conditions allow wider classes of complete bipartite graphs than just  $K_{\frac{3n}{2}, \frac{3n}{2}}$  for consideration. If we want to decompose  $K_{m,m}$  into  $(0, j)$ -prisms of order  $2n$ , then it follows that  $m^2 \equiv 0 \pmod{3n}$ , because the number of edges of the  $(0, j)$ -prism is  $3n$ . For example, when  $n = 50$ , the necessary conditions are met for  $K_{60,60}$  or  $K_{90,90}$ .

The remaining case of decomposition of  $K_{3n/2, 3n/2}$  satisfying the necessary conditions when  $n \equiv 0 \pmod{4}$  and  $j \equiv 2 \pmod{4}$  remains open. Also, the decompositions of other complete bipartite graphs  $K_{m,m}$ , satisfying the necessary conditions  $n \equiv j \equiv 0 \pmod{2}$  and  $m^2 \equiv 0 \pmod{3n}$  have not been investigated yet.

In this paper we take the first step in the latter direction and present a new class of complete bipartite graphs  $K_{m,m}$  that are decomposable into prisms (that is,  $(0, 0)$ -prisms).

## 2. TOOLS

We denote by  $G[H]$  the composition of graphs  $G$  and  $H$ , which is obtained by replacing every vertex of  $G$  by a copy of  $H$  and every edge of  $G$  by the complete bipartite graph  $K_{|H|, |H|}$ . We say that  $G[H]$  arose from  $G$  by *blowing up* by  $H$  and recall that  $tK_1$  is the graph consisting of  $t$  independent vertices. We will be repeatedly using graphs  $G[tK_1]$  in our constructions.

A labeling of a graph  $G$  is a function from  $V(G)$  into an Abelian group  $\Gamma$ . Rosa [7] introduced several types of graph labelings as tools for decompositions of complete graphs. By  $\mathbb{Z}_a$  we denote the cyclic group of order  $a$ . In this paper we will use a decomposition method based on certain vertex labeling, using an Abelian group  $\mathbb{Z}_a \times \mathbb{Z}_b$ .

**Definition 2.** Let  $G$  with vertex set  $V$  be a bipartite graph with  $k$  edges and partite sets  $V_0, V_1$  both of size  $k$  (that is,  $V = V_0 \cup V_1$ ,  $V_0 \cap V_1 = \emptyset$ , and  $|V_0| = |V_1| = k$ ), and  $a$  and  $b$  be positive integers such that  $ab = k$ . Let  $\lambda$  be an injection such that for  $i \in \{0, 1\}$  we have  $\lambda(V_i) \subseteq \{(u, v)_i : u \in \mathbb{Z}_a, v \in \mathbb{Z}_b\}$ . The dimension of an edge  $x_0y_1$  with  $x_0 \in V_0, y_1 \in V_1$  labeled  $\lambda(x_0) = (u, v)_0$  and  $\lambda(y_1) = (u', v')_1$ , respectively, is an element of  $\mathbb{Z}_a \times \mathbb{Z}_b$  defined as  $\dim(x_0y_1) = (u' - u, v' - v)$ .

The following result is based on a straightforward generalization of earlier results of Rosa [7] (see, e.g., [4]).

**Theorem 3.** *Let  $G$  be a bipartite graph with a labeling  $\lambda$  defined as above. If the set of dimensions of all edges of  $G$  is equal to the set of all elements of  $\mathbb{Z}_a \times \mathbb{Z}_b$ , then the complete bipartite graph  $K_{ab,ab}$  is  $G$ -decomposable.*

The following observation is quite straightforward yet very useful.

**Observation 4.** *Let  $G, H, J$  be graphs. If  $G$  is  $H$ -decomposable, and  $H$  is  $J$ -decomposable, then  $G$  is  $J$ -decomposable.*

In particular, we will be using the following special case.

**Observation 5.** *If  $K_{s,s}$  is  $G$ -decomposable, and  $G[tK_1]$  is  $H$ -decomposable, then  $K_{st,st} = K_{s,s}[tK_1]$  is  $H$ -decomposable.*

### 3. CONSTRUCTION

First we formally state the necessary conditions for the existence of a prism decomposition of  $K_{m,m}$ . We denote the prism with  $2n$  vertices by  $PR_n$ .

**Observation 6.** *If the complete bipartite graph  $K_{m,m}$  is  $PR_n$ -decomposable, then  $n$  is even,  $m^2 \equiv 0 \pmod{3n}$  and consequently,  $m \equiv 0 \pmod{6}$ .*

**Proof.** Obviously,  $n$  must be even, otherwise  $PR_n$  is not bipartite. Observe that  $PR_n$  is a cubic graph with  $3n$  edges. Hence, the number of edges of  $K_{m,m}$  must be divisible by  $3n$ . The condition  $m \equiv 0 \pmod{6}$  follows immediately. ■

A partial solution for the case when  $m \geq 3n/2$  was proved by Cichacz, Froncek, and Kovar [2].

**Theorem 7.** *The complete bipartite graph  $K_{m_1,m_2}$  is  $PR_n$ -decomposable if  $n$  is even,  $9n$  divides  $m_1m_2$ , both  $m_1, m_2 \geq 3n/2$  and  $6$  divides both  $m_1$  and  $m_2$ .*

For regular complete bipartite graphs, the result is as follows.

**Corollary 8.** *The complete bipartite graph  $K_{m,m}$  is  $PR_n$ -decomposable if  $n$  is even,  $m \geq 3n/2$  and  $m^2 \equiv 0 \pmod{9n}$ .*

A special case for  $m = 3n$  follows from a result by Frucht and Gallian [5]. They were investigating decompositions of complete graphs into prisms and proved that bipartite prisms have an  $\alpha$ -labeling, which can be easily transformed into the labeling described in Theorem 3. An  $\alpha$ -labeling of a graph  $G$  with the

vertex set  $V$  and edge set  $E$ , where  $|E| = q$  and  $|V| = 2q + 1$  is an injection  $\alpha : V \rightarrow \{0, 1, \dots, q\}$  satisfying  $\{|\alpha(x) - \alpha(y)| : xy \in E\} = \{1, 2, \dots, q\}$ .

We now show that for  $m \geq 3n/2$  the above results cover the whole spectrum of prism-decomposable regular complete bipartite graphs except for one case.

We write  $n = 3^t a$  and  $m = 3^s b$ , where both  $a, b \not\equiv 0 \pmod{3}$ . In other words,  $3^t$  and  $3^s$  are the highest powers of 3 contained in prime factorizations of  $n$  and  $m$ , respectively. It follows by Observation 6 that  $t \leq 2s - 1$ . Then by Corollary 8,  $K_{m,m}$  is decomposable into prisms of order  $2n$  for  $t < 2s - 1$ .

When  $t = 2s - 1$ , then

$$m^2 = 3^{2s} b^2 = 3 \cdot 3^t b^2,$$

and because  $b$  is not divisible by 3, we cannot write  $m^2$  as a multiple of  $9n$ . Hence, Corollary 8 does not apply to this case. We list this single remaining case separately as an open problem.

**Open Problem 9.** *Let  $n = 3^{2s-1}a$  and  $m = 3^s b$ , where both  $a, b \not\equiv 0 \pmod{3}$ . Is then the complete bipartite graph  $K_{m,m}$  decomposable into prisms of order  $2n$ ?*

Now we present a new family of prism-decomposable complete bipartite graphs  $K_{m,m}$  with  $m$  between  $n$  and  $3n/2$ . More specifically, we will have  $m = 6n/5$  for all  $n \equiv 0 \pmod{50}$ . We use Observation 5 and decompose first  $K_{m/3, m/3}$  into a union of  $C_{2m/3}$  and a matching  $(m/6)K_2$ . Then we show that the union blown-up by  $3K_1$  is prism-decomposable.

More formally, let  $s$  be even and  $H_{s,s}$  be a bipartite graph with vertices in partite sets  $V_0 = \{1_0, 2_0, \dots, s_0\}$  and  $V_1 = \{1_1, 2_1, \dots, s_1\}$  and edges  $i_0 i_1, i_0(i+1)_1$  for  $i = 1, 2, \dots, s$ , and  $i_0(i+2)_1$  for  $i = 1, 3, \dots, s-1$  (taken modulo  $s$  with 0 replaced by  $s$ ). Notice that  $H_{s,s}$  is a cycle of length  $2s$  with  $s/2$  chords of length 3 spaced evenly apart.

**Lemma 10.** *Let  $s \equiv 0 \pmod{20}$  and  $H_{s,s}$  be the graph defined above. Then  $H_{s,s}$  decomposes  $K_{s,s}$ .*

**Proof.** Denote the vertices in the partite sets of  $K_{s,s}$  by  $1_0, 2_0, \dots, s_0$  and  $1_1, 2_1, \dots, s_1$ , respectively. Let  $s = 20r$ . We want to construct  $8r = 2s/5$  copies of  $H_{s,s}$ , say  $H^1, H^2, \dots, H^{8r}$ .

The first two copies are constructed as follows. The cycle in  $H^1$  consists of edges  $i_0 i_1$  and  $i_0(i+1)_1$  for  $i = 1, 2, \dots, s$  and the chords are  $1_0 3_1, 3_0 5_1, \dots, (s-1)_0 1_1$ . Similarly, the cycle in  $H^2$  consists of edges  $i_0(i+3)_1$  and  $i_0(i+4)_1$ , and the chords are  $2_0 4_1, 4_0 6_1, \dots, s_0 2_1$ . If we now define the length of an edge  $i_0 j_1$  as  $j - i \pmod{s}$ , we observe that the first two copies use all edges of lengths from 0 to 4. For the next pair, we extend the lengths by five, and so on. To be precise, copy  $H^{2p-1}$  contains cycle edges  $i_0(5(p-1)+i)_1$  and  $i_0(5(p-1)+i+1)_1$  for  $i = 1, 2, \dots, s$

and chords  $1_0(5(p-1)+3)_1, 3_0(5(p-1)+5)_1, \dots, (s-1)_0(5(p-1)+1)_1$ , while  $H^{2p}$  contains cycle edges  $i_0(5(p-1)+i+3)_1$  and  $i_0(5(p-1)+i+4)_1$  for  $i = 1, 2, \dots, s$  and chords  $2_0(5(p-1)+4)_1, 4_0(5(p-1)+6)_1, \dots, s_0(5(p-1)+2)_1$ .

Hence, any two consecutive copies  $H^{2p-1}$  and  $H^{2p}$  consist of all edges of five consecutive lengths  $5(p-1), 5(p-1)+1, \dots, 5(p-1)+4$ , using each edge of  $K_{s,s}$  exactly once, and the collection  $H^1, H^2, \dots, H^{8r}$  forms an  $H_{s,s}$ -decomposition of  $K$ .  $\blacksquare$

Now we show that  $H_{s,s}[3K_1]$  is indeed  $PR_{5s/2}$ -decomposable.

**Lemma 11.** *Let  $s \equiv 0 \pmod{20}$ . Then the graph  $H_{s,s}[3K_1]$  described above is  $PR_{5s/2}$ -decomposable.*

**Proof.** We first split  $H_{s,s}$  into  $s/4$  mutually isomorphic segments induced by vertices  $i_0, (i+1)_0, \dots, (i+4)_0, (i+1)_1, \dots, (i+4)_1$  for  $i = 1, 5, 9, \dots, s-3$ . This can be done because  $s \equiv 0 \pmod{20}$  and each vertex  $1_0, 5_0, 9_0, \dots, (s-3)_0$  appears in two consecutive segments. We denote the segments (that is, induced subgraphs of  $H_{s,s}$ ) by  $H_{s,s}^i$ .

We need to show that the graph  $H_{s,s}^i[3K_1]$  arising from ten edges of  $H_{s,s}^i$  is decomposable into a thirty-edge segment of the prism.

For simplicity, rather than considering a general subgraph  $H_{s,s}^i$ , we look at the segment  $H_{s,s}^1$  induced by vertices  $1_0, 2_0, \dots, 5_0, 2_1, \dots, 5_1$  with edges  $1_02_1, 1_03_1, 2_03_1, 2_02_1, 3_03_1, 3_04_1, 3_05_1, 4_04_1, 4_05_1, 5_05_1$ . We want to blow it up into  $H_{s,s}^1[3K_1]$  so that every edge would blow up into three edges of different dimensions, and the graph would be a segment of the prism. We blow up each vertex  $i_j$  into the triple  $(i, 0)_j, (i, 1)_j, (i, 2)_j$  and show the correspondence between them and the prism vertices (using our previous prism notation) in Table 1. The first and last rows list the vertices of the first and second rim, respectively. The second row gives the corresponding labels of the first rim vertices in the notation used for  $H_{s,s}^1[3K_1]$ , the third row those of the second rim vertices.

$v_{n-1}^1$	$v_0^1$	$v_1^1$	$v_2^1$	$v_3^1$	$v_4^1$	$v_5^1$	$v_6^1$	$v_7^1$	$v_8^1$	$v_9^1$	$v_{10}^1$
$(1, 2)_0$	$(3, 0)_1$	$(1, 1)_0$	$(2, 1)_1$	$(2, 1)_0$	$(3, 1)_1$	$(3, 0)_0$	$(4, 2)_1$	$(4, 0)_0$	$(5, 2)_1$	$(5, 2)_0$	$(7, 0)_1$
$(1, 1)_1$	$(1, 0)_0$	$(2, 2)_1$	$(2, 0)_0$	$(3, 2)_1$	$(3, 2)_0$	$(5, 0)_1$	$(3, 1)_0$	$(4, 1)_1$	$(4, 1)_0$	$(5, 1)_1$	$(5, 0)_0$
$v_{n-1}^2$	$v_0^2$	$v_1^2$	$v_2^2$	$v_3^2$	$v_4^2$	$v_5^2$	$v_6^2$	$v_7^2$	$v_8^2$	$v_9^2$	$v_{10}^2$

Table 1. Prism segment vertex labels.

Now we view the original vertex names of type  $(i, t)_j$  as labels of the corresponding vertices  $v_k^l$  and inspect the edge dimensions computed according to Definition 2. We can check that for every pair of triples  $(i, 0)_0, (i, 1)_0, (i, 2)_0$  and

$(j, 0)_1, (j, 1)_1, (j, 2)_1$  the three edges between them have dimensions  $(j - i, 0), (j - i, 1)$  and  $(j - i, 2)$ .

For convenience, we list the edge dimensions below. The first row is the edge in  $H_{s,s}$  and the following rows list the edges arising from it in  $H_{s,s}[3K_0]$  along with their dimensions. Because all labels are single digit, to fit the data in two tables we write  $ij_k$  rather than  $(i, j)_k$ . The edge dimensions are still written as  $(i, j)$ .

$1_02_1$		$1_03_1$		$2_02_1$		$2_03_1$		$3_03_1$	
$10_022_1$	(1, 2)	$10_030_1$	(2, 0)	$20_021_1$	(0, 1)	$20_032_1$	(1, 2)	$30_031_1$	(0, 1)
$11_021_1$	(1, 0)	$11_030_1$	(2, 2)	$20_022_1$	(0, 2)	$21_031_1$	(1, 0)	$32_031_1$	(0, 2)
$11_022_1$	(1, 1)	$12_030_1$	(2, 1)	$21_021_1$	(0, 0)	$21_032_1$	(1, 1)	$32_032_1$	(0, 0)

$3_04_1$		$3_05_1$		$4_04_1$		$4_05_1$		$5_05_1$	
$30_042_1$	(1, 2)	$30_050_1$	(2, 0)	$40_041_1$	(0, 1)	$40_052_1$	(1, 2)	$50_051_1$	(0, 1)
$31_041_1$	(1, 0)	$31_050_1$	(2, 2)	$40_042_1$	(0, 2)	$41_051_1$	(1, 0)	$52_051_1$	(0, 2)
$31_042_1$	(1, 1)	$32_050_1$	(2, 1)	$41_041_1$	(0, 0)	$41_052_1$	(1, 1)	$52_052_1$	(0, 0)

Table 2. Prism segment vertex labels.

We denote the copy of  $PR_{5s/2}$  whose thirty-edge segment is described in Table 2 by  $G^0$ . We need to show now that there are two more copies,  $G^1$  and  $G^2$ , which along with  $G^0$  form the decomposition. Each edge  $a_0c_1$  of  $H_{s,s}$  is blown up into a copy of  $K_{3,3}$  in  $H_{s,s}[3K_0]$ . We produce the copy  $G^r$  by replacing every edge  $(a, b)_0(c, d)_1$  of dimension  $(c - a, d - b)$  by an edge  $(a, b + r)_0(c, d + r)_1$ . Clearly, the dimension of this edge is also  $(c - a, d - b)$ . It should be obvious that the graphs  $G^0, G^1$  and  $G^2$  are all mutually isomorphic.

Therefore, we need to show that each edge in every  $K_{3,3}$  arising from a particular edge  $a_0c_1$  is used in exactly one of the graphs  $G^0, G^1, G^2$ . Remember that the subtraction of the first entries is performed modulo  $s$  while the second entries are subtracted modulo 3. Obviously, the new edge has the same dimension  $(c - a, d - b)$  but different end vertices. More specifically, for a given edge  $(a, b)_0(c, d)_1$  of  $G^0$  we obtain  $(a, b + 1)_0(c, d + 1)_1$  in  $G^1$  and  $(a, b + 2)_0(c, d + 2)_1$  in  $G^2$ . Thus we have three independent edges of the same dimension. Because each  $K_{3,3}$  contains precisely three edges of the same dimension, the proof is now complete. ■

One can see that to prove Lemma 10, the assumption  $s \equiv 0 \pmod{20}$  could be relaxed to  $s \equiv 0 \pmod{10}$ . However, if we used the same approach in Lemma 11 for the case when  $n \equiv 10 \pmod{20}$ , we would obtain a Möbius ladder rather than a prism.

Our main result now follows immediately from Lemmas 10, 11, and Observation 5.

**Theorem 12.** *Let  $n \equiv 0 \pmod{50}$ . Then the complete bipartite graph  $K_{6n/5, 6n/5}$  is  $PR_n$ -decomposable.*

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### REFERENCES

- [1] S. Cichacz and D. Fronček, *Factorization of  $K_{n,n}$  into  $(0, j)$ -prisms*, Inform. Process. Lett. **109** (2009) 932–934.  
doi:10.1016/j.ipl.2009.04.024
- [2] S. Cichacz, D. Fronček and P. Kovář, *Note on decomposition of  $K_{n,n}$  into  $(0, j)$ -prisms*, Lect. Notes in Comput. Sci. **5874** (2009) 125–133.  
doi:10.1007/978-3-642-10217-2\_15
- [3] S. Cichacz, D. Fronček and P. Kovář, *Decomposition of complete bipartite graphs into generalized prisms*, European J. Combin. **34** (2013) 104–110.  
doi:10.1016/j.ejc.2012.07.018
- [4] D. Fronček, *Product rosy labeling*, Discuss. Math. Graph Theory **28** (2008) 431–440.  
doi:10.7151/dmgt.1417
- [5] R.W. Frucht and J.A. Gallian, *Labelling prisms*, Ars Combin. **26** (1988) 69–82.
- [6] R. Häggkvist, *Decompositions of complete bipartite graphs*, Surveys in Combinatorics, 1989 (Norwich, 1989), 115–147, London Math. Soc. Lecture Note Ser. **141** (Cambridge Univ. Press, Cambridge, 1989).
- [7] A. Rosa, *On certain valuations of the vertices of a graph*, in: Theory of Graphs, Internat. Symposium, Rome, July 1966 (Gordon and Breach, N.Y. and Dunod, Paris, 1967) 349–355.

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