

A NOTE ON NEIGHBOR EXPANDED SUM DISTINGUISHING INDEX¹

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Abstract

A total k -coloring of a graph G is a coloring of vertices and edges of G using colors of the set $[k] = \{1, \dots, k\}$. These colors can be used to distinguish the vertices of G . There are many possibilities of such a distinction. In this paper, we consider the sum of colors on incident edges and adjacent vertices.

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1. INTRODUCTION AND TERMINOLOGY

Let $G = (V, E)$ be a finite, undirected simple graph with vertex set V and edge set E .

Karoński *et al.* [4] introduced and investigated a coloring of the edges of a graph with positive integers so that adjacent vertices have different sums of incident edge colors. More precisely, let $c : E \rightarrow [k] = \{1, 2, \dots, k\}$ be an edge

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coloring of G (such a coloring is also called an edge k -coloring of G). For $x \in V$, we define

$$\sigma(x) = \sum_{e \ni x} c(e),$$

where the expression $e \ni x$ means that e contains x . An edge k -coloring c of G is called *neighbor sum distinguishing* if $\sigma(x) \neq \sigma(y)$ whenever $xy \in E$. In other words, the vertex coloring σ induced by c in the above must be proper. The minimum integer k for which there is a neighbor sum distinguishing coloring of a graph G will be denoted by $\text{gndi}_\Sigma(G)$.

In [4] Karoński *et al.* posed the following elegant problem, known as the 1-2-3 Conjecture.

Conjecture 1. *Let G be a connected graph, $G \neq K_2$. Then $\text{gndi}_\Sigma(G) \leq 3$.*

Thus far it is known that $\text{gndi}_\Sigma(G) \leq 5$ for any graph G without a connected component isomorphic to K_2 (see [3]).

In [5] the following problem related to the 1-2-3 Conjecture was introduced. Let $c : E \cup V \rightarrow \{1, 2, \dots, k\}$ be a total k -coloring of a graph G . For every vertex x , we denote by

$$t(x) := c(x) + \sum_{y \in N(x)} c(xy) = c(x) + \sigma(x),$$

where $N(x) = \{y \in V \mid xy \in E\}$ denotes an open neighborhood of x . Thus, $t(x)$ is the sum of edge colors of incident edges to x and the color of x . We say that c is a *total neighbor sum distinguishing* coloring of G if $t(x) \neq t(y)$ for all adjacent vertices x, y in G .

Similarly as above, the minimum value of k for which there exists a total neighbor sum distinguishing coloring of a graph G will be denoted by $\text{tgndi}_\Sigma(G)$.

In [5] Przybyło and Woźniak posed the following problem, known as the 1-2 Conjecture.

Conjecture 2. *Let G be a connected graph. Then $\text{tgndi}_\Sigma(G) \leq 2$.*

Thus far it is known that for every graph G , $\text{tgndi}_\Sigma(G) \leq 3$ (see [2]).

However, in the case of total coloring of G , there are also other possibilities to define the *palette* of colors i.e., the elements which we take into account. In this paper, for $x \in V$, we define

$$w(x) = \sum_{e \ni x} c(e) + \sum_{y \in N(x)} c(y),$$

where c is a total k -coloring of G . The value $w(x)$ will be called an *expanded sum at x* . A total k -coloring c of G is called *neighbor expanded sum distinguishing* (*NESD* for short) if

$$w(x) \neq w(y)$$

whenever $xy \in E$. In other words, the vertex coloring w induced by c in the above must be proper. The corresponding invariant, i.e., the minimum value of k for which such an NESD total k -coloring of G exists, is called the *neighbor expanded sum distinguishing index of G* or simply *expanded index of G* and denoted by $\text{egndi}_\Sigma(G)$.

We state the following conjecture.

Conjecture 3. *For every graph G , $\text{egndi}_\Sigma(G) \leq 2$.*

Remark 4. Another possibility would be to distinguish vertices by considering *full sums* defined for a vertex x by

$$\phi(x) = c(x) + \sum_{e \ni x} c(e) + \sum_{y \in N(x)} c(y),$$

where c is a total coloring of G . The corresponding parameter is denoted by $\text{fgndi}_\Sigma(G)$. The main reason why we consider expanded sums and not full sums is that the parameter $\text{egndi}_\Sigma(G)$ is well defined for each graph G , while the parameter $\text{fgndi}_\Sigma(G)$ does not exist for graphs containing K_2 as a component. Observe by the way that $\text{fgndi}_\Sigma(K_3) = 3$ while $\text{egndi}_\Sigma(K_3) = 2$. Thus, in general, we need three colors in order to distinguish adjacent vertices in such a way while, if the above conjecture is true, in the case of expanded sum two colors are sufficient. Therefore, in a sense, the parameter $\text{egndi}_\Sigma(G)$ is closer to $\text{tgndi}_\Sigma(G)$ than $\text{fgndi}_\Sigma(G)$.

Remark 5. As we continue to deal with only one parameter, namely $\text{egndi}_\Sigma(G)$, later on, we use the shorter notation, simply putting $\eta(G) := \text{egndi}_\Sigma(G)$.

Let G and G' be isomorphic graphs and c and c' be two total colorings of G and G' , respectively. We say that an isomorphism $\phi : V(G) \mapsto V(G')$ of G and G' is a *total isomorphism*, respect to c and c' , if $c'(\phi(x)) = c(x)$ for any $x \in V(G)$ and $c'(\phi(xy)) = c(xy)$ for any $xy \in E$.

If $a < b$, where a, b are natural numbers, then by $[a, b]$ we mean the integer interval of ends a and b , i.e., the set $\{a, a + 1, \dots, b\}$. Remind that $[n] = [1, n]$. We use Bondy and Murty's book [1] for terminologies and notations not defined here.

2. PATHS, CYCLES AND COMPLETE GRAPHS

The proof of the following proposition is left to the reader.

Proposition 6. *If P_m is the path of order $m \geq 2$, then $\eta(P_m) = 2$, if $m \neq 3$ and $\eta(P_3) = 1$.*

Proposition 7. For $m \geq 3$, $\eta(C_m) = 2$.

Proof. Let $C_m = x_1, \dots, x_m, x_1$ be the cycle of order $m \geq 4$. Put

$$c(x_{2i-1}) = 1, \text{ if } 1 \leq 2i - 1 \leq m;$$

$$c(x_{2i}) = 2, \text{ if } 2 \leq 2i \leq m;$$

$$c(x_i x_{i+1}) = 1, \text{ if } i = 1, \dots, m \text{ and } m \text{ even (indices of a cycle } C_m \text{ are taken modulo } m);$$

$$c(x_i x_{i+1}) = 1, \text{ if } i = 1, \dots, m, i \neq m - 1 \text{ and } m \text{ odd, } c(x_{m-1} x_m) = 2.$$

It can be easily seen that the above function is a neighbor expanded sum distinguishing total 2-coloring of C_m . For $m = 3$, the result follows from the next theorem. \blacksquare

Theorem 8. For every $n \geq 2$, $\eta(K_n) = 2$.

Proof. Denote by x_1, \dots, x_n the vertices of the complete graph K_n and let c_n be a total k -coloring of K_n . Let $w(c_n) = \{a \in \mathbb{N} \mid \text{there is an } i \in [n] \text{ such that } w(x_i) = a\}$ and let $f(c_n) = \sum_{i=1}^n c_n(x_i)$.

We claim that for every $n \geq 2$ there is an NESD total 2-coloring c_n of K_n such that

$$(1) \quad w(c_n) = \begin{cases} \left[\frac{5n-5}{2}, \frac{7n-7}{2} \right], & \text{if } n \text{ is odd,} \\ \left[\frac{5n-6}{2}, \frac{7n-8}{2} \right], & \text{if } n \text{ is even,} \end{cases}$$

and

$$(2) \quad f(c_n) = \begin{cases} \frac{3n-1}{2}, & \text{if } n \text{ is odd,} \\ \frac{3n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Note that for each $n \geq 2$ the set $w(c_n)$ contains exactly n different values, so if there exists a coloring c_n satisfying (1), then it is NESD.

Now, let c_2 be the total coloring of K_2 defined as follows: $c_2(x_1) = 1$, $c_2(x_2) = 2$ and $c_2(x_1 x_2) = 1$. Obviously, c_2 is an NESD total 2-coloring, where $w(c_2) = [2, 3] = \left[\frac{5n-6}{2}, \frac{7n-8}{2} \right]$ and $f(c_2) = 3 = \frac{3n}{2}$. The coloring c_n will be defined recursively as follows.

Suppose that there exists a total coloring c_{n-1} verifying the conditions (1) and (2) and color the subgraph of K_n induced by the set $\{x_1, x_2, \dots, x_{n-1}\}$ using the coloring c_{n-1} . If n is odd use 1 to color the vertex x_n , and 2 to color all $n-1$ edges incident to x_n . For n even use 2 to color x_n and 1 for all edges incident to x_n . We will denote by c_n the total coloring of K_n obtained in this way.

Observe that in the coloring c_n the weights $w(x_i)$ for $i = 1, \dots, n-1$ increase by 3 (with respect to the weights for c_{n-1}), and $w(x_n)$ is equal to $f(c_{n-1}) + 2(n-1)$ for n odd and $f(c_{n-1}) + (n-1)$ for n even. So, if n is odd, then

$$w(c_n) = \left[\frac{5(n-1)-6}{2} + 3, \frac{7(n-1)-8}{2} + 3 \right] \cup \{f(c_{n-1}) + 2(n-1)\}$$

$$= \left[\frac{5n-5}{2}, \frac{7n-9}{2} \right] \cup \left\{ \frac{3(n-1)}{2} + 2n - 2 \right\} = \left[\frac{5n-5}{2}, \frac{7n-7}{2} \right],$$

and

$$f(c_n) = \frac{3(n-1)}{2} + 1 = \frac{3n-1}{2}.$$

If n is even, then we have

$$\begin{aligned} w(c_n) &= \left[\frac{5(n-1)-5}{2} + 3, \frac{7(n-1)-7}{2} + 3 \right] \cup \{f(c_{n-1}) + (n-1)\} \\ &= \left[\frac{5n-4}{2}, \frac{7n-8}{2} \right] \cup \left\{ \frac{3(n-1)-1}{2} + n - 1 \right\} = \left[\frac{5n-6}{2}, \frac{7n-8}{2} \right] \end{aligned}$$

and

$$f(c_n) = \frac{3(n-1)-1}{2} + 2 = \frac{3n}{2}.$$

It follows that c_n is an NESD total 2-coloring of K_n . ■

Corollary 9. *There exist exactly two (up to total isomorphism) NESD total 2-colorings of K_n .*

Proof. We will use the notation of the proof of Theorem 8. Consider another NESD total 2-coloring c'_n of K_n defined recursively as follows. For $n = 2$ let $c'_2(x_1) = 1$, $c'_2(x_2) = 2$ and $c'_2(x_1x_2) = 2$. Suppose the coloring c'_{n-1} is given. Now color the subgraph of K_n induced by the set $\{x_1, x_2, \dots, x_{n-1}\}$ using the coloring c'_{n-1} . If n is odd use 2 to color the vertex x_n , and 1 to color all $n-1$ edges incident to x_n . A vertex x which is colored by 2 and such that its incident edges are colored by 1 will be called the *edge monochromatic vertex of type 2*.

For n even use 1 to color x_n and 2 for all edges incident to this vertex. Such a vertex is called the *edge monochromatic vertex of type 1*.

We denote the total coloring of K_n obtained in this way by c'_n . Note that, in each coloring c_n or c'_n of K_n there is exactly one edge monochromatic vertex of type 1 or 2, and if the coloring c_n contains one edge monochromatic vertex of type 1, then c'_n has one edge monochromatic vertex of type 2 and viceversa. Applying the method described in the proof of Theorem 8 one can easily show that c'_n is an NESD total 2-coloring of K_n .

Let l_n be another NESD total 2-coloring of K_n , $n \geq 2$. Let $d^n(x_i) = \sum_{e \ni x_i} l_n(e)$, $i = 1, \dots, n$.

Put $m = \min_i d^n(x_i)$, $M = \max_i d^n(x_i)$. We claim that $m = n-1$ or $M = 2n-2$ and there is exactly one edge monochromatic vertex of type 1 or 2 (so exactly one of these two equalities is true). Suppose for the sake of contradiction that $d^n(x_i) \geq n$ and $d^n(x_i) \leq 2n-3$ for all i . Without loss of generality, we may

assume that $d^n(x_1) = m \geq n$ and $d^n(x_2) = M \leq 2n - 3$. Therefore, the expanded sum associated to l_n , say w , satisfies

$$w(x_1) = d^n(x_1) + b \geq n + b,$$

where $b = \sum_{i \neq 1} l_n(x_i)$.

Moreover,

$$w(x_2) \leq d^n(x_2) + (b + 1) \leq b + 2n - 2,$$

so $w(x_2) - w(x_1) \leq n - 2$ and we cannot distinguish all the vertices of K_n by expanded sums $w(x_i)$, a contradiction.

Thus, we may assume that (for example) $d^n(x_1) = n - 1$, i.e., all edges incident to x_1 are painted using the color 1. It follows that $M \leq 2n - 3$ and $w(x_2) - w(x_1) \leq n - 1$. Therefore, $w(x_2) = b + 2n - 2 = \sum_{i \neq 2} l_n(x_i) + d^n(x_2) = \sum_{i \neq 1} l_n(x_i) + l_n(x_1) - l_n(x_2) + d^n(x_2) = b + l_n(x_1) - l_n(x_2) + d^n(x_2)$. Thus, $d^n(x_2) = l_n(x_2) - l_n(x_1) + 2n - 2 \leq 2n - 3$ and this implies $l_n(x_1) > l_n(x_2)$, so $l_n(x_1) = 2$. Hence, x_1 is the only vertex satisfying $d^n(x_1) = n - 1$ (the second one would have the same color so the same weight as x_1) and this is the edge monochromatic vertex of type 2. The proof of the case when $d^n(x_2) = M = 2n - 2$ is analogous, so our claim is true.

Now applying the induction on n , we will show that every NESD total 2-coloring l_n of K_n ($n \geq 2$) is identical with c_n or c'_n (up to total isomorphism). Clearly, this assertion is evident for $n = 2$. Assume that it is true for $n' < n$ and consider an NESD total 2-coloring l_n of K_n . Deleting an unique edge monochromatic vertex of type 2 (type 1, respectively), we get the graph K_{n-1} together with an NESD total 2-coloring c_{n-1} or c'_{n-1} having edge monochromatic vertex of type 1 (type 2, respectively), so our assertion is true. ■

3. BIPARTITE GRAPHS

Theorem 10. *Let T be a tree of order $n \geq 2$. Then $\eta(T) \leq 2$.*

Proof. The proof is by induction on n . Observe that the theorem is trivial if T is a star $K_{1,n-1}$, hence, in particular, for every tree of order $n \in \{2, 3\}$, $\eta(T) \leq 2$.

Suppose our assertion is true for all trees of order $n - 1 \geq 3$ and let T be a tree of order n . We may assume that T is not isomorphic to $K_{1,n-1}$. Let x be an end-vertex of a longest path $P = xyz \cdots$ in T and let T' denote the tree $T \setminus \{x\}$. By the choice of x and T , z is the only neighbor of y having the degree greater than or equal to 2 in T . Let $d_{T'}(t) = d'(t)$ for any vertex $t \in V(T')$. The degree in T' of any vertex t is the same as in T , except for $t = y$ for which $d'(y) = d_T(y) - 1$.

By induction hypothesis, there is an NESD total 2-coloring c' of T' . We will color the edge xy and the vertex x by a and b , respectively, $a, b \in \{1, 2\}$, so that the coloring c of T defined as follows

$$(3) \quad c(\alpha) = \begin{cases} c'(\alpha), & \text{if } \alpha \in V(T') \cup E(T'), \\ a, & \text{if } \alpha = xy, \\ b, & \text{if } \alpha = x, \end{cases}$$

would be an NESD total 2-coloring of T . We prove that this is always possible.

Let $w'(v)$ denote the expanded sum at $v \in V(T')$ with respect to the coloring c' .

Suppose now that the degree $d_T(y)$ of y in T is at least three and observe that for any total 2-coloring c of T and for any $t \in N_T(y) \setminus \{z\}$, we have $w(t) = c(y) + c(yt) \leq 4$ and $w(y) \geq 6$, so the vertices t and y are distinguished. Therefore, we can choose a and b such that $w(z) = w'(z) \neq w'(y) + a + b = w(y)$ and the new total coloring c of T defined by (3) will distinguish all vertices of T .

If $d_T(y) = 2$, we can also choose a and b such that $w(x) = a + c'(y) \neq w'(y) + a + b = w(y)$ and $w(z) = w'(z) \neq w'(y) + a + b = w(y)$, so the total coloring c distinguishes all vertices of T . \blacksquare

Proposition 11. *Let $G = (X, Y, E)$ be a connected bipartite graph with bipartition classes X and Y such that $|X|$ is even or G has a vertex of odd degree. Then $\eta(G) \leq 2$.*

Proof. Suppose that $|X|$ is even. We will follow the idea presented in [4] and show that there exists a coloring of vertices and edges of G with the elements of the group \mathbb{Z}_2 such that all vertices of X have expanded sums 1 and the expanded sum at any vertex of Y is 0. Let $X = \{x_1, \dots, x_{2k}\}$ and let P_j denote a path of end-vertices x_{2j-1} and x_{2j} , $j = 1, \dots, k$. Clearly, each P_j is of even length. Begin now with color 0 on all vertices and edges of G and modify this coloring along the consecutive paths P_j in the following way: start with P_1 and add 1 (in \mathbb{Z}_2) to the color of every edge of P_1 , then add 1 to the color of every edge of P_2 and so on. Obviously, in j -th step this operation maintains the expanded sums at internal vertices of P_j , so that of Y , and change the expanded sums at end-vertices of P_j . After k steps we obtain the desired coloring with the elements of \mathbb{Z}_2 .

Replacing the color 0 by 2 and applying the addition in \mathbb{N} we get an NESD total 2-coloring of G .

Now, assume that $|X|$ and $|Y|$ are odd (otherwise we could apply the first part of the proof), $X = \{x_1, \dots, x_{2k}, x_{2k+1}\}$, and $d(x_{2k+1}) = 2l + 1$ is odd. Color the edges and the vertices of G with the elements of \mathbb{Z}_2 using the same method as in the first part of the proof, taking the set $X' = \{x_1, \dots, x_{2k}\}$ as the set of end-vertices of paths P_j . Perhaps some paths P_j contain the vertex x_{2k+1} . Now the weight of every vertex of $Y \cup \{x_{2k+1}\}$ is 0 and all remaining weights are equal

to 1. Put 1 on the vertex x_{2k+1} and add 1 to the color of each edge incident to the vertex x_{2k+1} . Now the weights of vertices which are not adjacent to x_{2k+1} remain unchanged, we add $1+1$ to the weight of every neighbor of x_{2k+1} and $(2l+1) \cdot 1$ to the weight of the vertex x_{2k+1} . Thus the weight of every vertex of X is 1 and all weights of vertices of Y are equal to 0. Now we change the color 0 for 2, apply the addition in \mathbb{N} and get an NESD total 2-coloring of G . ■

The following proposition is obvious.

Proposition 12. *If every two adjacent vertices of G have different degrees, then $\eta(G) = 1$.*

In some cases, the value of $\eta(G)$ can be determined exactly.

Corollary 13. *For any integers $n, p \geq 1$, $\eta(K_{n,p}) = 2$ for $n = p$ and $\eta(K_{n,p}) = 1$ for $n \neq p$.*

Proof. Suppose that $K_{n,n}$ has bipartition (X, Y) . If we color the vertices of X by 1 and other vertices and edges of $K_{n,n}$ by 2, we get an NESD total 2-coloring. For $n \neq p$ our result follows from Proposition 12. ■

4. SOME OTHER RESULTS

In [4] Karoński *et al.* proved the following result.

Theorem 14. *Let Γ be a finite abelian group of odd order k and let G be a k -colorable graph on $n \geq 3$ vertices. Then there exists a coloring c of the edges of G with the elements of Γ such that the resulting vertex coloring σ induced by c is a proper coloring of G .*

Corollary 15. *Let k be an odd integer and let G be a connected k -colorable graph. Then $\eta(G) \leq k$.*

Proof. If we color the edges of G using the method described in [4] with the elements of $\mathbb{Z}_k = \mathbb{Z}_{2l+1}$ and put 0 on the vertices of G , then we get an NESD total coloring of G with the elements of $\Gamma = \mathbb{Z}_k$. Now we can obtain an NESD total k -coloring of G by replacing 0 by k and applying the addition in \mathbb{N} . ■

Thus the following corollary is true.

Corollary 16. *If G is a connected k -colorable graph, then $\eta(G) \leq k + 1$.*

As already mentioned, in [3], Kalkowski *et al.* showed that for every graph G without components isomorphic to K_2 there exists a coloring of the edges of G with the elements of $\{1, \dots, 5\}$ such that the resulting vertex weighting is a proper vertex coloring of G . This implies at once the following corollary.

Corollary 17. *If G is a connected regular graph, then $\eta(G) \leq 5$.*

Proof. Color the edges of G with 5 colors in such a way that the obtained vertex coloring is proper. Afterwards put 1 on the vertices of G . All weights will increase by a constant. ■

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