

*Discussiones Mathematicae*  
*Graph Theory* 37 (2017) 13–28  
doi:10.7151/dmgt.1916

**A FINITE CHARACTERIZATION AND RECOGNITION  
OF INTERSECTION GRAPHS OF HYPERGRAPHS WITH  
RANK AT MOST 3 AND MULTIPLICITY AT MOST 2  
IN THE CLASS OF THRESHOLD GRAPHS**

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**Abstract**

We characterize the class  $L_3^2$  of intersection graphs of hypergraphs with rank at most 3 and multiplicity at most 2 by means of a finite list of forbidden induced subgraphs in the class of threshold graphs. We also give an  $O(n)$ -time algorithm for the recognition of graphs from  $L_3^2$  in the class of threshold graphs, where  $n$  is the number of vertices of a tested graph.

**Keywords:** intersection graph, hypergraph rank, hypergraph multiplicity, forbidden induced subgraph, threshold graph.

**2010 Mathematics Subject Classification:** 05C62, 05C75, 05C70, 05C65, 05C85.

## 1. INTRODUCTION

In this paper, we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively;  $N(v) = N_G(v)$  is the neighborhood of a vertex  $v$  in  $G$  and  $\deg(v)$  is the degree of  $v$ ; the subgraph of  $G$  induced by a set  $X \subseteq V(G)$  is denoted by  $G(X)$ . A vertex  $v$  of a graph  $G$  is called *dominating* if  $N(v) \cup \{v\} = V(G)$ .

The *intersection graph*  $L(\mathcal{H})$  of a hypergraph  $\mathcal{H}$  is defined as follows:

- (1) the vertices of  $L(\mathcal{H})$  are in a bijective correspondence with the edges of  $\mathcal{H}$ ;
- (2) two vertices are adjacent in  $L(\mathcal{H})$  if and only if the corresponding edges have a non-empty intersection.

The *rank* of a hypergraph  $\mathcal{H}$  is the maximum size of its edges. The *multiplicity of a pair of vertices*  $u, v$  of  $\mathcal{H}$  is the number of edges in  $\mathcal{H}$  containing both  $u$  and  $v$ ; the *multiplicity*  $m(\mathcal{H})$  of  $\mathcal{H}$  is the maximum multiplicity among all pairs of vertices in  $\mathcal{H}$  (see for example [15]).

Denote by  $L_r^m$  the class of intersection graphs of hypergraphs with rank at most  $r$  and multiplicity at most  $m$ . So, we refer to  $L_r^\infty$  as the class of intersection graphs of hypergraphs with rank at most  $r$ . The class  $L_r^m$ , where  $r \geq 1$ ,  $m \geq 1$  or  $m = \infty$ , is hereditary (i.e., every induced subgraph of a graph in  $L_r^m$  is also in  $L_r^m$ ). Therefore, it can be characterized by means of a list (finite or not) of forbidden induced subgraphs.

A non-trivial characterization of the class  $L_r^m$  is known only for  $r \leq 2$ . These are:

- Beineke's finite characterization of the class  $L_2^1$  of line graphs (i.e., intersection graphs of simple graphs) [1];
- a finite characterization of the class  $L_2^\infty$  of intersection graphs of multigraphs by Bermond and Meyer [2];
- a finite characterization of the class  $L_2^m$  by Tashkinov [22].

Such finite characterizations of the classes above imply that there exist polynomial algorithms for recognizing graphs from these classes. (For efficient algorithms for recognizing graphs from  $L_2^1$  see, e.g., [4, 11, 17, 19].) It is also known that for any  $r \geq 3$  and  $m$ , where  $m \geq 1$  or  $m = \infty$ , there does not exist a finite characterization for the class  $L_r^m$  (see [6, 15, 16, 10]).

Poljak, Rödl and Turzik [18] proved that the problem of determining whether a graph belongs to  $L_r^\infty$  is NP-complete for an arbitrary  $r$ . Moreover, they proved that for every fixed  $r \geq 4$ , the analogous problem remains NP-complete. The question whether or not the class  $L_3^\infty$  can be recognized in polynomial time is still open, but recognizing intersection graphs of hypergraphs without multiple edges with rank at most 3 is NP-complete as well [18]. The following result generalizing one from [18] was obtained in [7]: For every fixed  $m \geq 1$  and an arbitrary  $r$ , the problem of determining whether a graph belongs to  $L_r^m$  is NP-complete.

Hliněný and Kratochvíl [8] proved that for every fixed  $r \geq 3$ , the problem of determining whether a graph belongs to  $L_r^1$  is NP-complete. The class  $L_3^1$  was studied in different papers, and several graph classes were found, where the problem of recognizing graphs from the class is polynomially solvable or remains NP-complete ([7, 9, 14, 15, 16, 21]).

A graph  $G$  is called *split* [5] if there exists a partition of its vertex set  $V(G) = A \cup B$  into a clique  $A$  and a stable set  $B$  (*bipartition*  $(A, B)$ ). It was proved in [12] that for every fixed  $r$ , there exists a finite characterization of the graphs from  $L_r^1$  in the class of split graphs. In [13] (see also [7]), this result was generalized to the class  $L_r^m$  for every fixed  $m$ .

A split graph with the bipartition  $(A, B)$  is called *threshold* [3] if the vertices from  $B$  can be numbered as  $b_1, b_2, \dots, b_k$  so that  $N(b_1) \supseteq N(b_2) \supseteq \dots \supseteq N(b_k)$ . In [20], the problem of determining the Krausz dimension of a graph (the minimum  $r$  such that the graph belongs to the class  $L_r^1$ ) was solved in the subclass of threshold graphs of the form  $K_n - E(K_p)$ .

In Section 2 of this paper, we give some preliminary facts (e.g., a so-called Krausz type characterization of the class  $L_3^2$  in terms of clique coverings), prove some technical lemmas and formulate Theorem 2 that gives a finite characterization of the class  $L_3^2$  (consisting of 15 graphs) in the class of threshold graphs. In Sections 3 and 4, we prove the necessity and sufficiency of Theorem 2, respectively. In Section 5 we give an  $O(n)$ -time algorithm for the recognition of graphs from  $L_3^2$  in the class of threshold graphs, where  $n$  is the number of vertices of a tested graph.

## 2. SOME PRELIMINARIES AND THE FORMULATION OF THEOREM 2

A finite family  $\mathcal{C} = (C_1, C_2, \dots, C_q)$  of cliques of the graph  $G$  is called a *covering* of  $G$  if every vertex as well as every edge of  $G$  is contained in some  $C_i$ . The cliques  $C_i$  are the *clusters* of  $\mathcal{C}$ . For a vertex  $v \in V(G)$ , denote by  $\mathcal{C}(v)$  the subfamily of all clusters of  $\mathcal{C}$  that contain  $v$ . A covering  $\mathcal{C}$  of the graph  $G$  is called an  $(r, m)$ -*covering* if any vertex of  $G$  belongs to at most  $r$  clusters of  $\mathcal{C}$ , and any two clusters of  $\mathcal{C}$  have at most  $m$  vertices in common.

**Theorem 1** [7, 13]. *A graph  $G$  belongs to the class  $L_3^2$  if and only if there exists a  $(3, 2)$ -covering of  $G$ .*

A clique of a graph  $G$  is called *maximal* if it is not contained in some other clique of  $G$ .

Let a threshold graph with the bipartition  $(A, B)$  be given, where  $B = \{b_1, b_2, \dots, b_k\}$  and  $N(b_1) \supseteq N(b_2) \supseteq \dots \supseteq N(b_k)$ . We denote such a graph by  $G(p, q_1, q_2, \dots, q_k)$  if  $|A| = p$  and  $\deg(b_i) = q_i$  for any  $i = 1, 2, \dots, k$ . Without loss of generality (W.l.o.g.), we assume below that any threshold graph

$G(p, q_1, q_2, \dots, q_k)$  with the bipartition  $(A, B)$  satisfies the conditions  $A = \{a_1, a_2, \dots, a_p\}$ ,  $B = \{b_1, b_2, \dots, b_k\}$ ,  $p > q_1$  and  $N(b_i) = \{a_1, a_2, \dots, a_{q_i}\}$  for any  $i = 1, 2, \dots, k$  (see Figure 1).

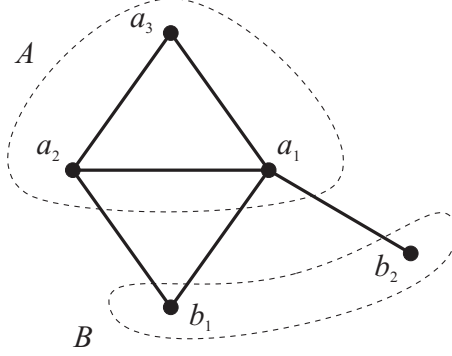


Figure 1. The graph  $G(3, 2, 1)$  and its bipartition  $(A, B)$ .

In this paper, we characterize the class  $L_3^2$  by means of a finite list of forbidden induced subgraphs in the class of threshold graphs:

**Theorem 2.** *A threshold graph  $H$  belongs to the class  $L_3^2$  if and only if it contains none of the graphs  $K_{1,4}$ ,  $G(12, 7)$ ,  $G(11, 10)$ ,  $G(10, 9, 5)$ ,  $G(10, 9, 7)$ ,  $G(10, 9, 9)$ ,  $G(10, 7, k)$ ,  $k = 1, 2, \dots, 7$ ,  $G(9, 8, 1)$ ,  $G(9, 8, 2)$  as induced subgraphs.*

Now we formulate some technical statements that will be used for proving Theorem 2.

A  $(3, 2)$ -covering  $\mathcal{C} = (C_1, C_2, \dots, C_t)$  of a complete graph  $G$  is called a *decomposition*  $(3, 2)$ -covering if  $C_i \neq V(G)$  for any  $i = 1, 2, \dots, t$ .

**Lemma 3.** *Let  $\mathcal{C} = (C_1, C_2, \dots, C_t)$  be a decomposition  $(3, 2)$ -covering of a complete graph  $G$ . Then the following statements hold:*

- (i)  $|C_i| \leq 6$  for any  $i = 1, 2, \dots, t$ .
- (ii) If  $C_i \setminus C_j \neq \emptyset$  for some  $i, j \in \{1, 2, \dots, t\}$ , then  $|C_j \setminus C_i| \leq 4$ .
- (iii) If  $(C_i \cap C_j) \setminus C_k \neq \emptyset$  for some different  $i, j, k \in \{1, 2, \dots, t\}$ , then  $|C_k \setminus (C_i \cup C_j)| \leq 2$ .

**Proof.** (i) Let, to the contrary,  $C_i = \{a_1, a_2, \dots, a_7, \dots\}$  for some  $i \in \{1, 2, \dots, t\}$ . Consider a vertex  $v \in V(G) \setminus C_i$ . By the definition of a  $(3, 2)$ -covering, each cluster of  $\mathcal{C}$  contains at most two edges of  $va_s$ ,  $s = 1, 2, \dots, 7$ . Hence, the edges  $va_s$ ,  $s = 1, 2, \dots, 7$ , are covered by at least four clusters of  $\mathcal{C}$ , and, therefore, the vertex  $v$  is contained in at least four clusters of  $\mathcal{C}$ , which is a contradiction to the definition of  $\mathcal{C}$ .

(ii) Assume, to the contrary, that for a vertex  $v \in V(G)$ , we have  $v \in C_i \setminus C_j$  and  $C_j \setminus C_i = \{a_1, a_2, a_3, a_4, a_5, \dots\}$ . By the definition of a (3,2)-covering, the edges  $va_s$ ,  $s = 1, 2, \dots, 5$ , are covered by at least three clusters of  $\mathcal{C}$ , different from  $C_i$ . So, taking into account the cluster  $C_i$ , the vertex  $v$  is contained in at least four clusters of  $\mathcal{C}$ , which is a contradiction to the definition of  $\mathcal{C}$ .

(iii) Let, instead,  $v \in (C_i \cap C_j) \setminus C_k \neq \emptyset$  and  $C_k \setminus (C_i \cup C_j) = \{a_1, a_2, a_3, \dots\}$ . By the definition of a (3,2)-covering, the edges  $va_1, va_2, va_3$  are covered by at least two clusters of  $\mathcal{C}$ , different from  $C_i$  and  $C_j$ . So, together with the clusters  $C_i, C_j$ , the vertex  $v$  is contained in at least four clusters of  $\mathcal{C}$ , which is a contradiction. ■

**Lemma 4.** *Let  $\mathcal{C} = (C_1, C_2, \dots, C_t)$  be a decomposition (3,2)-covering of a complete graph  $G$ . Then the following statements hold:*

- (i) *If  $G$  has order 11, then it contains no cluster of size at most 2.*
- (ii) *If  $G$  has order 12, then it contains no cluster of size at most 3.*

**Proof.** (i) Let  $V(G) = \{a_1, a_2, \dots, a_{11}\}$ ,  $C_1 \in \mathcal{C}(a_1)$  and  $|C_1| \leq 2$ . W.l.o.g., assume that  $\{a_3, a_4, \dots, a_{11}\} \subseteq V(G) \setminus C_1$ . By the definition of  $\mathcal{C}$ , there exists a cluster  $C_2 \in \mathcal{C}(a_1)$  of size at least 6 among the clusters covering some of the nine edges  $a_1a_i$ ,  $i = 3, 4, \dots, 11$ . By Lemma 3(i),(ii),  $|C_2| = 6$  and  $C_1 \subseteq C_2$ . Hence,  $|V(G) \setminus (C_1 \cup C_2)| = 5$  and there exists a cluster  $C_3 \in \mathcal{C}(a_1) \setminus \{C_1, C_2\}$  of size at least 6 containing the set  $V(G) \setminus (C_1 \cup C_2)$ . By Lemma 3(i),  $C_3 = \{a_1\} \cup (V(G) \setminus (C_1 \cup C_2))$ . We have  $|C_2| = |C_3| = 6$  and  $|C_2 \cap C_3| = 1$ , which is a contradiction to Lemma 3(ii).

The statement (ii) of the lemma follows immediately from the statement (i). ■

### 3. PROOF OF NECESSITY OF THEOREM 2

By heredity of the class  $L_3^2$ , one has to show that none of the graphs  $K_{1,4}$ ,  $G(12, 7)$ ,  $G(11, 10)$ ,  $G(10, 9, 5)$ ,  $G(10, 9, 7)$ ,  $G(10, 9, 9)$ ,  $G(10, 7, k)$ ,  $k = 1, 2, \dots, 7$ ,  $G(9, 8, 1)$  and  $G(9, 8, 2)$  belongs to this class. Obviously, there exists no (3,2)-covering for the star  $K_{1,4}$ . Therefore,  $K_{1,4} \notin L_3^2$  by Theorem 1.

Furthermore, let  $G$  be one of the graphs  $G(12, 7)$ ,  $G(11, 10)$ ,  $G(10, 9, 5)$ ,  $G(10, 9, 7)$ ,  $G(10, 9, 9)$ ,  $G(10, 7, k)$ ,  $k = 1, 2, \dots, 7$ ,  $G(9, 8, 1)$ ,  $G(9, 8, 2)$  with the bipartition  $(A, B)$ . Suppose, to the contrary, that there exists a (3,2)-covering  $\mathcal{D} = (D_1, D_2, \dots, D_t)$  of  $G$ .

W.l.o.g., we will assume that no cluster of  $\mathcal{D}$  is contained in some other cluster of  $\mathcal{D}$ . By Theorem 1, it can be easily seen that  $D_i \neq A$  for any  $i = 1, 2, \dots, t$ , since  $\deg(b_1) \geq 7$ .

Put  $\mathcal{C} = (C_1, C_2, \dots, C_t)$ , where  $C_i = D_i \cap A$ ,  $i = 1, 2, \dots, t$ . Then  $\mathcal{C}$  is a decomposition (3,2)-covering of the subgraph  $G(A)$ , since  $N(b_i) \neq A$  for each

$b_i \in B$ . A cluster  $C \in \mathcal{C}$  is called  $b_i$ -reduced with  $b_i \in B$ , if  $C \cup \{b_i\} \in \mathcal{D}$ . A cluster  $C \in \mathcal{C}$  is called simply reduced if it is  $b_i$ -reduced for some  $b_i \in B$ . By Lemma 3(i),  $\mathcal{C}$  contains two or three  $b_1$ -reduced clusters, since  $\deg(b_1) \geq 7$ .

**Lemma 5.** *The following statements hold:*

- (i) *If  $C_1, C_2 \in \mathcal{C}$  are two different  $b_i$ -reduced clusters with  $b_i \in B$ , then  $|C_1 \cap C_2| \leq 1$ .*
- (ii) *If  $C_1, C_2 \in \mathcal{C}$  are two different  $b_i$ -reduced clusters with  $b_i \in B$ , then  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ .*
- (iii) *If  $C_1, C_2, C_3 \in \mathcal{C}$  are three different reduced clusters, then  $C_1 \cap C_2 \cap C_3 = \emptyset$ .*

**Proof.** (i) The validity of the statement follows immediately from the definition of  $\mathcal{C}$ .

(ii) The statement follows from the above assumption that no cluster of  $\mathcal{D}$  is contained in some other cluster of  $\mathcal{D}$ .

(iii) If, to the contrary,  $a \in C_1 \cap C_2 \cap C_3$ , then the edge  $aa_p$  is not covered by a cluster from  $\mathcal{C}(a) = \{C_1, C_2, C_3\}$ , which is a contradiction to the definition of  $\mathcal{C}$ . ■

We consider the following separate cases and come to a contradiction in each of them.

(1)  $G = G(12, 7)$ .

(a) Assume that there exist exactly two  $b_1$ -reduced clusters  $C_1, C_2 \in \mathcal{C}$ . By Lemma 4(ii),  $|C_1| \geq 4$  and  $|C_2| \geq 4$ . Hence, by Lemma 5(i) and the equality  $|C_1 \cup C_2| = 7$ , we obtain  $|C_1| = |C_2| = 4$  and  $|C_1 \cap C_2| = 1$ . W.l.o.g., assume that  $C_1 \cap C_2 = \{a_1\}$ . Consider the cluster  $C_3 \in \mathcal{C}(a_1) \setminus \{C_1, C_2\}$ . Then  $\{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\} \subseteq C_3$ . By Lemma 3(i),  $C_3 = \{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\}$  (see Figure 2). We have  $|C_3 \setminus C_1| = 5$ , which is a contradiction to Lemma 3(ii).

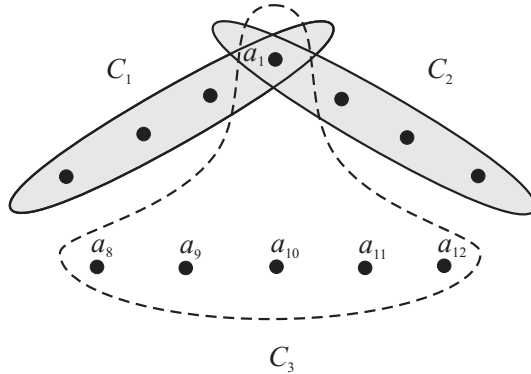


Figure 2. The clusters  $C_1, C_2$  and  $C_3$  of the covering  $\mathcal{C}$  in the case (1).

(b) Suppose that there exist exactly three  $b_1$ -reduced clusters  $C_1, C_2, C_3 \in \mathcal{C}$ . Taking into account Lemmas 5(i) and 4(ii), we obtain that  $|C_1 \cup C_2| \geq 7$  and, therefore,  $|C_1 \cup C_2 \cup C_3| \geq 9 > 7 = \deg(b_1)$ , which is a contradiction.

(2)  $G = G(11, 10)$ .

(a) Assume that there exist exactly two  $b_1$ -reduced clusters  $C_1, C_2 \in \mathcal{C}$ . By Lemma 5(i),  $|C_1 \cap C_2| \leq 1$ . By Lemmas 5(ii) and 3(ii),  $|C_1 \setminus C_2| \leq 4$  and  $|C_2 \setminus C_1| \leq 4$ . Therefore,  $\deg(b_1) = |C_1 \cup C_2| \leq 9$ , which is a contradiction.

(b) Let  $\mathcal{C}$  contain three  $b_1$ -reduced clusters  $C_1, C_2$  and  $C_3$ .

First, we suppose that  $C_1, C_2$  and  $C_3$  are pairwise disjoint. By Lemmas 3(ii) and 4(i), we have  $3 \leq |C_i| \leq 4$  for any  $i = 1, 2, 3$ . W.l.o.g., assume that  $C_1 = \{a_1, a_2, a_3\}$ ,  $C_2 = \{a_4, a_5, a_6\}$ ,  $C_3 = \{a_7, a_8, a_9, a_{10}\}$ . By the definition of  $\mathcal{C}$  and Lemma 3(i), we have  $|\mathcal{C}(a_1)| = 3$ , since  $|A \setminus C_1| = 8$ .

Let  $C_4$  and  $C_5$  be two clusters in  $\mathcal{C}(a_1) \setminus \{C_1\}$ . Each of the clusters  $C_4$  and  $C_5$  has at least one common vertex with any of the clusters  $C_2, C_3$ . If, for example,  $C_4 \cap C_2 = \emptyset$ , then  $a_1 \in (C_1 \cap C_4) \setminus C_2$  and  $|C_2 \setminus (C_1 \cup C_4)| = |C_2| = 3$ , which is a contradiction to Lemma 3(iii). Since  $C_3 \subseteq C_4 \cup C_5$  by the definition of  $\mathcal{C}$  and  $|C_3| = 4$ , then each of the clusters  $C_4$  and  $C_5$  has exactly two common vertices with the cluster  $C_3$ .

The inequalities  $|C_4| \geq 5$  and  $|C_5| \geq 5$  hold. Otherwise, let, for example,  $|C_4| \leq 4$ . Then  $|C_5| \geq 6$ , since  $|C_4 \cup C_5| \geq 9$ . Hence, by Lemma 3(i),  $|C_5| = 6$ . Therefore,  $C_4 \cap C_5 = \{a_1\}$  and  $|C_5 \setminus C_4| = 5$ , which is a contradiction to Lemma 3(ii).

W.l.o.g., assume that  $\{a_4, a_7, a_8\} \subseteq C_4$ ,  $\{a_6, a_9, a_{10}, a_{11}\} \subseteq C_5$ . Since  $|C_5 \setminus C_1| \leq 4$  by Lemma 3(ii), then  $a_5 \notin C_5$ . Hence,  $a_5 \in C_4$ . We have  $a_5 \in (C_2 \cap C_4) \setminus C_5$ . By Lemma 3(iii),  $|C_5 \setminus (C_2 \cup C_4)| \leq 2$ . Then  $a_{11} \in C_4$  and, by Lemma 3(i),  $C_4 = \{a_1, a_4, a_5, a_7, a_8, a_{11}\}$  (see Figure 3). Therefore,  $|C_4 \setminus C_1| = 5$ , which is a contradiction to Lemma 3(ii).

Now, w.l.o.g., assume that  $a_1 \in C_1 \cap C_2$ . By Lemma 5(i),  $C_1 \cap C_2 = \{a_1\}$ . By Lemmas 5(ii) and 3(ii),  $|C_1| \leq 5$  and  $|C_2| \leq 5$ . Each of the clusters  $C_1, C_2$  has size at least 4. If not, then  $a_1 \in (C_1 \cap C_2) \setminus C_3$  by Lemma 5(iii), and  $|C_3 \setminus (C_1 \cup C_2)| \geq 10 - (3 + 5 - 1) = 3$ , which is a contradiction to Lemma 3(iii).

Furthermore, assume that at least one of the clusters  $C_1, C_2$ , say  $C_1$ , has size 5. Then  $|C_1 \setminus C_3| \leq 4$  by Lemmas 5(ii) and 3(ii), and so  $|C_1 \cap C_3| = 1$  by Lemma 5(i). Let  $C_1 \cap C_3 = \{a_2\}$ . Then  $a_2 \in (C_1 \cap C_3) \setminus C_2$  by Lemma 5(iii). We obtain that  $|C_2 \setminus (C_1 \cup C_3)| \leq 2$  by Lemma 3(iii). Therefore,  $|C_2 \cap C_3| = 1$ . Let  $C_2 \cap C_3 = \{a_3\}$ . We have  $a_3 \in (C_2 \cap C_3) \setminus C_1$  and  $|C_1 \setminus (C_2 \cup C_3)| = 3$ , contradicting Lemma 3(iii).

Thus,  $|C_1| = |C_2| = 4$ . Let, w.l.o.g.,  $C_1 = \{a_1, a_2, a_3, a_4\}$ ,  $C_2 = \{a_1, a_5, a_6, a_7\}$ . Then  $\{a_8, a_9, a_{10}\} \subseteq C_3$ , since  $\{a_1, a_2, \dots, a_{10}\} = N(b_1)$ . By Lemma 5(iii),  $a_1 \in (C_1 \cap C_2) \setminus C_3$ . However, then  $|C_3 \setminus (C_1 \cup C_2)| = 3$ , which is a contradiction to Lemma 3(iii).

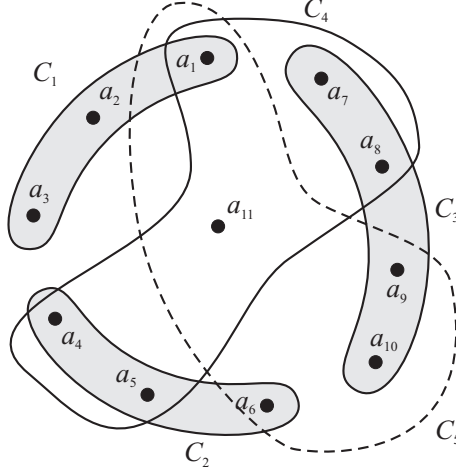


Figure 3. The clusters  $C_1, C_2, C_3, C_4$  and  $C_5$  of the covering  $\mathcal{C}$  in the case (2).

(3)  $G = G(10, 9, 5)$ .

Each vertex  $a_i$ , where  $i = 1, 2, \dots, 5$ , belongs to one  $b_1$ - and one  $b_2$ -reduced clusters. Therefore, by Lemma 5(iii), each two of the  $b_2$ -reduced clusters have no common vertices. By Lemma 5(iii), if a vertex belongs to two of the  $b_1$ -reduced clusters, then this vertex belongs to the set  $\{a_6, a_7, a_8, a_9\}$ .

(a) Let  $\mathcal{C}$  contain exactly two  $b_1$ -reduced clusters  $C_1, C_2$ . Since  $|C_1 \cup C_2| = 9$ , we get  $|C_1 \cap C_2| = 1$  and  $|C_1| = |C_2| = 5$  by Lemmas 5(i),(ii) and 3(ii). Let, w.l.o.g.,  $C_1 \cap C_2 = \{a_9\}$ . By the definition of  $\mathcal{C}$ , any vertex  $a_i$ , where  $i = 1, 2, \dots, 8$ , belongs to exactly two clusters from  $\mathcal{C}(a_i) \setminus \{C_1, C_2\}$ . Moreover, it is easy to obtain that, for any vertex  $a_i$ , where  $i = 1, 2, \dots, 8$ , each cluster  $C \in \mathcal{C}(a_i) \setminus \{C_1, C_2\}$  satisfies the equalities  $|C \cap (C_1 \setminus C_2)| = 2$  and  $|C \cap (C_2 \setminus C_1)| = 2$ . Since every  $b_2$ -reduced cluster is a subset of  $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$  and belongs to  $\mathcal{C}(a_i) \setminus \{C_1, C_2\}$ , it has size 4, which is a contradiction.

(b) Let  $\mathcal{C}$  contain three pairwise non-intersecting  $b_1$ -reduced clusters  $C_1, C_2$  and  $C_3$ . By Lemma 3(ii),  $|C_i| \leq 4$  for every  $i = 1, 2, 3$ .

(b1) First, suppose that  $|C_1| = 1$ ,  $|C_2| = 4$  and  $|C_3| = 4$ . Put  $C_1 = \{a_1\}$ . Consider the clusters  $C_4, C_5 \in \mathcal{C}(a_1) \setminus \{C_1\}$ . By the definition of  $\mathcal{C}$ ,  $|C_i \cap C_j| = 2$  for any  $i = 2, 3$  and  $j = 4, 5$ . In particular,  $(C_4 \cap C_5) \cap (C_2 \cup C_3) = \emptyset$ . Since  $(C_2 \cap C_4) \setminus C_5 \neq \emptyset$ , then  $|C_5 \setminus (C_2 \cup C_4)| \leq 2$  by Lemma 3(iii). Similarly,  $|C_4 \setminus (C_2 \cup C_5)| \leq 2$ . Therefore,  $a_{10} \in C_4 \cap C_5$ . We obtain that there does not exist a  $b_2$ -reduced cluster in  $\mathcal{C}(a_1)$ , which is a contradiction.

Now, let  $C_1 \subset \{a_6, a_7, a_8, a_9\}$ . W.l.o.g., put  $C_1 = \{a_9\}$ . Note that each  $b_2$ -reduced cluster  $C$  in  $\mathcal{C}$  has size at most 4. If not (i.e.,  $|C| = \deg(b_2) = 5$ ), then the inclusion  $C \subseteq C_2 \cup C_3$  implies that  $|C \cap C_2| \geq 3$  or  $|C \cap C_3| \geq 3$ , which is



a contradiction to the definition of  $\mathcal{C}$ . Let  $C_4$  be a  $b_2$ -reduced cluster in  $\mathcal{C}$  with size at most 2. Let  $a_1 \in C_4 \cap C_2$ . Consider the cluster  $C_5 \in \mathcal{C}(a_1) \setminus \{C_2, C_4\}$ . By the definition of  $\mathcal{C}$ , we have  $C_3 \setminus C_4 \subseteq C_5$ . Since  $|C_4| \leq 2$  and  $C_4 \cap C_2 \neq \emptyset$ , we have  $|C_3 \setminus C_4| \geq 3$ . Therefore,  $|C_3 \cap C_5| \geq 3$ , which is a contradiction.

(b2) Suppose that  $|C_1| = 2$ ,  $|C_2| = 3$  and  $|C_3| = 4$ . Let  $a \in C_1$ , where  $a \in \{a_1, a_2, \dots, a_9\}$ . Consider the clusters  $C_4, C_5 \in \mathcal{C}(a) \setminus \{C_1\}$ . By the definition of  $\mathcal{C}$ ,  $1 \leq |C_i \cap C_2| \leq 2$  and  $|C_i \cap C_3| = 2$  for any  $i = 4, 5$ . Moreover, at least one of the clusters  $C_4, C_5$ , say  $C_5$ , has exactly two common vertices with  $C_2$ . Clearly,  $(C_4 \cap C_5) \cap C_3 = \emptyset$  and  $|(C_4 \cap C_5) \cap C_2| \leq 1$ . If  $a_{10} \in C_5$ , then  $|C_5| = 6$  by Lemma 3(i). We have  $C_1 \setminus C_5 \neq \emptyset$  and  $|C_5 \setminus C_1| = 5 > 4$ , which is a contradiction to Lemma 3(ii). Therefore,  $a_{10} \in C_4 \setminus C_5$ . By Lemma 3(i), at least one vertex  $a'$  of the set  $C_5 \cap C_2$  does not belong to  $C_4$ . We obtain that  $a' \in (C_2 \cap C_5) \setminus C_4$  and  $|C_4 \setminus (C_2 \cup C_5)| \geq 3$ , which is a contradiction to Lemma 3(iii).

(b3) Let  $|C_1| = |C_2| = |C_3| = 3$ . Assume that there exists a  $b_2$ -reduced cluster in  $\mathcal{C}$  with size at most 2. Therefore, this cluster does not intersect with some of the clusters  $C_1, C_2$  and  $C_3$ , which is a contradiction to the definition of  $\mathcal{C}$ .

Now, let  $C_4 = N(b_2)$  be the only  $b_2$ -reduced cluster in  $\mathcal{C}$ . W.l.o.g., assume that  $C_1 = \{a_1, a_6, a_7\}$ ,  $C_2 = \{a_2, a_3, a_8\}$  and  $C_3 = \{a_4, a_5, a_9\}$ . Consider the clusters  $C' \in \mathcal{C}(a_2) \setminus \{C_2, C_4\}$  and  $C'' \in \mathcal{C}(a_3) \setminus \{C_2, C_4\}$ . By the definition of  $\mathcal{C}$ , we have  $a_6, a_7, a_9, a_{10} \in C' \cap C''$ . Therefore,  $C' = C''$ . Put  $C_5 = C'$ . Then  $C_3 \setminus C_5 \neq \emptyset$  and  $|C_5 \setminus C_3| = 5 > 4$ , which is a contradiction to Lemma 3(ii).

(c) Let  $\mathcal{C}$  contain three  $b_1$ -reduced clusters  $C_1, C_2, C_3$  and  $C_1 \cap C_2 \neq \emptyset$ . W.l.o.g., assume that  $|C_1| \geq |C_2|$ . By Lemma 5(iii), we obtain that  $(C_1 \cap C_2) \setminus C_3 \neq \emptyset$ . It follows from Lemma 3(iii) that  $|C_3 \setminus (C_1 \cup C_2)| \leq 2$ . Hence,  $|C_1 \cup C_2| \geq 7$ . Then  $|C_1| \geq 4$ . Moreover, by Lemmas 5(ii) and 3(ii), we have  $|C_1| \leq 5$ .

(c1) Let  $|C_1| = 5$ . Then  $C_1 \cap C_3 \neq \emptyset$  by Lemmas 5(ii) and 3(ii). Furthermore,  $C_2 \cap C_3 = \emptyset$  by Lemmas 5(iii) and 3(iii). Since  $(C_1 \cap C_3) \setminus C_2 \neq \emptyset$  and, by Lemma 3(iii),  $|C_2 \setminus (C_1 \cup C_3)| \leq 2$ , we have  $|C_1| = 5$ ,  $|C_2| = 3$  and  $|C_3| = 3$ . Recall that  $C_1 \cap C_2, C_1 \cap C_3 \subseteq \{a_6, a_7, a_8, a_9\}$ . W.l.o.g., assume that  $C_1 \cap C_2 = \{a_8\}$ ,  $C_1 \cap C_3 = \{a_9\}$ . Consider the clusters  $C_4 \in \mathcal{C}(a_8) \setminus \{C_1, C_2\}$  and  $C_5 \in \mathcal{C}(a_9) \setminus \{C_1, C_3\}$ . By the definition of  $\mathcal{C}$ , we have  $|C_4 \cap (C_1 \setminus \{a_8, a_9\})| \leq 1$  and  $|C_5 \cap (C_1 \setminus \{a_8, a_9\})| \leq 1$ . Note that  $C_4 \cap (C_2 \setminus \{a_8\}) = \emptyset$ . If, to the contrary,  $a \in C_4 \cap (C_2 \setminus \{a_8\})$ , then  $\mathcal{C}(a) = \{C_2, C_4, C_5\}$  and some vertex of the set  $C_1 \setminus \{a_8, a_9\}$  does not belong to the set  $C_2 \cup C_4 \cup C_5$ , contradicting the definition of  $\mathcal{C}$ . Analogously,  $C_5 \cap (C_3 \setminus \{a_9\}) = \emptyset$ . At least one of the clusters  $C_2, C_3$ , say  $C_3$ , contains a vertex  $a' \in \{a_1, a_2, \dots, a_5\}$ , since  $|\{a_1, a_2, \dots, a_5\} \cap C_1| \leq 3$ . Let  $a''$  be another vertex in the set  $C_3 \setminus \{a_9\}$ . Consider the clusters  $C' \in \mathcal{C}(a') \setminus \{C_3, C_4\}$  and  $C'' \in \mathcal{C}(a'') \setminus \{C_3, C_4\}$ . Each of them contains the set  $(C_2 \setminus \{a_8\}) \cup (C_1 \setminus (C_3 \cup C_4))$  of size at least 4. Therefore,  $C' = C'' = C_6$  is a cluster of  $\mathcal{C}$  of size at least 6. By Lemma 3(i),  $|C_6| = 6$  (see Figure 4). Since  $a' \in \{a_1, a_2, \dots, a_5\}$ , then  $C_6$  is a  $b_2$ -reduced cluster in  $\mathcal{C}$ , which is a contradiction.

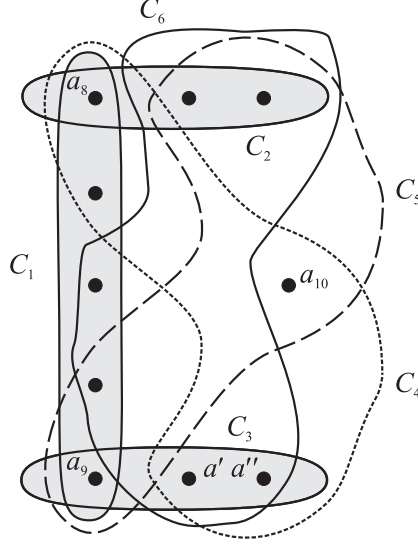


Figure 4. The clusters  $C_1, C_2, C_3, C_4, C_5$  and  $C_6$  of the covering  $\mathcal{C}$  in the case (3).

(c2) Now, let  $|C_1| = 4$ . Then, taking into consideration the inequalities  $|C_1 \cup C_2| \geq 7$  and  $|C_1| \geq |C_2|$ , we have  $|C_2| = 4$ .

Let  $C_3$  intersect with  $C_1$  or  $C_2$ . Then, by Lemma 3(iii),  $C_3$  intersects with both  $C_1$  and  $C_2$ . By Lemma 5(i),(iii), we can assume, w.l.o.g., that  $C_1 = \{a_1, a_2, a_7, a_8\}$ ,  $C_2 = \{a_3, a_4, a_7, a_9\}$  and  $C_3 = \{a_5, a_6, a_8, a_9\}$ . Consider the cluster  $C_4 \in \mathcal{C}(a_7) \setminus \{C_1, C_2\}$ . By the definition of  $\mathcal{C}$ , we have  $a_5, a_6, a_{10} \in C_4$ . Initially, let  $C_4 = \{a_5, a_6, a_7, a_{10}\}$ . Consider the cluster  $C_5 \in \mathcal{C}(a_5) \setminus \{C_3, C_4\}$ . By the definition of  $\mathcal{C}$ , we have  $a_1, a_2, a_3, a_4 \in C_5$ . Both clusters  $C_3, C_4$  are not  $b_2$ -reduced since each of them contains at least one of the vertices  $a_6, a_7, a_8, a_9, a_{10}$ . Hence  $C_5$  is a  $b_2$ -reduced cluster. It follows from the inclusion  $N(b_2) \subseteq C_5$  that  $C_5 = N(b_2) = \{a_1, a_2, \dots, a_5\}$ . Consider the cluster  $C_6 \in \mathcal{C}(a_6) \setminus \{C_3, C_4\}$ . By the definition of  $\mathcal{C}$ , we have  $a_1, a_2, a_3, a_4 \in C_6$ . Thus  $C_6 \neq C_5$  and  $|C_6 \cap C_5| \geq 4 > 2$ , which is a contradiction to the definition of  $\mathcal{C}$ . If the cluster  $C_4$  has a non-empty intersection with the set  $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ , for example  $a_1 \in C_4$ , then at least one of the vertices  $a_3, a_4$  also belongs to  $C_4$ . Otherwise, by the definition of  $\mathcal{C}$ , the cluster  $C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\}$  contains the vertices  $a_3, a_4$  and  $a_9$ . We obtain that  $C_5 \neq C_2$  and  $|C_5 \cap C_2| \geq 3 > 2$ , which is a contradiction. Let  $a_3 \in C_4$  and  $C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\}$ . Then  $a_4, a_9 \in C_5$ . We obtain that none of the clusters  $C_1, C_4, C_5 \in \mathcal{C}(a_1)$  is  $b_2$ -reduced, which is a contradiction.

Assume that the cluster  $C_3$  does not intersect with  $C_1$  and  $C_2$ . Then  $|C_3| = 2$ . One of the vertices  $a_6, a_7, a_8$  and  $a_9$ , say  $a_9$ , belongs to  $C_1 \cap C_2$ . Consider the cluster  $C_4 \in \mathcal{C}(a_9) \setminus \{C_1, C_2\}$ . Clearly,  $C_3 \cup \{a_{10}\} \subseteq C_4$ . We show that  $|C_4 \cap$

$(C_1 \setminus C_2) = 1$  and  $|C_4 \cap (C_2 \setminus C_1)| = 1$ . Indeed, if  $C_4$  has no common vertices with one of the sets  $C_1 \setminus C_2$  or  $C_2 \setminus C_1$ , say with  $C_1 \setminus C_2$ , then  $(C_3 \cap C_4) \setminus C_1 \neq \emptyset$  and  $|C_1 \setminus (C_3 \cup C_4)| = 3$ , contradicting Lemma 3(iii). Let  $C_3 = \{a', a''\}$ . Consider the clusters  $C' \in \mathcal{C}(a') \setminus \{C_3, C_4\}$  and  $C'' \in \mathcal{C}(a'') \setminus \{C_3, C_4\}$ . We have  $(C_1 \setminus C_4) \cup (C_2 \setminus C_4) \subseteq C' \cap C''$ . Since  $|(C_1 \setminus C_4) \cup (C_2 \setminus C_4)| = 4$ , we obtain that  $C' = C''$  by the definition of  $\mathcal{C}$ . Denote the cluster  $C'$  by  $C_5$ . It can be easily obtained by the definition of  $\mathcal{C}$  that there are two clusters  $C_6$  and  $C_7$  in  $\mathcal{C}$  such that  $((C_1 \setminus C_2) \cap C_4) \cup (C_2 \cap C_5) \cup \{a_{10}\} \subseteq C_6$  and  $((C_2 \setminus C_1) \cap C_4) \cup (C_1 \cap C_5) \cup \{a_{10}\} \subseteq C_7$ . Each vertex from the set  $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$  belongs to exactly three of the non- $b_2$ -reduced clusters  $C_1, C_2, C_4, C_5, C_6, C_7$ . Clearly, at least three of the vertices  $a_1, a_2, \dots, a_5$  belong to the set  $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ , which is a contradiction.

(4) We can come to a contradiction for each of the graphs  $G = G(10, 9, 9)$  and  $G = G(10, 9, 7)$  analogously to the graph  $G = G(10, 9, 5)$ .

(5)  $G = G(10, 7, k)$ ,  $k = 1, 2, \dots, 7$ .

(a) First, assume that  $4 \leq k \leq 7$ . For any  $i = 1, 2, 3, 4$ , denote by  $C_{i1}$  and  $C_{i2}$ , respectively,  $b_1$ - and  $b_2$ -reduced clusters from  $\mathcal{C}(a_i)$ . Consider the cluster  $C_{i3} \in \mathcal{C}(a_i) \setminus \{C_{i1}, C_{i2}\}$ . Since  $C_{i1}, C_{i2} \subseteq \{a_1, a_2, \dots, a_7\}$ , we have  $\{a_8, a_9, a_{10}\} \subseteq C_{i3}$  for any  $i = 1, 2, 3, 4$ . By the definition of  $\mathcal{C}$ , we obtain  $C_{13} = C_{23} = C_{33} = C_{43}$  and  $\{a_1, a_2, a_3, a_4, a_8, a_9, a_{10}\} \subseteq C_{i3}$  for any  $i = 1, 2, 3, 4$ , which is a contradiction to Lemma 3(i).

(b) Put  $k = 1$ . Let  $C_1$  and  $C_2$ , respectively, be  $b_1$ - and  $b_2$ -reduced clusters from  $\mathcal{C}(a_1)$ . Then  $C_1 \subseteq \{a_1, a_2, \dots, a_7\}$ ,  $C_2 = \{a_1\}$ . By Lemma 3(i),  $|C_1| \leq 6$ . Consider the cluster  $C_3 \in \mathcal{C}(a_1) \setminus \{C_1, C_2\}$ . The equality  $C_1 \cup C_2 \cup C_3 = C_1 \cup C_3 = A$  implies that  $|C_1| \geq 5$  by Lemma 3(i).

W.l.o.g., assume that  $C_1 = \{a_1, a_2, \dots, a_5\}$ . Then  $C_3 = \{a_1, a_6, a_7, \dots, a_{10}\}$  by Lemma 3(i). We obtain  $C_1 \setminus C_3 \neq \emptyset$  and  $|C_3 \setminus C_1| = 5$ , contradicting Lemma 3(ii). Now, w.l.o.g. put  $C_1 = \{a_1, a_2, \dots, a_6\}$ . Then  $\{a_1, a_7, a_8, a_9, a_{10}\} \subseteq C_3$ . By Lemma 3(ii),  $|C_1 \setminus C_3| \leq 4$ . Therefore, one of the vertices  $a_2, a_3, \dots, a_6$ , say  $a_2$ , belongs to  $C_3$ . By Lemma 3(i),  $C_3 = \{a_1, a_2, a_7, a_8, a_9, a_{10}\}$ . Let  $C_4$  be a  $b_1$ -reduced cluster from  $\mathcal{C}(a_7)$ . We get  $C_3 \neq C_4$ , since  $C_3 \not\subseteq N(b_1)$ . By Lemma 5(i),  $|C_4 \cap C_1| \leq 1$ . We obtain that  $a_7 \in (C_3 \cap C_4) \setminus C_1$  and  $|C_1 \setminus (C_3 \cup C_4)| \geq 3$ , which is a contradiction to Lemma 3(iii).

(c) Put  $k = 2$ . Let  $C_1$  and  $C_2$ , respectively, be  $b_1$ - and  $b_2$ -reduced clusters from  $\mathcal{C}(a_1)$ . Taking into account the case (b), we can assume that  $C_2 = \{a_1, a_2\}$ . Then we can proceed analogously to the case (b).

(d) Finally, we assume that  $k = 3$ . For any  $i = 1, 2, 3$ , denote by  $C_{i1}$  and  $C_{i2}$ , respectively,  $b_1$ - and  $b_2$ -reduced clusters from  $\mathcal{C}(a_i)$ . Taking into account the cases (b) and (c), we can assume that  $C_{12} = \{a_1, a_2, a_3\}$ . Consider the cluster  $C_{i3} \in \mathcal{C}(a_i) \setminus \{C_{i1}, C_{i2}\}$ . Since  $C_{i1}, C_{i2} \subseteq \{a_1, a_2, \dots, a_7\}$ , we have  $\{a_8, a_9, a_{10}\} \subseteq C_{i3}$  for any  $i = 1, 2, 3$ . By the definition of  $\mathcal{C}$ ,  $C_{13} = C_{23} = C_{33}$

and  $\{a_1, a_2, a_3, a_8, a_9, a_{10}\} \subseteq C_{i3}$  for any  $i = 1, 2, 3$ . By Lemma 3(i),  $C_{i3} = \{a_1, a_2, a_3, a_8, a_9, a_{10}\}$ . We obtain that  $C_{12} \neq C_{13}$  and  $|C_{12} \cap C_{13}| = 3$ , which is a contradiction to the definition of  $\mathcal{C}$ .

(6)  $G = G(9, 8, 1)$ .

(a) Assume that there exist exactly two  $b_1$ -reduced clusters  $C_1, C_2 \in \mathcal{C}$ . Clearly,  $\mathcal{C}$  contains a unique  $b_2$ -reduced cluster  $C_3 = \{a_1\}$ . If  $C_1 \cap C_2 = \emptyset$ , then  $|C_1| = |C_2| = 4$  by Lemma 3(ii). W.l.o.g., assume that  $a_1 \in C_1$ . Thus,  $a_1 \in (C_1 \cap C_3) \setminus C_2$  and  $|C_2 \setminus (C_1 \cup C_3)| = 4 > 2$ , which is a contradiction to Lemma 3(iii).

Let  $C_1 \cap C_2 \neq \emptyset$ . It follows from Lemma 5(i) that  $|C_1 \cap C_2| = 1$ . Then  $C_1 \cap C_2 \neq \{a_1\}$  by Lemma 5(iii). Let  $C_1 \cap C_2 = \{a_2\}$  and  $a_1 \in C_1$ . Since  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ , we have  $|C_1| \leq 5$  and  $|C_2| \leq 5$  by Lemma 3(ii). The equality  $|C_1 \cup C_2| = 8$  implies  $|C_1| \geq 4$  and  $|C_2| \geq 4$ . We have  $a_1 \in (C_1 \cap C_3) \setminus C_2$  and  $|C_2 \setminus (C_1 \cup C_3)| \geq 3$ , which is a contradiction to Lemma 3(iii).

(b) Now, let  $C_1, C_2, C_3$  and  $C_4 = \{a_1\}$ , respectively, be three  $b_1$ - and a unique  $b_2$ -reduced clusters in  $\mathcal{C}$ . W.l.o.g., assume that  $a_1 \in C_1$ . By Lemma 5(iii),  $C_1 \cap C_i \neq \{a_1\}$  for any  $i = 2, 3$ .

Furthermore, we have  $|C_1| \geq 5$ . Otherwise,  $|A \setminus C_1| \geq 5$  and, by the definition of  $\mathcal{C}$ , there exists a cluster  $C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\}$  such that  $(A \setminus C_1) \cup \{a_1\} \subseteq C_5$ . By Lemma 3(i), it follows that  $|A \setminus C_1| = 5$ , i.e.,  $|C_1| = 4$ . We have  $C_1 \setminus C_5 \neq \emptyset$  and  $|C_5 \setminus C_1| \geq 5$ , which is a contradiction to Lemma 3(ii). Therefore, by the same lemma,  $C_1 \cap C_2 \neq \emptyset$  and  $C_1 \cap C_3 \neq \emptyset$ . By Lemmas 5(ii) and 3(ii), we have  $|C_1 \setminus C_2| \leq 4$  and, consequently,  $|C_1| = 5$ .

The equality  $C_2 \cap C_3 = \emptyset$  holds. Otherwise, by Lemma 5(i) and (iii), we have  $(C_2 \cap C_3) \setminus C_1 \neq \emptyset$  and  $|C_1 \setminus (C_2 \cup C_3)| = 3$ , which is a contradiction to Lemma 3(iii).

Let  $C_5 \in \mathcal{C}(a_1) \setminus \{C_1, C_4\}$ . Since  $A \setminus (C_1 \cup C_4) \subset C_5$ , we have  $|C_5| \geq 5$ . Since  $|C_1 \cap C_i| = 1$  for any  $i = 2, 3$ ,  $C_2 \cap C_3 = \emptyset$  and  $|(C_2 \cup C_3) \setminus C_1| = 3$ , one of the clusters  $C_2, C_3$ , say  $C_2$ , has size 2. So, we have  $(C_2 \cap C_5) \setminus C_1 \neq \emptyset$  and  $|C_1 \setminus (C_2 \cup C_5)| \geq 3$  both in the case  $|C_5| = 6$  (since  $C_2 \subseteq C_5$  by Lemma 3(ii)) and in the case  $|C_5| = 5$ , which is a contradiction to Lemma 3(iii).

(7) We can come to a contradiction for the graph  $G = G(9, 8, 2)$  analogously to the graph  $G = G(9, 8, 1)$ .

#### 4. PROOF OF SUFFICIENCY OF THEOREM 2

Let a threshold graph  $H = G(p, q_1, q_2, \dots, q_k)$  with the bipartition  $(A, B)$  not contain any of the graphs  $K_{1,4}, G(12, 7), G(11, 10), G(10, 9, 9), G(10, 9, 7), G(10, 9, 5), G(10, 7, k), k = 1, 2, \dots, 7, G(9, 8, 2), G(9, 8, 1)$  as an induced subgraph. By Theorem 1, we have to show that there exists a  $(3, 2)$ -covering of  $H$ .

W.l.o.g., assume that  $H$  is a connected non-complete graph. Therefore,  $H$

has a dominating vertex by the definition of  $H$ . Furthermore,  $|B| \leq 2$ , since  $H$  does not contain  $K_{1,4}$  as an induced subgraph. Thus, we have  $H = G(p, q_1)$  or  $H = G(p, q_1, q_2)$ .

First, we suppose that  $|A| = p \geq 14$ . Then  $q_1 \leq 6$ , since  $H$  does not contain any of the graphs  $G(11, 10)$  and  $G(12, 7)$  as an induced subgraph. For any vertex  $b \in B$ , partition the set  $N(b)$  into  $n_b \leq 3$  pairwise disjoint cliques  $C_i^b$  each having size at most 2. Obviously, the list of cliques  $(C_i^b \cup \{b\} : b \in B, i = 1, \dots, n_b)$  together with the clique  $A$  gives a desired  $(3, 2)$ -covering of  $H$ .

If  $|A| \leq 7$ , then  $q_1 \leq 6$  by the maximality of the clique  $A$ . Therefore, a desired  $(3, 2)$ -covering of  $H$  can be constructed as above.

Now, let  $8 \leq |A| \leq 13$ . Taking into account the above considerations, we can assume that  $q_1 \geq 7$ .

Let  $H = G(p, q_1)$ . Since  $H$  does not contain any of the graphs  $G(12, 7)$  and  $G(11, 10)$  as an induced subgraph, it is isomorphic to one of the graphs  $G(13, 9)$ ,  $G(12, 9)$ ,  $G(12, 8)$ ,  $G(11, 9)$ ,  $G(11, 8)$ ,  $G(11, 7)$ ,  $G(10, 9)$ ,  $G(10, 8)$ ,  $G(10, 7)$ ,  $G(9, 8)$ ,  $G(9, 7)$ ,  $G(8, 7)$ . Clearly, the set of cliques

$$\begin{aligned} \mathcal{C} = \{ & \{a_1, a_2, a_3, a_4, a_5, b_1\}, \{a_1, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_{10}, a_{11}, a_{12}, a_{13}\}, \\ & \{a_2, a_3, a_6, a_7, a_{10}, a_{11}\}, \{a_2, a_3, a_8, a_9, a_{12}, a_{13}\}, \{a_4, a_5, a_6, a_7, a_{12}, a_{13}\}, \\ & \{a_4, a_5, a_8, a_9, a_{10}, a_{11}\} \} \end{aligned}$$

of the graph  $G(13, 9)$  is one of its  $(3, 2)$ -coverings. Each of the graphs  $G(12, 9)$ ,  $G(12, 8)$ ,  $G(11, 9)$ ,  $G(11, 8)$ ,  $G(11, 7)$ ,  $G(10, 9)$ ,  $G(10, 8)$ ,  $G(10, 7)$ ,  $G(9, 8)$ ,  $G(9, 7)$  and  $G(8, 7)$  is an induced subgraph of  $G(13, 9)$ . Therefore, a desired  $(3, 2)$ -covering for each of these graphs can be obtained from the covering  $\mathcal{C}$ .

Now, let  $H = G(p, q_1, q_2)$ . Since  $H$  does not contain any of the graphs  $G(12, 7)$ ,  $G(11, 10)$ ,  $G(10, 9, 9)$ ,  $G(10, 9, 7)$ ,  $G(10, 9, 5)$ ,  $G(10, 7, k)$ ,  $k = 1, 2, \dots, 7$ ,  $G(9, 8, 2)$  and  $G(9, 8, 1)$  as an induced subgraph, it is isomorphic to one of the graphs  $G(11, 9, 8)$ ,  $G(11, 9, 6)$ ,  $G(11, 9, 4)$ ,  $G(10, 9, 8)$ ,  $G(10, 9, 6)$ ,  $G(10, 9, 4)$ ,  $G(10, 8, 8)$ ,  $G(10, 8, 7)$ ,  $G(10, 8, 6)$ ,  $G(10, 8, 5)$ ,  $G(10, 8, 4)$ ,  $G(10, 8, 3)$ ,  $G(9, 8, 8)$ ,  $G(9, 8, 7)$ ,  $G(9, 8, 6)$ ,  $G(9, 8, 5)$ ,  $G(9, 8, 4)$ ,  $G(9, 8, 3)$ ,  $G(9, 7, 7)$ ,  $G(9, 7, 6)$ ,  $G(9, 7, 5)$ ,  $G(9, 7, 4)$ ,  $G(9, 7, 3)$ ,  $G(9, 7, 2)$ ,  $G(9, 7, 1)$ ,  $G(8, 7, 7)$ ,  $G(8, 7, 6)$ ,  $G(8, 7, 5)$ ,  $G(8, 7, 4)$ ,  $G(8, 7, 3)$ ,  $G(8, 7, 2)$ ,  $G(8, 7, 1)$ . Some of the desired  $(3, 2)$ -coverings  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ ,  $\mathcal{C}_3$ ,  $\mathcal{C}_4$  for the graphs  $G(11, 9, 8)$ ,  $G(11, 9, 6)$ ,  $G(11, 9, 4)$ ,  $G(9, 7, 1)$ , respectively, are given below:

$$\begin{aligned} \mathcal{C}_1 = \{ & \{a_1, a_2, a_3, a_4, a_9, b_1\}, \{a_5, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_2, a_7, a_8, b_2\}, \\ & \{a_3, a_4, a_5, a_6, b_2\}, \{a_1, a_2, a_5, a_6, a_{10}, a_{11}\}, \{a_3, a_4, a_7, a_8, a_{10}, a_{11}\}, \\ & \{a_9, a_{10}, a_{11}\} \}, \\ \mathcal{C}_2 = \{ & \{a_1, a_2, a_7, a_9, b_1\}, \{a_3, a_4, a_7, a_8, b_1\}, \{a_5, a_6, a_8, a_9, b_1\}, \\ & \{a_1, a_2, a_3, a_4, a_5, a_6, b_2\}, \{a_5, a_6, a_7, a_{10}, a_{11}\}, \{a_1, a_2, a_8, a_{10}, a_{11}\}, \\ & \{a_3, a_4, a_9, a_{10}, a_{11}\} \}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_3 &= \{\{a_1, a_2, a_3, a_4, a_9, b_1\}, \{a_5, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_2, a_7, a_8, b_2\}, \\ &\quad \{a_3, a_4, a_5, a_6\}, \{a_1, a_2, a_5, a_6, a_{10}, a_{11}\}, \{a_3, a_4, a_7, a_8, a_{10}, a_{11}\}, \\ &\quad \{a_9, a_{10}, a_{11}\}\}, \\ \mathcal{C}_4 &= \{\{a_1, a_2, a_3, a_4, a_5, b_1\}, \{a_5, a_6, a_7, b_1\}, \{a_1, b_2\}, \{a_1, a_2, a_6, a_7, a_8, a_9\}, \\ &\quad \{a_3, a_4, a_6, a_7\}, \{a_3, a_4, a_8, a_9\}, \{a_5, a_6, a_7\}, \{a_5, a_8, a_9\}\}. \end{aligned}$$

Each of the remaining graphs  $G(10, 9, 8)$ ,  $G(10, 9, 6)$ ,  $G(10, 9, 4)$ ,  $G(10, 8, 8)$ ,  $G(10, 8, 7)$ ,  $G(10, 8, 6)$ ,  $G(10, 8, 5)$ ,  $G(10, 8, 4)$ ,  $G(10, 8, 3)$ ,  $G(9, 8, 8)$ ,  $G(9, 8, 7)$ ,  $G(9, 8, 6)$ ,  $G(9, 8, 5)$ ,  $G(9, 8, 4)$ ,  $G(9, 8, 3)$ ,  $G(9, 7, 7)$ ,  $G(9, 7, 6)$ ,  $G(9, 7, 5)$ ,  $G(9, 7, 4)$ ,  $G(9, 7, 3)$ ,  $G(9, 7, 2)$ ,  $G(8, 7, 7)$ ,  $G(8, 7, 6)$ ,  $G(8, 7, 5)$ ,  $G(8, 7, 4)$ ,  $G(8, 7, 3)$ ,  $G(8, 7, 2)$ ,  $G(8, 7, 1)$  is an induced subgraph for some of the graphs  $G(11, 9, 8)$ ,  $G(11, 9, 6)$ ,  $G(11, 9, 4)$ ,  $G(9, 7, 1)$ . Therefore, a desired  $(3, 2)$ -covering for each of the remaining graphs can be obtained from one of the coverings  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ ,  $\mathcal{C}_3$ ,  $\mathcal{C}_4$ .

## 5. RECOGNITION ALGORITHM

The proof of sufficiency of Theorem 2 implies the following linear algorithm for recognizing graphs from  $L_3^2$  in the class of threshold graphs.

### Algorithm

**Input:** a connected threshold graph  $H$  with bipartition  $(A, B)$ , where  $A$  is a maximal clique in  $H$ .

**Output:** 1 if  $H \in L_3^2$ , and 0 otherwise.

1. **begin**
2.   **if**  $B = \emptyset$ , i.e., the graph  $H$  is complete,
3.     **return** 1;
4.   **if**  $|B| \geq 3$
5.     **return** 0;
6.   **if**  $\deg(b) \leq 6$  for every  $b \in B$
7.     **return** 1;
8.   **if**  $|A| \geq 14$
9.     **return** 0;
10. **if**  $H$  contains some of the graphs  $G(12, 7)$ ,  $G(11, 10)$ ,  $G(10, 9, 9)$ ,  $G(10, 9, 7)$ ,  $G(10, 9, 5)$ ,  $G(10, 7, k)$ ,  $k = 1, 2, \dots, 7$ ,  $G(9, 8, 2)$ ,  $G(9, 8, 1)$  as an induced subgraph
11.    **return** 0;
12.    **return** 1;
13. **end.**

The complexity of the algorithm in lines 1–9 is at most  $O(n)$ , where  $n = |V(H)|$ . Since the order of the graph  $H$  in line 10 is at most 13, this line takes  $O(1)$  time.

So, the total complexity of the recognition algorithm is  $O(n)$ .

### Acknowledgements

This work is partially supported by BRFFR (Projects F11OB-064 and F13MLD-012), and by DAAD.

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Received 26 February 2015

Revised 29 January 2016

Accepted 29 January 2016