

ALL TIGHT DESCRIPTIONS OF 3-STARS IN 3-POLYTOPES WITH GIRTH 5

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Abstract

Lebesgue (1940) proved that every 3-polytope P_5 of girth 5 has a path of three vertices of degree 3. Madaras (2004) refined this by showing that every P_5 has a 3-vertex with two 3-neighbors and the third neighbor of degree at most 4. This description of 3-stars in P_5 s is tight in the sense that no its parameter can be strengthened due to the dodecahedron combined with the existence of a P_5 in which every 3-vertex has a 4-neighbor.

We give another tight description of 3-stars in P_5 s: there is a vertex of degree at most 4 having three 3-neighbors. Furthermore, we show that there are only these two tight descriptions of 3-stars in P_5 s.

Also, we give a tight description of stars with at least three rays in P_5 s and pose a problem of describing all such descriptions. Finally, we prove a structural theorem about P_5 s that might be useful in further research.

Keywords: 3-polytope, planar graph, structure properties, k -star.

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1. INTRODUCTION

Let δ be the minimum vertex degree, and g be the girth (the length of a shortest cycle) in a given 3-polytope. We recall that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to Steinitz's famous theorem [22].

The degree of a vertex v or a face f , which is the number of edges incident with v or f in a 3-polytope, is denoted by $d(v)$ or $d(f)$, respectively. A k -vertex is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k , respectively. Hence, a k^+ -vertex v satisfies $d(v) \geq k$, etc.

Let \mathbf{P}_5 be the set of (finite) 3-polytopes with $g = 5$, and \mathbf{P}_5^* be the 3-polytopes with $\delta = 5$. We note that \mathbf{P}_5 and \mathbf{P}_5^* are in 1–1 correspondence due to the vertex-face duality, so structural results on \mathbf{P}_5 are easily translated to the language of \mathbf{P}_5^* and vice versa.

The early interest of researchers to the structure of \mathbf{P}_5^* was motivated by the Four Color Problem. Already in 1904, Wernicke [23] proved that every graph in \mathbf{P}_5^* contains a 5-vertex adjacent to a 6^- -vertex, and Franklin [12] in 1922 strengthened this to the existence of at least two 6^- -neighbors. Franklin's result is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

In 1940, Lebesgue [17] gave, in particular, an approximate description of the neighborhoods of 5-vertices in \mathbf{P}_5^* and proved that every 3-polytope in \mathbf{P}_5 has a 5-face incident with four 3-vertices and the fifth 5^- -vertex, which face includes a path of three 3-vertices. In 2004, Madaras [19] refined the last mentioned result by Lebesgue as follows.

Theorem 1 (Madaras [19]). *Every 3-polytope with girth 5 has a 3-vertex adjacent to two 3-vertices and another vertex of degree at most 4, which is tight.*

In dual terms, Theorem 1 reads equivalently as follows.

Theorem 2 (Madaras [19]). *Every 3-polytope with minimum degree 5 has a 3-face adjacent to two 3-faces and another face of degree at most 4, which is tight.*

Nowadays, a lot of structural results on \mathbf{P}_5 and \mathbf{P}_5^* can be found in the literature; for example, see [1–9, 13–16, 18–21].

We need a few definitions. A k -star $S_k(v; v_1, \dots, v_k)$ in a 3-polytope consists of the central vertex v and its neighbor vertices v_1, \dots, v_k , in no particular order. A k^+ -star has at least k rays. In this note, we deal with 3^+ -stars in \mathbf{P}_5 .

We say that $S_k(v; v_1, \dots, v_k)$ is an $(a; b_1, \dots, b_k)$ -star, or a star of type $(a; b_1, \dots, b_k)$, where $b_1 \geq \dots \geq b_k$, if $d(v) = a$ and $d(v_i) = b_i$ whenever $1 \leq i \leq k$.

A set $D = \{T_1, \dots, T_n\}$ of star-types is a *description* for \mathbf{P}_5 if every graph in \mathbf{P}_5 has a star of one of the types from D . A description D is *tight* if all descriptions $D - T_i$ with $1 \leq i \leq n$ are invalid, which means that for every i there is a graph in \mathbf{P}_5 that has no stars of types from $D - T_i$, but it has a star of type T_i .

Madaras [19] constructed a polytope in \mathbf{P}_5 in which every 3-vertex has at least one 4^+ -neighbor (see Figure 1). In what follows, this construction is called M_{04} . The tightness of the description of 3-stars in Theorem 1 is implied by the dodecahedron (which has no $(3; 4, 3, 3)$ -stars) together with M_{04} .

One of the purposes of our paper is to augment Theorem 1 by giving another tight description of 3-stars in \mathbf{P}_5 .

Theorem 3. *Every 3-polytope with girth 5 has a vertex of degree at most 4 having three 3-neighbors, which is a tight description of 3-stars in \mathbf{P}_5 .*

Here, the tightness follows from the facts that the dodecahedron has no stars of type $(4; 3, 3, 3)$, while M_{04} avoids the $(3; 3, 3, 3)$ -star.

Our next result is that there are only two tight descriptions of 3-stars in \mathbf{P}_5 .

Theorem 4. *In \mathbf{P}_5 , there are precisely two tight descriptions of 3-stars:*

- (a) $D_{04} = \{(3; 3, 3, 3), (3; 4, 3, 3)\}$, given by Theorem 1 (Madaras [19]), and
- (b) $D_{15} = \{(3; 3, 3, 3), (4; 3, 3, 3)\}$, given by Theorem 3.

A 3-vertex is *weak* if it has two 3-neighbors and a 4-neighbor. For further attempts to find tight descriptions of 3^+ -stars in \mathbf{P}_5 , the following structural result seems useful.

Theorem 5. *Every 3-polytope of girth 5 has one of the following configurations (see Figure 2):*

- (a) a 3-vertex with three 3-neighbors;
- (b) a 4-vertex with four 3-neighbors, at least one of which is weak;
- (c) a 4-vertex with a 4-neighbor and three 3-neighbors, at least two of which are weak.

It is easy to see that Theorem 5 implies Theorems 1 and 3, as well as the next fact.

Corollary 6. *The following tight descriptions of 3^+ -stars in \mathbf{P}_5 hold:*

- (i) $D_{04} = \{(3; 3, 3, 3), (3; 4, 3, 3)\}$ (Madaras [19]);
- (ii) $D_{15} = \{(3; 3, 3, 3), (4; 3, 3, 3)\}$;
- (iii) $D_1 = \{(3; 3, 3, 3), (4; 3, 3, 3, 3), (4; 4, 3, 3, 3)\}$.

The tightness of D_{04} and D_{15} follows from the dodecahedron combined with M_{04} . In Figure 3, we see a half of a graph H_1 whose every vertex has a 4-neighbor. (Note that in M_{04} only each 3-vertex has a 4-neighbor.) So the type $(4; 4, 3, 3, 3)$ cannot be dropped from D_1 , while the first and second types cannot be dropped due to the dodecahedron and M_{04} , respectively.

It looks like the following tempting problem is hard.

Problem 7. Find all tight descriptions of 3^+ -stars in \mathbf{P}_5 .

In fact, we were not able to solve even the following two much more modest problems.

Problem 8. Is it true that $D_2 = \{(3; 3, 3, 3), (3; 4, 4, 4), (4; 3, 3, 3, 3)\}$ is a tight description of 3^+ -stars in \mathbf{P}_5 ?

We note that if D_2 is a description, then it is tight due to the dodecahedron, H_1 , and M_{04} .

Problem 9. Is it true that $\{(3; 3, 3, 3), (3; 4, 4, 3), (3; 4, 4, 4), (4; 3, 3, 3, 3), (4; 4, 4, 3, 3), (4; 4, 4, 4, 3)\}$ is a description of 3^+ -stars in \mathbf{P}_5 ?

In Section 2, we illustrate the constructions M_{04} and H_1 and configurations in Theorem 5. Section 3 contains proofs of Theorems 5 and 4. Note that Theorem 5 implies Theorem 1 and is proved shorter than Theorem 1 in Madaras [19].

2. CONSTRUCTIONS M_{04} AND H_1 AND CONFIGURATIONS IN THEOREM 5

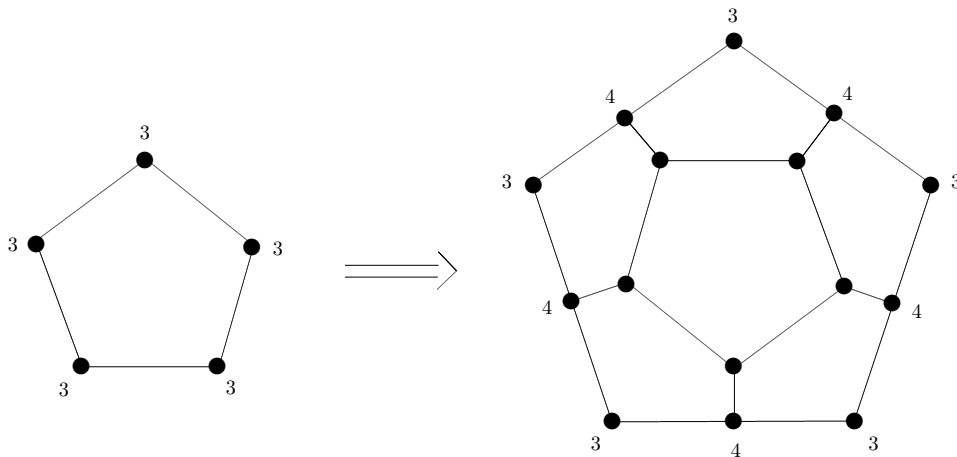


Figure 1. (Madaras [19]) M_{04} : every 3-vertex has a 4-neighbor.

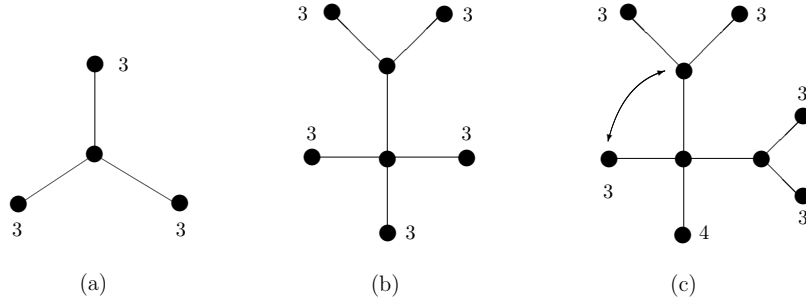


Figure 2. Configurations in Theorem 5.

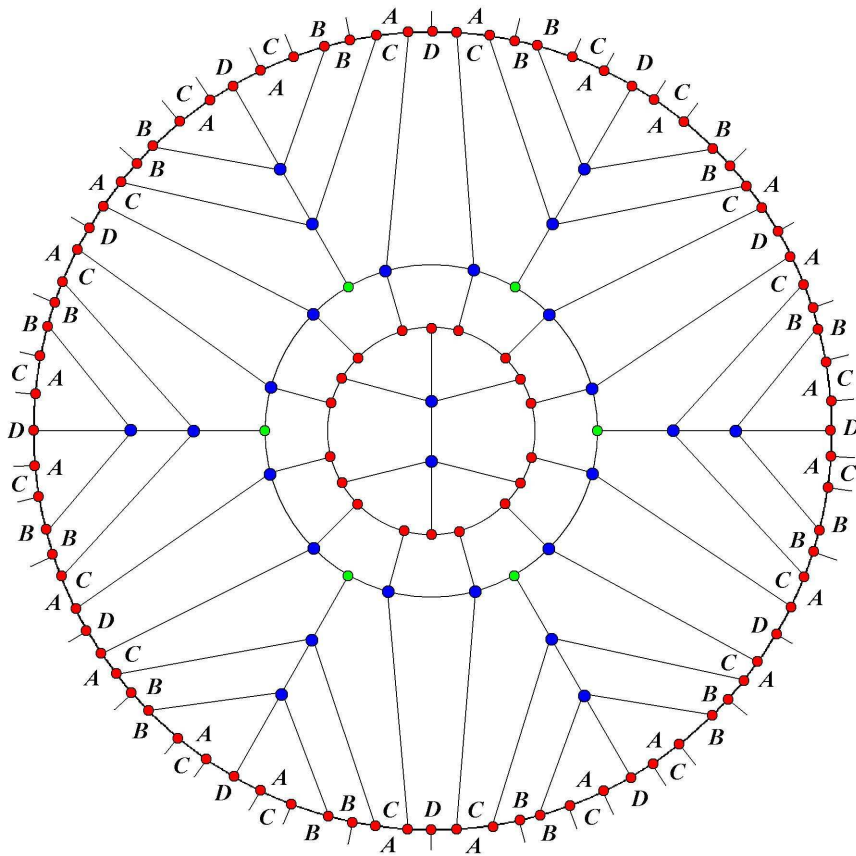


Figure 3. (A half of) H_1 : every vertex has a 4-neighbor.

3. PROOFS

3.1. Proving Theorem 5

Suppose that P is a counterexample to Theorem 5. Euler's formula $|V| - |E| + |F| = 2$ for P may be written as

$$(1) \quad \sum_{v \in V} \left(\frac{3d(v)}{2} - 5 \right) \leq \sum_{v \in V} \left(\frac{3d(v)}{2} - 5 \right) + \sum_{f \in F} (d(f) - 5) = -10,$$

where V , E , and F are the sets of vertices, edges and faces of P , respectively.

Let us assign a *charge* $\mu(v) = \frac{3d(v)}{2} - 5$ to every vertex v in V , so that the charge of vertices, depending on their degree, is $-\frac{1}{2}$, 1 , $\frac{5}{2}$, and so on. Using the properties of P as a counterexample, we define a local redistribution of μ 's, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is at most -10 , according to (1). Our rules of discharging are:

R1. Every 3-vertex v receives from every adjacent 4^+ -vertex either $\frac{1}{2}$ if v is weak or $\frac{1}{4}$ otherwise.

R2. Every 4-vertex receives $\frac{1}{2}$ from every adjacent 5^+ -vertex.

To complete the proof of Theorem 5, we first observe that every 3-vertex v receives from its 4^+ -neighbors either $\frac{1}{2}$ if v is weak, or at least $2 \times \frac{1}{4}$ otherwise, so $\mu'(v) \geq -\frac{1}{2} + \frac{1}{2} = 0$.

Now if $d(v) = 4$, then $\mu(v) = 1$, and we are easily done unless v has at least three 3-neighbors but no 5^+ -neighbors. If so, then v either gives $4 \times \frac{1}{4}$ to its four 3-neighbors, or at most $\frac{1}{2} + 2 \times \frac{1}{4}$ to its three 3-neighbors, which yields $\mu'(v) \geq 1 - 1 = 0$.

Finally, for $d(v) \geq 5$ we have $\mu'(v) \geq \frac{3d(v)}{2} - 5 - d(v) \times \frac{1}{2} = d(v) - 5 \geq 0$, as desired.

3.2. Proving Theorem 4

Suppose that D is a tight description of 3-stars for \mathbf{P}_5 . Since the dodecahedron has 3-stars only of the type $(3; 3, 3, 3)$, it follows that $(3; 3, 3, 3) \in D$. Now we look at the graph M_{04} ; it has 3-stars only of the types $(3; 4, 3, 3)$ and $(4; 3, 3, 3)$. As M_{04} obeys D , at least one of these types should appear in D .

Case 1. $(3; 4, 3, 3) \in D$. Since $D' = \{(3; 3, 3, 3), (3; 4, 3, 3)\}$ is a tight description by Theorem 1, we have $D = \{(3; 3, 3, 3), (3; 4, 3, 3)\}$ due to the minimality of D .

Case 2. $(4; 3, 3, 3) \in D$. Since $D' = \{(3; 3, 3, 3), (4; 3, 3, 3)\}$ is a tight description by Theorem 3, we have $D = \{(3; 3, 3, 3), (4; 3, 3, 3)\}$, as desired.

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