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A NOTE ON NON-DOMINATING SET PARTITIONS IN GRAPHS

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Abstract

A set S of vertices of a graph G is a dominating set if every vertex not in S is adjacent to a vertex of S and is a total dominating set if every vertex of G is adjacent to a vertex of S. The cardinality of a minimum dominating (total dominating) set of G is called the domination (total domination) number. A set that does not dominate (totally dominate) G is called a non-dominating (non-total dominating) set of G. A partition of the vertices of G into non-dominating (non-total dominating) sets is a non-dominating (non-total dominating) set partition. We show that the minimum number of sets in a non-dominating set partition of a graph G equals the total domination number of its complement $\overline{G}$ and the minimum number of sets in a non-total dominating set partition of G equals the domination number of $\overline{G}$. This perspective yields new upper bounds on the domination and total domination numbers. We motivate the study of these concepts with a social network application.

Keywords: domination, total domination, non-dominating partition, non-total dominating partition.

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1. Introduction

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. The degree of a vertex v is $|N(v)|$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of the vertices of G, respectively. For a graph G of order n, a vertex of degree $n-1$ is called a *universal vertex*. For any $S \subseteq V$, we denote the subgraph of G induced by S as $G[S]$. The *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood of a set* S is the set $N[S] = N(S) \cup S = \bigcup_{v \in S} N[v]$. The S*-closed private neighborhood* of v, denoted by $\text{pn}[v, S]$, consists of all vertices in the closed neighborhood of v but not in $N[S \setminus \{v\}]$. The S*-open private neighborhood* of v, denoted by $\text{pn}(v, S)$, consists of all vertices in the open neighborhood of v but not in $N(S \setminus \{v\})$. We use the standard notation $[k] = \{1, \ldots, k\}$.

A set S of vertices of G is a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to at least one vertex in S, that is, $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of any dominating set of G. A set S of vertices of G is a *total dominating set* of G if every vertex in V is adjacent to at least one vertex in S, that is, $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of any total dominating set of G. If S is a dominating set of G, we simply write that S *dominates* G, while if S is a total dominating set of G, we write that S *totally dominates* G. For more details on domination and total domination, we refer the reader to the books [2, 3, 7].

A set that does not dominate G is called a *non-dominating set* of G. In other words, for any non-dominating set S, there exists a vertex in $V \setminus S$ that has no neighbor in S. A partition of the vertices of G into non-dominating sets is a *non-dominating set partition*. We let $\psi(G)$ be the minimum cardinality of a non-dominating set partition of G. A set that does not totally dominate G is called a *non-total dominating* set of G. In other words, for any non-total dominating set S, either S is a non-dominating set or there is an isolate vertex in $G[S]$. *Non-total dominating set partitions* are defined as expected, and we let $\psi_t(G)$ denote the minimum cardinality of a non-total dominating set partition of G.

A partition of the vertices of a graph into dominating sets is called a *domatic partition* and has been well-studied in the literature, for example, see [4, 6, 8, 9, 10, 11]. However, as far as we know, non-dominating partitions have not been previously investigated, so we initiate their study here.

Let $\pi = \{A_1, A_2, \ldots, A_k\}$ be a non-dominating set partition of G with minimum cardinality $\psi(G) = k$. We note that for each $i \in [k]$, there exists at least one vertex $v_i \in V \setminus A_i$ with no neighbor in A_i. Moreover, since π has minimum cardinality, v_i is dominated by the set A_j for every $j \in [k] \setminus \{i\}$; otherwise, the partition π' formed from π by removing A_i and A_j and adding $A_i \cup A_j$ is a non-dominating set partition of G with cardinality less than $\psi(G)$, a contradiction. Hence, the vertices v_i are distinct for each $i \in [k]$.

If the domination number of a graph G is one, then clearly, G has a universal vertex, and hence, no non-dominating set partition. Moreover, notice that the identity partition, $\pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ is a non-dominating set partition for every graph G with $\gamma(G) \geq 2$. Also, since V is a dominating set of G, any non-dominating set partition must have a least two sets. Hence, we have the following observation.

Observation 1. *A graph G of order n has a non-dominating set partition if and only if G has no universal vertex. Further, if G has no universal vertex, then $2 \leq \psi(G) \leq n$.*

For an application, consider a factory with a large number of employees and a need to implement a quality assurance checking system of their workers. The factory manager decides to designate an internal committee to do this, that is, the manager will select a subset of the workers to form a quality assurance team to inspect the work of their co-workers. The manager wants to keep this team as small as possible in order to minimize costs (inspectors' extra pay) and to protect privacy (keep the identity of inspectors secret). To avoid bias, an inspector should neither be close friends nor enemies with any of the workers he/she is responsible for inspecting. To model the situation, a social network graph can be constructed, where each worker is represented by a vertex and an edge between two workers represents possible bias, that is, if the two workers are either close friends or enemies. Ideally, an inspector should not be adjacent to any worker under his/her inspection. If we desire a situation where every worker including the inspectors themselves has his/her work inspected, then the minimum cardinality of a non-dominating set partition of G gives the minimum number of inspectors needed. Such a partition provides that a vertex v_i exists outside of each set A_i of the partition that is a qualified inspector for A_i, that is, v_i is not adjacent to any vertex in A_i. Thus, selecting one v_i for each A_i gives $\psi(G)$ inspectors.

If we are not concerned whether the inspectors' work is inspected, then the minimum cardinality of a non-total dominating set partition of G gives the minimum number of inspectors needed. Such a partition provides that a vertex exists either inside or outside of each set of the partition that is not adjacent to any other vertex in the set. As we will see in the closing of this article, not insisting that the work of the inspectors themselves is inspected can result in as much as a fifty percent savings in the cost of hiring inspectors.

Our aim in this paper is twofold. Our first aim is to show that for a graph G and its complement $\overline{G}$, $\psi(G) = \gamma_t(\overline{G})$ and $\psi_t(G) = \gamma(\overline{G})$. Our second aim is to show that using these identities, we can establish new upper bounds on the domination and total domination numbers of the complement $\overline{G}$ of a graph G, given the minimum and maximum degree of G.

2. Main Results

We first show that the minimum order of a non-dominating set partition of G is the total domination number of its complement $\overline{G}$.

Theorem 2. *If G is a graph with no universal vertex, then $\psi(G) = \gamma_t(\overline{G})$.*

Proof. Let G be a graph with no universal vertex, and let $\pi = \{A_1, A_2, \ldots, A_k\}$ be a non-dominating set partition of G with cardinality $\psi(G) = k$. By Observation 1, $2 \le \psi(G) \le n$. We first show that $\gamma_t(\overline{G}) \le \psi(G)$. Since π is a non-dominating set partition of G, the set A_i does not dominate G for each $i \in [k]$, implying that there exists a vertex $a_i \in V \setminus A_i$ such that $N(a_i) \cap A_i = \emptyset$. Note that $A = \bigcup_{i=1}^{k}\{a_i\}$ is a dominating set of $\overline{G}$ of cardinality k. If A is a total dominating set of $\overline{G}$, then $\gamma_t(\overline{G}) \le k$. If not, then it follows that there exists an $a_i \in A$ such that a_i is an isolated vertex of $\overline{G}[A]$. By our selection of a_i, it follows that $a_i \notin A_i$. Hence, $a_i \in A_j$ for some $j \in [k] \setminus \{i\}$. Since the vertex $a_j \in A$ has no neighbor in A_j in G, we note, in particular, that a_j is not adjacent to a_i in G. Hence, in $\overline{G}$, a_i and a_j are adjacent, contradicting the fact that a_i is an isolate in $\overline{G}[A]$. Thus, A is a total dominating set for $\overline{G}$, and so $\gamma_t(\overline{G}) \le |A| = k = \psi(G)$.

Since G has no universal vertex, $\overline{G}$ has no isolated vertex, that is, the total domination number of $\overline{G}$ is defined. To see that $\gamma_t(\overline{G}) \ge \psi(G)$, let $S = \{v_1, v_2, \ldots, v_\ell\}$ be a total dominating set of $\overline{G}$ with $\ell = \gamma_t(\overline{G})$. For $i \in [\ell]$, let $B_i = N_{\overline{G}}(v_i)$. Since S is a total dominating set of $\overline{G}$, every vertex of V belongs to some B_i. Moreover, since S is a minimum total dominating set, $\mathrm{pn}(v_i, S) \neq \emptyset$ and $\mathrm{pn}(v_i, S) \subseteq B_i$ for each $i \in [\ell]$. We partition the vertices of V as follows: let $B'_1 = B_1$. For each $j \ge 2$, form B'_j by removing the vertices from B_j that are contained in $\bigcup_{i=1}^{j-1} B_i$. Note that $\mathrm{pn}(v_i, S) \subseteq B'_i$, and so $B'_i \neq \emptyset$ for $i \in [\ell]$. Note also that the vertex $v_i \notin B'_i$ and v_i is not dominated by B'_i in G for $i \in [\ell]$. Hence, each B'_i is a non-dominating set of G. Thus, $\pi = \{B'_1, B'_2, \ldots, B'_\ell\}$ is a partition of V into non-dominating sets of G, implying that $\psi(G) \le |\pi| = \ell = \gamma_t(\overline{G})$. Consequently, $\gamma_t(\overline{G}) = \psi(G)$. ■

As a consequence of Theorem 2, we have the following upper bounds on the total domination number of a graph.

Corollary 3. *Let G be any graph of order n with no universal vertex. If k is the smallest positive integer such that $n > 1 + \Delta(G)\left\lceil \frac{\delta(G)}{k} \right\rceil$, then $\gamma_t(\overline{G}) \le k + 2$ and this bound is sharp.*

Proof. Let k be the smallest positive integer such that $n > 1 + \Delta(G) \left\lceil \frac{\delta(G)}{k} \right\rceil$. Note that $k \le \delta(G)$. Suppose, for purposes of contradiction, that $\gamma_t(\overline{G}) > k + 2$. By Theorem 2, $\psi(G) = \gamma_t(\overline{G}) > k+2$. Let v be a vertex of minimum degree in G and $B = V \setminus N[v]$. Since G has no universal vertex, that is, $\delta(G) \le \Delta(G) \le n-2$, we have that $B \ne \emptyset$. Let $\delta(G) \equiv x \pmod k$, where $0 \le x \le k-1$, and let $\pi = \{A_1, A_2, \ldots, A_k\}$ be a partition of $N(v)$ such that $|A_i| = \left\lceil \frac{\delta(G)}{k} \right\rceil$ for $i \in [x]$ and $|A_i| = \left\lfloor \frac{\delta(G)}{k} \right\rfloor$ for $i \in [k] \setminus [x]$. If for every $i \in [k]$ the set A_i does not dominate G, then $\pi' = \pi \cup \{B, \{v\}\}$ is a non-dominating set partition of G, implying that $\psi(G) \le k + 2$, a contradiction. Hence, for some $i \in [k]$, the set A_i dominates G. We note that A_i can dominate at most $(\Delta(G) - 1)|A_i|$ vertices of $V \setminus (A_i \cup \{v\})$. Thus, $n \le 1 + |A_i| + (\Delta(G) - 1)|A_i| = 1 + \Delta(G)|A_i| \le 1 + \Delta(G) \left\lceil \frac{\delta(G)}{k} \right\rceil$, a contradiction to our supposition. Hence, $\gamma_t(\overline{G}) \le k + 2$. This establishes the desired upper bound.

That the upper bound is sharp, may be seen as follows. For $r \ge 2$, let G be obtained from a complete graph K_{2r} of order $n = 2r$ by removing the edges of a perfect matching. Then, G is an $(n-2)$-regular graph, and so $\delta(G) = \Delta(G) = n-2$. Further, the smallest positive integer k such that $n > 1 + \Delta(G) \left\lceil \frac{\delta(G)}{k} \right\rceil = 1 + (n-2) \left\lceil \frac{n-2}{k} \right\rceil$ is $k = n-2$, implying, by Corollary 3, that $\gamma_t(\overline{G}) \le k + 2 = n$. However, $\overline{G} = rK_2$ consists of r vertex disjoint copies of K_2, and so $\gamma_t(\overline{G}) = n = k + 2$. ■

A partitioning of the vertices of a graph G into independent sets is called a *proper coloring* of the vertices of G. The cardinality of a minimum proper coloring of G is the *chromatic number* of G and is denoted by $\chi(G)$. A complete subgraph of G is called a *clique*, and a clique whose vertices dominate G is called a *dominating clique* in G.

Corollary 4. *If a graph G has no dominating clique, then $\gamma_t(\overline{G}) \le \chi(\overline{G})$.*

Proof. Let G be a graph with no dominating clique. In particular, we note that G has no universal vertex, and so, by Theorem 2, $\gamma_t(\overline{G}) = \psi(G)$. Let π be (minimum) proper coloring of $\overline{G}$ using $\chi(\overline{G})$ colors. Each color class of π is a clique in G. By our assumption that G has no dominating clique, the partition π is therefore a non-dominating set partition of G, implying that $\gamma_t(\overline{G}) = \psi(G) \le \chi(\overline{G})$. ■

That the upper bound on the total domination number established in Corollary 4 is sharp, may be seen by taking, for example, the graph $G = K_r \cup K_s$, where r and s are positive integers. Since G is disconnected, it has no dominating clique. The complement, $\overline{G}$, of G is the complete bipartite graph $K_{r,s}$, implying that $\gamma_t(\overline{G}) = 2 = \chi(\overline{G})$. Therefore, the upper bound in Corollary 4 is sharp.

Using similar proof techniques to the ones used to prove Theorem 2 and Corollary 3, we obtain the following analogous results for the domination number.

Theorem 5. *For any graph G, $\psi_t(G) = \gamma(\overline{G})$.*

Proof. Let $\pi = \{A_1, A_2, \ldots, A_k\}$ be a non-total dominating set partition of G with cardinality $\psi_t(G) = k$. We first show that $\gamma(\overline{G}) \le \psi_t(G)$. Since π is a non-total dominating set partition of G, there exists some vertex $a_i \in V$ for each $i \in [k]$ such that $N(a_i) \cap A_i = \emptyset$. Note that $A = \bigcup_{i=1}^{k}\{a_i\}$ is a dominating set of $\overline{G}$ of cardinality k. Hence, $\gamma(\overline{G}) \le |A| = k = \psi_t(G)$.

To see that $\gamma(\overline{G}) \ge \psi_t(G)$, let $S = \{v_1, v_2, \ldots, v_\ell\}$ be a dominating set of $\overline{G}$ with $\ell = \gamma(\overline{G})$. For $i \in [\ell]$, let $B_i = N_{\overline{G}}[v_i]$. Since S is a dominating set of $\overline{G}$, every vertex of V belongs to some B_i. Moreover, since S is a minimum dominating set of $\overline{G}$, $\mathrm{pn}[v_i, S] \neq \emptyset$ and $\mathrm{pn}[v_i, S] \subseteq B_i$ for each $i \in [\ell]$. We partition the vertices of V as follows: let $B_1' = B_1$. For each $j \ge 1$, form B_j' by removing the vertices from B_j that are contained in $\bigcup_{i=1}^{j-1} B_i$. Note that $\mathrm{pn}[v_i, S] \subseteq B_i'$, and so $B_i' \neq \emptyset$ for $i \in [\ell]$. Thus, $\pi = \{B_1', B_2', \ldots, B_\ell'\}$ is a partition of V. Note further that $N(v_i) \cap B_i = \emptyset$ for each $i \in [\ell]$. Hence, π is a non-total dominating set partition of V and $\psi_t(G) \le \ell = \gamma(\overline{G})$. Consequently, $\gamma(\overline{G}) = \psi_t(G)$. ■

Corollary 6. *Let G be any graph of order n. If k is the smallest positive integer such that $n > 1 + (\Delta(G) - 1)\left\lceil \frac{\delta(G)}{k} \right\rceil$, then $\gamma(\overline{G}) \le k + 1$ and this bound is sharp.*

Proof. Let k be the smallest positive integer such that $n > 1 + (\Delta(G) - 1)\left\lceil \frac{\delta(G)}{k} \right\rceil$. Note that $k \le \delta(G)$. Suppose, for purposes of contradiction, that $\gamma(\overline{G}) > k + 1$. By Theorem 5, $\psi_t(G) = \gamma(\overline{G}) > k + 1$. Let v be a vertex of minimum degree in G and $B = V \setminus N[v]$. Let $\delta(G) \equiv x \pmod{k}$, where $0 \le x \le k - 1$, and let $\pi = \{A_1, A_2, \ldots, A_k\}$ be a partition of $N(v)$ such that $|A_i| = \left\lceil \frac{\delta(G)}{k} \right\rceil$ for $i \in [x]$ and $|A_i| = \left\lfloor \frac{\delta(G)}{k} \right\rfloor$ for $i \in [k] \setminus [x]$. If for every $i \in [k]$ the set A_i does not totally dominate G, then $\pi' = \pi \cup \{B \cup \{v\}\}$ is a non-total dominating set partition of G, implying that $\psi_t(G) \le k + 1$, a contradiction. Hence, for some $i \in [k]$, the set A_i totally dominates G. Each vertex in A_i is adjacent to the vertex v and to at least one vertex in A_i, and is therefore adjacent to at most $(\Delta(G) - 2)|A_i|$ vertices of $V \setminus (A_i \cup \{v\})$. Thus, $n \le 1 + |A_i| + (\Delta(G) - 2)|A_i| = 1 + \Delta(G)|A_i| - |A_i| \le 1 + (\Delta(G) - 1)\left\lceil \frac{\delta(G)}{k} \right\rceil$, a contradiction to our supposition. Hence, $\gamma(\overline{G}) \le k + 1$. This establishes the desired upper bound. The upper bound is sharp, as may be seen by taking G to be a cycle C_n, where $n \ge 4$. The smallest positive integer k such that $n > 1 + (\Delta(G) - 1)\left\lceil \frac{\delta(G)}{k} \right\rceil = 1 + \left\lceil \frac{2}{k} \right\rceil$ is $k = 1$, implying, by Corollary 6, that $\gamma(\overline{G}) \le k + 1 = 2$. However, $\gamma(\overline{G}) = 2 = k + 1$. ■

3. Concluding Remarks

In this paper, we introduce the concepts of non-dominating set partitions and non-total dominating set partitions, and show that they give us a new perspective on domination and total domination in graphs. Further, using these concepts, we establish new upper bounds on the domination and total domination numbers.

Returning to our factory example, it can be seen that if one desired to hire a minimum number of non-biased inspectors such that everyone's work is inspected, then it is necessary to appoint $\gamma_t(\overline{G})$ inspectors. If management does not insist that the work of the inspectors themselves be subject to inspection, then only $\gamma(\overline{G})$ inspectors need be appointed. As first observed by Bollobás and Cockayne [1], if G is an isolate-free graph, then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$. We remark that there are infinitely many (connected) graphs G satisfying $\gamma_t(G) = 2\gamma(G)$, as shown, for example, in [5]. Hence, depending on the properties of the complement $\overline{G}$ of the social network graph for our hypothetical factory, it may be possible to hire as few as 1/2 the number of inspectors if management loosens the restriction that every inspector's work is also examined by an inspector.

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