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A NOTE ON PATH DOMINATION

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Abstract

We study domination between different types of walks connecting two non-adjacent vertices u and v of a graph (shortest paths, induced paths, paths, tolled walks). We succeeded in characterizing those graphs in which every uv-walk of one particular kind dominates every uv-walk of other specific kind. We thereby obtained new characterizations of standard graph classes like chordal, interval and superfragile graphs.

Keywords: domination, paths, geodesics, chordal graphs, interval graphs.

2010 Mathematics Subject Classification: 05C38, 05C75, 05C69, 05C12.

1. Introduction

An *interval representation* of a graph G is a family $(I_w)_{w\in V(G)}$ of intervals of the real line satisfying that two vertices of G are adjacent if and only if the corresponding intervals have nonempty intersection. Graphs admitting an interval representation are called *interval graphs* [2, 9, 16]. A simple idea arising from the topology of the line is that if P and P' are induced paths between two non-adjacent vertices of an interval graph, then every internal vertex of P is adjacent to some internal vertex of P', and vice versa. This property is not enough to characterize interval graphs, a counterexample is the graph F_2 in Figure 1 which is not an interval graph. We wonder if interval graphs can be characterized in terms of domination between paths. In a wider sense, we are interested in understanding the structure of those graphs in which for every pair of non-adjacent vertices u and v, and every pair of uv-walks W and W', each internal vertex of W' is adjacent to some internal vertex of W.

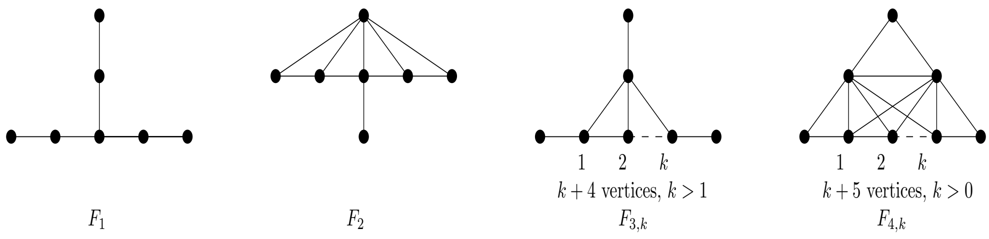


Figure 1. Chordal forbidden induced subgraphs for interval graphs.

Inspired by such ideas we studied domination between different types of walks connecting two non-adjacent vertices u and v of a graph G, not necessarily interval. We succeeded in characterizing the graphs in which every uv-walk of one particular kind inside tolled walks (which are introduced in the present work), paths, induced paths or shortest paths dominates every uv-walk of other specific kind. We thereby obtained new characterizations of standard graph classes like chordal, interval and superfragile graphs [2].

In the context of convexity theory, several graph convexity spaces arise when intervals are defined using different types of walks: geodesic convexity, monophonic convexity [14], all-paths convexity [3], triangle-path convexity [4], longest-path convexity [5], and others [10]. As a by-product, we prove that every geodesic interval (monophonic interval) of a graph G is chordal if and only if in G there exists domination between shortest paths (induced paths).

The main results are stated and proved in Section 3. Conclusions and some remarks on related topics that may be motivating for future works are developed in Section 4.

2. Definitions and Basic Results

Let G be a finite, simple and connected graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. We write $N(v)$ for the set of *neighbors* of the vertex v and $N[v]$ for the *closed neighborhood*. A *clique* is a subset of pairwise adjacent vertices. A vertex v is *simplicial* if $N(v)$ is a clique. The subgraph induced in G by a subset $S \subseteq V(G)$ is denoted by $G[S]$.

A *walk* in G is a sequence $W : v_1, v_2, \ldots, v_k$ whose terms are vertices of G, not necessarily distinct, such that v_i is adjacent to v_{i+1} for $i \in \{1, 2, \ldots, k-1\}$. If $v_1 = u$ and $v_k = v$, we say that W connects u to v and refer to W as an *uv-walk*. The vertices u and v are called the *ends* of the walk; the vertices $v_2, v_3, \ldots, v_{k-1}$ are its *internal vertices*. The integer $k-1$ is the *length* of the walk. The *distance* $d(u, v)$ between the vertices u and v is the length of a shortest uv-walk.

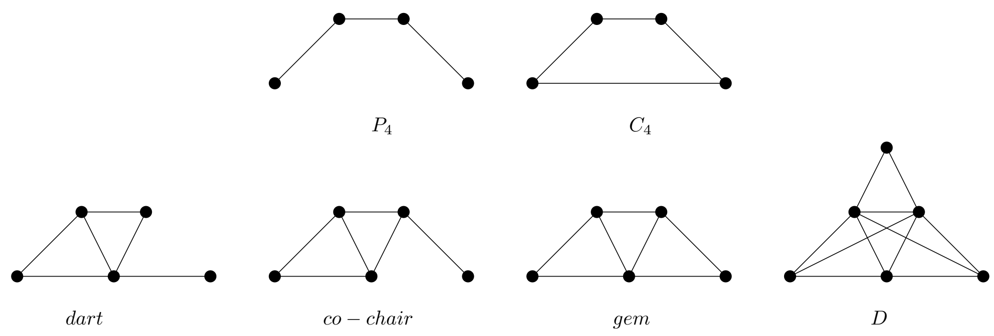


Figure 2. Graphs used to describe the graph classes considered in our results. They are named as in [2].

An *uv-tolled-walk* is an uv-walk satisfying that the only internal vertex adjacent to v_1 is v_2 and the only internal vertex adjacent to v_k is v_{k-1}. An *uv-path* is an uv-walk with all its vertices distinct. An *uv-induced-path* (or *chordless path*, or *monophonic path*) is an uv-path such that two of its vertices are adjacent if and only if they are consecutive. The chordless path of length k is denoted by P_k. An *uv-shortest-path* (or *geodesic*) is an uv-path of length $d(u, v)$.

Notice that every shortest-path is an induced-path and every induced-path is a tolled-walk. However, paths and tolled-walks are incomparable: the sequence $u, 4, 2, 3, v$ of vertices of F_2 in Figure 4 is an uv-path which is not an uv-tolled-walk. In the same graph, the sequence $u, 1, 2, w, 2, 3, v$ is an uv-tolled-walk that is not an uv-path.

The walk $W : v_1, v_2, \ldots, v_k$ *contains* the walk $W' : v'_1, v'_2, \ldots, v'_\ell$ if there exists an strict increasing function $\Phi : \{1, 2, \ldots, \ell\} \to \{1, 2, \ldots, k\}$ such that $v'_i = v_{\Phi(i)}$ for $1 \leq i \leq \ell$. Notice that this is a transitive relation between walks.

It is well known that every uv-walk contains some uv-path and that every uv-path contains some uv-induced-path [17]. However, not every uv-induced-path contains some uv-shortest-path.

Definition. The uv-walk $W : v_1, v_2, \ldots, v_k$ *dominates* the uv-walk $W' : v'_1, v'_2, \ldots, v'_\ell$ if every internal vertex of W' is adjacent to some internal vertex of W or belongs to W, i.e., for every $i \in \{2, \ldots, \ell - 1\}$ there exists $j \in \{2, \ldots, k - 1\}$ such that either v'_i is adjacent to v_j or $v'_i = v_j$.

In order to simplify the statement of the main results in the next section, we introduce the following notation.

$\mathbf{W_1}(u, v) = \{W : W$ is an uv-shortest-path$\}$,
$\mathbf{W_2}(u, v) = \{W : W$ is an uv-induced-path$\}$,
$\mathbf{W_3}(u, v) = \{W : W$ is an uv-path$\}$,

$\widehat{\mathbf{W_3}}(u, v) = \{W : W \text{ is an } uv\text{-tolled-walk}\}$,
$\mathbf{W_4}(u, v) = \{W : W \text{ is an } uv\text{-walk}\}$.

The following two remarks summarize the relation between the different types of walks we have considered.

Remark 1.

$$\mathbf{W_1}(u, v) \subseteq \mathbf{W_2}(u, v) \subseteq \mathbf{W_3}(u, v) \subseteq \mathbf{W_4}(u, v).$$

$$\mathbf{W_1}(u, v) \subseteq \mathbf{W_2}(u, v) \subseteq \widehat{\mathbf{W_3}}(u, v) \subseteq \mathbf{W_4}(u, v).$$

Remark 2. If $W \in \mathbf{W_4}(u, v)$, then W contains some $W' \in \mathbf{W_2}(u, v)$.

A *cycle* of *length* k in a graph G is a path $C : v_1, v_2, \ldots, v_k$ plus and edge between v_1 and v_k. The edges v_iv_{i+1} for $i \in \{1, 2, \ldots, k-1\}$ and v_kv_1 are the edges of the cycle; any other edge of G between two vertices of C is called a *chord*. The cycle of length k without chords is denoted by C_k.

Chordal graphs, defined as those graphs in which every cycle of length greater than three has a chord, have been widely study and admit different characterizations. As intersection graphs, chordal graphs are described as the graphs admitting a representation by subtrees of a tree. Thus, clearly, every interval graph is chordal. In terms of vertex elimination orders, chordal graphs are seen as those graphs whose vertices can be totally ordered $v_1, v_2, \ldots, v_n$ in such a way that every v_i is a simplicial vertex of $G[\{v_i, v_{i+1}, \ldots, v_n\}]$. See [2, 9] for more on interval graphs, chordal graphs and related classes of graphs.

A *distance-hereditary* graph is a graph in which every induced path is a geodesic [6]. A graph is *Ptolemaic* if for every four vertices v_1, v_2, v_3 and v_4,

$$d(v_1, v_2) \cdot d(v_3, v_4) \leq d(v_1, v_3) \cdot d(v_2, v_4) + d(v_1, v_4) \cdot d(v_2, v_3).$$

In [7], it was proved that Ptolemaic graphs are exactly the distance-hereditary chordal graphs. The graph F_1 in Figure 1 is Ptolemaic but it is not interval. The graph *gem* in Figure 2 is an interval graph but it is not Ptolemaic.

A graph is *superfragile* [2, 15] if it has a vertex elimination order with respect to the two rules below, such that at each stage every vertex is eligible for elimination. Recall that P_3 denotes the induced path with three vertices.

Rule 1. If v does not appear as an end vertex in an induced P_3, then v may be removed.

Rule 2. If v does not appear as an internal vertex in an induced P_3, then v may be removed.

Chordal, interval, Ptolemaic and superfragile graphs have been characterized by forbidden induced subgraphs.

Theorem 3. *A graph is chordal if and only if it does not contain a chordless cycle C_k with $k \geq 4$ as induced subgraph.*

Theorem 4 [8]**.** *A graph is interval if and only if it is chordal and it does not contain any one of the graphs F_1, F_2, $F_{3,k}$ or $F_{4,k}$ in Figure* 1 *as induced subgraphs.*

Theorem 5 [7]**.** *A graph is Ptolemaic if and only if it is chordal and it does not contain the graph gem in Figure* 2 *as induced subgraph.*

Theorem 6 [15]**.** *A graph is superfragile if and only if it contains none of the graphs C_4, P_4 or dart in Figure* 2 *as induced subgraph.*

Denote by **Chordal**, **Interval**, and **Superfragile** to the classes of chordal, interval and superfragile graphs, respectively.

3. Main Results

Let G be any graph and $i, j \in \{1, 2, 3, 4\}$. We say that $G \in \mathbf{W_i}/\mathbf{W_j}$ if for every pair of non-adjacent vertices u and v of G, every $W \in \mathbf{W_i}(u, v)$ dominates every $W' \in \mathbf{W_j}(u, v)$, i.e.,

$$W \in \mathbf{W_i}(u, v) \text{ and } W' \in \mathbf{W_j}(u, v) \text{ implies } W \text{ dominates } W'.$$

In an analogous way, we define $\mathbf{W_i}/\widehat{\mathbf{W}}_\mathbf{3}$ and $\widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_j}$.

The aim of the present paper is to describe the graph classes $\mathbf{W_i}/\mathbf{W_j}$. Our main results are summarized in Table 1.

	$\mathbf{W_1}$	$\mathbf{W_2}$	$\mathbf{W_3}$	$\widehat{\mathbf{W}}_\mathbf{3}$	$\mathbf{W_4}$
$\mathbf{W_1}$	**g-Chordal**	**Chordal**	**Ptolemaic$^-$**		**Superfragile**
$\mathbf{W_2}$	**Chordal**	**Chordal**	**Ptolemaic$^-$**	**Interval**	**Superfragile**
$\mathbf{W_3}$	**Chordal**	**Chordal**	**Ptolemaic$^-$**	**Interval**	**Superfragile**
$\widehat{\mathbf{W}}_\mathbf{3}$	**Chordal**	**Chordal**	**Ptolemaic$^-$**	**Interval**	**Superfragile**
$\mathbf{W_4}$	**Chordal**	**Chordal**	**Ptolemaic$^-$**	**Interval**	**Superfragile**

Table 1. With W_i in the first column and W_j in the first row, the table describe each one of the graph classes $\mathbf{W_i}/\mathbf{W_j}$ except $\mathbf{W_1}/\widehat{\mathbf{W}}_\mathbf{3}$. Recall that $\mathbf{W_1}$: shortest-paths; $\mathbf{W_2}$: induced-paths; $\mathbf{W_3}$: paths; $\widehat{\mathbf{W}}_\mathbf{3}$:tolled-walks; $\mathbf{W_4}$: walks. The classes **Ptolemaic$^-$** and **g-Chordal** are defined in 3 and 3. Theorem 15 provides a partial characterization of $\mathbf{W_1}/\widehat{\mathbf{W}}_\mathbf{3}$. The classes $\mathbf{W_1}/\mathbf{W_1}$ and $\mathbf{W_1}/\widehat{\mathbf{W}}_\mathbf{3}$ are not closed under taking induced subgraphs.

Lemma 7. *For every $i, j \in \{1,2,3,4\}$, the following statements hold.*

1. $\mathbf{W_i/W_1} \supseteq \mathbf{W_i/W_2} \supseteq \mathbf{W_i/W_3} \supseteq \mathbf{W_i/W_4}$.
2. $\mathbf{W_i/W_1} \supseteq \mathbf{W_i/W_2} \supseteq \mathbf{W_i}/\widehat{\mathbf{W}}_\mathbf{3} \supseteq \mathbf{W_i/W_4}$.
3. $\widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_1} \supseteq \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_2} \supseteq \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_3} \supseteq \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_4}$.
4. $\widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_1} \supseteq \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_2} \supseteq \widehat{\mathbf{W}}_\mathbf{3}/\widehat{\mathbf{W}}_\mathbf{3} \supseteq \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_4}$.
5. $\mathbf{W_4/W_j} = \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_j} = \mathbf{W_3/W_j} = \mathbf{W_2/W_j} \subseteq \mathbf{W_1/W_j}$.
6. $\mathbf{W_4}/\widehat{\mathbf{W}}_\mathbf{3} = \widehat{\mathbf{W}}_\mathbf{3}/\widehat{\mathbf{W}}_\mathbf{3} = \mathbf{W_3}/\widehat{\mathbf{W}}_\mathbf{3} = \mathbf{W_2}/\widehat{\mathbf{W}}_\mathbf{3} \subseteq \mathbf{W_1}/\widehat{\mathbf{W}}_\mathbf{3}$.

Proof. Statements 1, 2, 3 and 4 follow in a straightforward way from Remark 1.

Also by Remark 1, we have $\mathbf{W_4/W_j} \subseteq \mathbf{W_3/W_j} \subseteq \mathbf{W_2/W_j} \subseteq \mathbf{W_1/W_j}$. And by Remark 2, $\mathbf{W_2/W_j} \subseteq \mathbf{W_3/W_j} \subseteq \mathbf{W_4/W_j}$. Thus, $\mathbf{W_4/W_j} = \mathbf{W_3/W_j} = \mathbf{W_2/W_j} \subseteq \mathbf{W_1/W_j}$. In an analogous way, we have $\mathbf{W_4/W_j} = \widehat{\mathbf{W}}_\mathbf{3}/\mathbf{W_j} = \mathbf{W_2/W_j} \subseteq \mathbf{W_1/W_j}$, which completes the proof of statement 5.

Statement 6 has an identical proof to that of statement 5 replacing $\mathbf{W_j}$ by $\widehat{\mathbf{W}}_\mathbf{3}$. ■

Notice that Lemma 7 implies that the last four rows of Table 1 must be the same. The following theorems addresses the characterization of the classes in the first two columns.

Theorem 8. $\mathbf{W_2/W_1} = \mathbf{W_2/W_2} = \mathbf{W_1/W_2} = \mathbf{Chordal}$.

Proof. Let G be a chordal graph and assume, in order to derive a contradiction, that $G \notin \mathbf{W_2/W_2}$. Then there exist two non-adjacent vertices u and v, and two uv-induced-paths $W : w_1, \ldots, w_m$ and $W' : w'_1, \ldots, w'_\ell$ such that W does not dominate W'. It follows that there is some internal vertex w'_k of W', which is neither a vertex of W nor adjacent to an internal vertex of W. Denote by r the greatest $i < k$ such that w'_i is adjacent to some vertex of W. Notice that $1 \leq r < k$. Denote by s the smallest $i > k$ such that w'_i is adjacent to some vertex of W. Observe that $k < s \leq \ell$. We can choose the nearest vertices $w_{r'}$ and $w_{s'}$ of W adjacent to w'_r and w'_s respectively (by nearest we mean minimizing $\mid r' - s' \mid$). Notice it could be $r' = s'$. By the concatenation of the subpath of W' between w'_r and w'_s and the subpath of W between $w_{s'}$ and $w_{r'}$, we obtain the induced cycle $w'_r, \ldots, w'_k, \ldots, w'_s, w_{s'}, \ldots, w_{r'}$ with at least four vertices, which contradicts Theorem 3.

On the other hand, if a graph G has an induced cycle C_k with $k \geq 4$, then any two vertices u and v of the cycle at distance 2 determine on the cycle an uv-shortest-path and an uv-induced-path such that neither of them dominates the other. Thus $\mathbf{W_1/W_2}$ and $\mathbf{W_2/W_1}$ are contained in **Chordal**. Lemma 7 completes the proof. ■

Definition. The class of Ptolemaic graphs which contain none of the graphs *co-chair* or D in Figure 2 as induced subgraph is denoted by **Ptolemaic**$^{-}$. In other words,

$$\begin{aligned}\mathbf{Ptolemaic}^{-} &= \mathbf{Ptolemaic} \cap \{\mathbf{co\text{-}chair,\ D}\}\text{-}\mathbf{free}\\ &= \mathbf{Chordal} \cap \{\mathbf{gem,\ co\text{-}chair,\ D}\}\text{-}\mathbf{free}.\end{aligned}$$

Theorem 9. $\mathbf{W_2/W_3} = \mathbf{W_1/W_3} = \mathbf{Ptolemaic}^{-}$.

Proof. Let G be a chordal graph with no induced subgraph isomorphic to a gem, a co-chair or the graph D in Figure 2. Assume, in order to derive a contradiction, that $G \notin \mathbf{W_2/W_3}$. Then there exist two non-adjacent vertices u and v, an uv-induced-path $W : w_1, \ldots, w_m$ and an uv-path $W' : w'_1, \ldots, w'_\ell$ satisfying that W does not dominate W'. Thus, there is some internal vertex w'_k of W' that is neither a vertex of W nor adjacent to an internal vertex of W.

Without loss of generality, we can assume that W' is an uv-path with minimum length between the ones that are not dominated by W. This implies that the subpaths $w'_1, \ldots, w'_k$ and $w'_k, \ldots, w'_\ell$ of W' are induced-paths. Since G is chordal, by Theorem 8, W' is not an induced-path, thus there is some w'_i with $i < k$ which is adjacent to some w'_j with $j > k$; moreover, w'_{k-1} must be adjacent to w'_{k+1}.

Notice that w'_{k-1} and w'_{k+1} are not internal vertices of W, and the uv-path $w'_1, \ldots, w'_{k-1}, w'_{k+1}, \ldots, w'_\ell$ is dominated by W since it is shorter than W'.

We will deal with two cases. First assume $w'_{k-1} \neq u$ and $w'_{k+1} \neq v$. Then both vertices, w'_{k-1} and w'_{k+1}, are adjacent to some internal vertex w_h of W.

We claim that w_{h-1} is non-adjacent to w'_k. Indeed, if it were then w_{h-1} would not be an internal vertex of W, thus $w_{h-1} = w_1 = w'_1 = u$, which contradicts the fact that $w'_1, \ldots, w'_{k-1}, w'_k$ is an induced-path. Therefore, w_{h-1} must be adjacent to both vertices w'_{k-1} and w'_{k+1} because in other case the vertices w_{h-1}, w_h, w'_{k-1}, w'_k and w'_{k+1} induce a subgraph isomorphic to a gem or to a co-chair.

In an analogous way, we prove that w_{h+1} is non-adjacent to w'_k and adjacent to w'_{k-1} and w'_{k+1}.

It follows that the vertices w_{h-1}, w_h, w_{h+1}, w'_{k-1}, w'_k and w'_{k+1} induce a subgraph isomorphic to the graph D in Figure 2, in contradiction with the hypothesis.

Now we consider the case $w'_{k-1} = u$ or $w'_{k+1} = v$. By symmetry considerations, it is sufficient to address the case $w'_{k-1} = u$. Observe that $w'_{k-1} = u$ implies w'_{k+1} is adjacent to u and to w_2. Since w'_k is adjacent to no interval vertex of W and it is not adjacent to v because, as we said previously, $w'_k, \ldots, w'_\ell$ is a chordless path, it follows that w'_k is adjacent neither to w_2 nor to w_3. Therefore, if w_3 is non-adjacent to w'_{k+1}, then there is an induced co-chair, and if w_3 is adjacent to w'_{k+1}, then there is an induced gem, both cases contradict our assumptions.

On the other hand, by Lemma 7 and Theorem 8, $\mathbf{W_1/W_3} \subseteq \mathbf{W_1/W_2} =$ **Chordal**. Morover, as it is shown in Figure 3, each induced forbidden subgraph

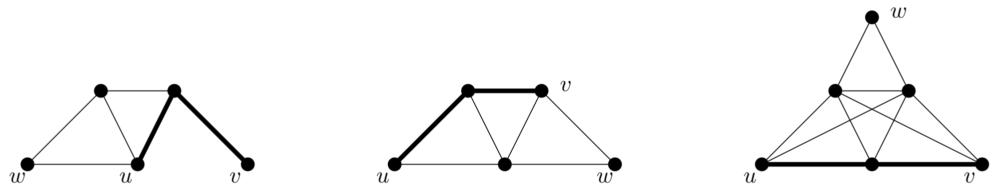


Figure 3. In each graph above, the vertex labelled w belongs to an uv-path and it is adjacent to no internal vertex of the bold uv-shortest-path.

for the class **Ptolemaic**$^-$ (gem, co-chair and D) has a pair of non-adjacent vertices u and v, and an uv-path which is not dominated by an uv-shortest-path. Thus the class $\mathbf{W_1/W_3}$ is contained in **Ptolemaic**$^-$. Lemma 7 completes the proof. ■

Theorem 10. $\mathbf{W_2/\widehat{W}_3} =$ **Interval**.

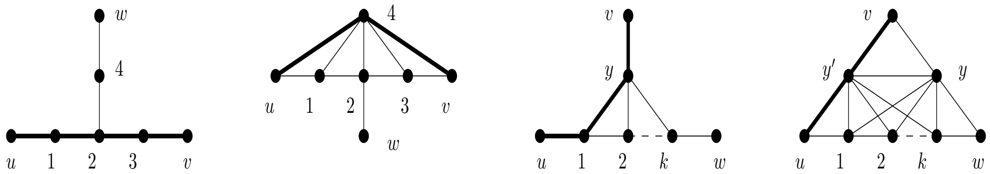
Proof. Let G be an interval graph and assume, in order to derive a contradiction, that $G \notin \mathbf{W_2/\widehat{W}_3}$. Then there exist two non-adjacent vertices u and v, an uv-induced-path $W : w_1, \ldots, w_m$ and an uv-tolled-walk $W' : w'_1, \ldots, w'_\ell$ such that W does not dominate W'. It follows that there is some internal vertex w'_k of W', which is neither a vertex of W nor adjacent to an internal vertex of W.

Let $I_u = [x_u, y_u]$ and $I_v = [x_v, y_v]$ with $x_u < y_u < x_v < y_v$ be the intervals corresponding to vertices u and v in a given interval representation of G. It is clear that the segment of line $[y_u, x_v]$ is contained in the union of the intervals corresponding to the internal vertices of W, then we can assume that the interval $I_{w'_k}$ is contained in $(-\infty, y_u)$. This implies that there is a vertex w'_i with $i > k \geq 2$ adjacent to u, which contradicts the fact that W' is an uv-tolled-walk.

On the other hand, by Lemma 7 and Theorem 8, $\mathbf{W_2/\widehat{W}_3} \subseteq \mathbf{W_2/W_2} =$ **Chordal**. Moreover, as it is shown in Figure 4, each induced forbidden subgraph for the class **Interval** has a pair of non-adjacent vertices u and v, and an uv-tolled-walk which is not dominated by an uv-induced-path. Thus the class $\mathbf{W_2/\widehat{W}_3}$ is contained in **Interval**. Lemma 7 completes the proof. ■

Theorem 11. $\mathbf{W_1/W_4} = \mathbf{W_2/W_4} =$ **Superfragile**.

Proof. Let G be superfragile and assume, in order to derive a contradiction, that $G \notin \mathbf{W_2/W_4}$. Then there exist two non-adjacent vertices u and v, an uv-induced-paths $W : w_1, \ldots, w_m$ and an uv-walk $W' : w'_1, \ldots, w'_\ell$ such that W does not dominate W'. It follows that there is some internal vertex w'_k of W', which is neither a vertex of W nor adjacent to an internal vertex of W.



F_1 F_2 $F_{3,k}$ $F_{4,k}$

Figure 4. In each graph above, the vertex labelled w belongs to an uv-tolled-walk W' and it is adjacent to no internal vertex of the bold uv-induced-path. For F_1 take W' : $u, 1, 2, 4, w, 4, 2, 3, v$; for F_2, W' : $u, 1, 2, w, 2, 3, v$; for $F_{3,k}$, W' : $u, 1, 2, \ldots, k, w, k, y, v$; and for $F_{4,k}$ take W' : $u, 1, 2, \ldots, k, w, y, v$.

Since G has no induced P_4, we have that W must be a P_3: u,w,v for some vertex w with $w \neq w'_k$ and w non-adjacent to w'_k, and $d(w'_k, u) \leq 2$.

If $d(w'_k, u) = 1$ and w'_k is adjacent to v, then there is an induced C_4. If $d(w'_k, u) = 1$ and w'_k is non-adjacent to v, then there is an induced P_4. Both cases contradict Theorem 6.

If $d(w'_k, u) = 2$, let w'_k,x,u be a shortest path. Notice that $x \neq w$ and $x \neq v$. In addition, w'_k is not adjacent to v because otherwise there will be an induced $P_4 : w'_k, v, w, u$. Moreover, x is adjacent to w because otherwise there will be an induced $P_4 : w'_k, x, u, w$.

Thus, either x is adjacent to v and there is an induced subgraph isomorphic to the graph dart in Figure 2, or x is non-adjacent to v and there is an induced $P_4 : w'_k, x, w, v$. Both cases contradict again Theorem 6.

On the other hand, it is easy to see that each induced forbidden subgraph for the class **Superfragile** (C_4, P_4, dart) in Figure 2 has a pair of non-adjacent vertices u and v, and an uv-walk which is not dominated by an uv-shortest-path. Notice that in the case of $P_4 : v_1, v_2, v_3, v_4$, we can considerer $u = v_1$, $v = v_3$ and the uv-walk $u = v_1, v_2, v_3, v_4, v_3 = v$ which is not dominated by the uv-shortest-path $u = v_1, v_2, v_3 = v$.

It follows that class $\mathbf{W_1/W_4}$ is contained in **Superfragile**. Lemma 7 completes the proof. ■

Unlike the graph classes described by the preceding theorems, we will see that $\mathbf{W_1/W_1}$ and $\mathbf{W_1}/\widehat{\mathbf{W_3}}$ are not *hereditary* classes of graphs, i.e., they are not closed under taking induced subgraphs.

Indeed, it is easy to see that every cycle C_{2k} with $k > 2$ plus an universal vertex belongs to $\mathbf{W_1/W_1}$, but the cycle C_{2k} does not. Notice that the class of C_4-free diameter 2 graphs is contained in $\mathbf{W_1/W_1}$ and that

Remark 12. If $G \in \mathbf{W_1/W_1}$, then G is C_4-free.

Since the class $\mathbf{W_1/W_1}$ is not hereditary, it cannot be characterized by forbidden induced subgraphs. Instead, we present in Theorem 14 a characterization based on geodesic intervals.

The *closed geodesic interval* $I_g[u,v]$ for two vertices u and v of a graph G is the set of all vertices lying on some uv-shortest-path of G. Geodesic intervals were studied and characterized by Nebeský [12, 13] and play an important role in the study of metric and convexity properties of graphs [2, 11].

The vertices of $I_g[u,v]$ can be partitioned into level sets L_i for $i \in \{0,1,\dots, d(u,v)\}$ according to the distance to u by doing

$$L_i = \{x \in I_g[u,v] : d(u,x) = i\}.$$

Lemma 13. *If L_i is a level set of a closed geodesic interval $I_g[u,v]$ of a graph $G \in \mathbf{W_1/W_1}$, then L_i is a clique.*

Proof. First we will prove the proposition for $i = 1$. Assume, in order to obtain a contradiction, that there exist two non-adjacent vertices v_1 and v_1' in the level set L_1 of $I_g[u,v]$. Let $P : u, v_1, v_2, \dots, v$ and $P' : u, v_1', v_2', \dots, v$ be two uv-shortest-paths. Since v_1' must be adjacent to some internal vertex of P, we have that v_1' is adjacent to v_2, which implies the existence of the induced $C_4 : u, v_1, v_2, v_1'$ in contradiction with Remark 12.

Now let $i > 1$ and assume, in order to obtain a contradiction, that v_i and v_i' are two non-adjacent vertices in L_i. As before, let $P : u, v_1, \dots, v_i, \dots, v$ and $P' : u, v_1', \dots, v_i', \dots, v$ be uv-shortest-paths. Since v_i' must be adjacent to some internal vertex of P, we have that v_i' is adjacent to v_{i-1} or to v_{i+1}; without loosing generality let v_i' be adjacent to v_{i-1}. It follows that v_i and v_i' belong to the level set L_1 of $I_g[v_{i-1}, v]$, thus v_i and v_i' are adjacent, in contradiction with our assumption. ■

Definition. We let **g-Chordal** denote the class of graphs G in which any closed geodesic interval induces a chordal subgraph.

Notice that the class of geodetic graphs (for every pair of its vertices there is a unique shortest path between them [2]) is contained in **g-Chordal**.

Theorem 14. $\mathbf{W_1/W_1} =$ **g-Chordal**.

Proof. Let u and v be vertices of a graph $G \in \mathbf{W_1/W_1}$. In order to obtain a contradiction, assume that $k \geq 4$ and $C_k : v_1, v_2, \dots, v_k$ is an induced cycle in the closed geodesic interval $I_g[u,v]$. For every $j \in \{1,\dots,k\}$, let L_{j_i} be the level set of $I_g[u,v]$ containing v_i. We can clearly assume that $j_1 \leq j_i$ for $2 \leq i \leq k$ and $j_2 = j_1 + 1$. By Lemma 13, two vertices of C_k in a same level are adjacent; thus $j_k \neq j_2$. This implies $j_k = j_1$ and $j_{k-1} = j_2$; so v_{k-1} and v_2 are adjacent. Therefore, $k = 4$ in contradiction with Remark 12.

Now let P and P' be uv-shortest-paths with u and v two non-adjacent vertices of a graph $G \in$ **g-Chordal**. We have to prove that P dominates P'. By definition of **g-Chordal**, the subgraph $G[I_g[u,v]]$ induced by $I_g[u,v]$ is chordal. Since P and P' are induced paths in $G[I_g[u,v]]$, by Theorem 8, P dominates P'. ■

The rest of the paper is devoted to the study of the more intricate class $\mathbf{W_1}/\widehat{\mathbf{W_3}}$. Observe that the graph F_1 in Figure 4 does not belong to $\mathbf{W_1}/\widehat{\mathbf{W_3}}$ (the bold uv-induced-path is also an uv-shortest-path). However, F_1 plus an *universal vertex* (i.e., a vertex adjacent to every vertex of F_1) belongs to $\mathbf{W_1}/\widehat{\mathbf{W_3}}$ since it is C_4-free and has diameter 2. Consequently $\mathbf{W_1}/\widehat{\mathbf{W_3}}$ is not hereditary. Theorem 15 presents a partial characterization of the graphs in this class.

Definition. A graph G belongs to the class **Interval$^+$** if G is chordal, contains none of the graphs F_2 or $F_{4,k}$ in Figure 1 as induced subgraph and satisfies the following condition.

> If G has an induced subgraph H isomorphic to F_1 ($F_{3,k}$), then the distance in G between the vertices of F_1 ($F_{3,k}$) labelled u and v in Figure 4 is 2, and any vertex of G adjacent to both u and v is universal to F_1 ($F_{3,k}$).

Notice that every interval graph belongs to **Interval$^+$**.

Theorem 15. $\mathbf{W_1}/\widehat{\mathbf{W_3}} \subseteq$ **Interval$^+$**.

Proof. Let $G \in \mathbf{W_1}/\widehat{\mathbf{W_3}}$. By Lemma 7 and Theorem 8, $\mathbf{W_1}/\widehat{\mathbf{W_3}} \subseteq \mathbf{W_1}/\mathbf{W_2} =$ **Chordal**, therefore G is chordal.

Assume, in order to obtain a contradiction, that G has an induced subgraph isomorphic to the graph F_2 in Figure 1. In Figure 4, we show two non-adjacent vertices u and v of F_2 and an uv-induced-path W which does not dominate an uv-tolled-walk W'. Since the length of W is 2 and u and v are non-adjacent, we have that such W is also an uv-shortest-path in G. It contradicts the fact that every shortest-path dominates every tolled-walk.

In analogous way it can be proved that G has no induced subgraph isomorphic to the graph $F_{4,k}$.

Notice that using the same argument we can prove that if G has an induced subgraph isomorphic to $F_{3,k}$, then the distance in G between the vertices of $F_{3,k}$ labelled u and v in Figure 4 cannot be 3, then it must be 2. Let x be the internal vertex of some uv-shortest-path; clearly x is not a vertex of $F_{3,k}$ and is adjacent to u and v. Since $G \in \mathbf{W_1}/\widehat{\mathbf{W_3}}$ and $W' : u, 1, 2, \ldots, k, w, k, y, v$ is an uv-tolled-walk (see Figure 4), it follows that x is adjacent to every vertex of F_1.

A reasoning analogous to the one applied in the case of $F_{3,k}$ shows that if G has an induced subgraph isomorphic to F_1, then the distance in G between the vertices of F_1 labelled u and v in Figure 4 is at most 3, and also resolves the case $d(u, v) = 2$. We claim that $d(u, v) = 3$ leads to a contradiction. Indeed, assume that u, x, z, v is an uv-shortest-path. Notice that neither x nor z may be the vertex of T_1 labelled w in Figure 4. In addition, since w is an internal vertex of an uv-tolled-walk, x or z must be adjacent to w. Without loss of generality, let x be adjacent to w. Thus u, x, w is an uw-shortest-path. Since $G \in \mathbf{W_1}/\widehat{\mathbf{W_3}}$, and $u, 1, 2, 3, v, 3, 2, 4, w$ is an uw-tolled-walk, we have that x must be adjacent to v, which contradicts that u, x, z, v is an uv-shortest-path. ■

4. Conclusions

We have obtained characterization of the graphs in which, for every pair of non-adjacent vertices u and v, every uv-walk, tolled-walk, path, induced-path or shortest-path dominates every uv-walk, tolled-walk, path, induced-path or shortest-path, with the exception of those in which every uv-shortest-path dominates every uv-tolled-walk. We let open the problem of determining if such graphs are exactly the ones in **Interval**$^+$.

Conjecture 16. Interval$^+ \subseteq \mathbf{W_1}/\widehat{\mathbf{W_3}}$.

We have proved that the classes $\mathbf{W_1}/\mathbf{W_1}$ and $\mathbf{W_1}/\widehat{\mathbf{W_3}}$ are not hereditary (closed under taking induced subgraphs), but $\mathbf{W_1}/\mathbf{W_2}$, $\mathbf{W_1}/\mathbf{W_3}$ and $\mathbf{W_1}/\mathbf{W_4}$ are.

Regarding to convexity theory, we propose the study of the convexity space obtained by considering tolled-walk intervals.

We have proved that $\mathbf{W_1}/\mathbf{W_1}$ is the class of graphs in which every geodesic interval is chordal, while $\mathbf{W_1}/\mathbf{W_2}$ is the class of graphs in which every monophonic interval is chordal, we wonder what other graph classes can be characterized using this approach.

Finally, we observe the following property of **g-Chordal**. According to [11], the interval function of a graph G is the mapping $f : V(G) \times V(G) \to 2^{V(G)}$ given by $f(u, v) = I_g[u, v]$ (the closed geodesic interval). It is clear that any hereditary class of graphs is closed under the interval function, in the sense that the subgraph induced by $f(u, v)$ also belongs to the class. However, this is not necessarily true for non-hereditary graph classes. We observe that the class **g-Chordal** is closed for the interval function. Other non-hereditary graph class for which this property holds is the class of median graphs [2, 10, 11].

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Note added in proof: Reference [2] in [1] is the present paper; the notion of tolled-walk introduced in the current work was used there to develop the toll convexity.

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