

INTEGRAL CAYLEY SUM GRAPHS AND GROUPS

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Abstract

For any positive integer k , let \mathcal{A}_k denote the set of finite abelian groups G such that for any subgroup H of G all Cayley sum graphs $\text{CayS}(H, S)$ are integral if $|S| = k$. A finite abelian group G is called Cayley sum integral if for any subgroup H of G all Cayley sum graphs on H are integral. In this paper, the classes \mathcal{A}_2 and \mathcal{A}_3 are classified. As an application, we determine all finite Cayley sum integral groups.

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1. INTRODUCTION

A graph is *integral* if all its eigenvalues are integers. Harary and Schwenk [13] introduced integral graphs, and proposed the problem of classifying integral graphs. There are some constructions of graphs from groups in the literature; for example, Cayley graphs, which are integral were studied in [1, 2, 3, 11, 14, 15].

Let G be a finite abelian group. A subset S of G is said to be *square-free* if $x + x \notin S$ for each $x \in G$. The *Cayley sum graph* of G with respect to a square-free subset S of G , denoted by $\text{CayS}(G, S)$, is a simple graph with vertex set G and two distinct vertices x and y form an edge if $x + y \in S$. Some results on Cayley sum graphs can be found in [4, 5, 9, 12, 17].

For any positive integer k , let \mathcal{A}_k denote the set of finite abelian groups G such that for any subgroup H of G all Cayley sum graphs $\text{CayS}(H, S)$ are integral if $|S| = k$. A finite abelian group G is called *Cayley sum integral* if for any subgroup H of G all Cayley sum graphs on H are integral.

In the paper we classify the classes \mathcal{A}_2 and \mathcal{A}_3 . As an application, all finite Cayley sum integral groups are determined. Our main results are the following.

Theorem 1. *The class \mathcal{A}_2 consists of the groups:*

$$(1) \quad \mathbb{Z}_2^n, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_3^m, \quad n \geq 2, \quad m \geq 1.$$

Theorem 2. *The class \mathcal{A}_3 consists of the groups:*

$$(2) \quad \mathbb{Z}_2^n, \mathbb{Z}_6, \mathbb{Z}_8, \quad n \geq 2.$$

Theorem 3. *All finite Cayley sum integral groups are represented by*

$$(3) \quad \mathbb{Z}_2^n, \mathbb{Z}_4, \mathbb{Z}_6, \quad n \geq 1.$$

2. CAYLEY SUM GRAPHS

In this section we recall some results on Cayley sum graphs.

For a finite abelian group G of odd order, since $G = \{2x : x \in G\}$, there exists no Cayley sum graph of G . In fact, an abelian group G has square-free elements if and only if $|G|$ is even, where $|G|$ is the order of G . Thus, in this paper we always consider the finite abelian groups of even order. Observe that \mathcal{A}_1 is the set of all finite abelian groups of even order.

Suppose that X is a set. Let $\Omega = \{X_1, X_2, \dots, X_n\}$ be a family of subsets of X , and f be a complex valued function on X . We denote the sets of integers and complex numbers by \mathbb{Z} and \mathbb{C} , respectively. A subset M of X is called *f-integral* if

$$f(M) = \sum_{m \in M} f(m) \in \mathbb{Z}.$$

The *Boolean algebra* generated by Ω in X is the smallest system of subsets of X that contains Ω , and is obtained by arbitrary finite intersections, unions, and complements of the sets. Let G be a finite abelian group. Denote by $\mathbb{B}(G)$ the Boolean algebra generated by all subgroups of G . A *character* of G is a homomorphism from G into the multiplicative group of complex numbers $\mathbb{C} \setminus \{0\}$.

In [10] the authors studied the eigenvalues of a Cayley sum graph.

Proposition 4 [10, Theorem 2.1]. *The multiset of eigenvalues of $\text{CayS}(G, S)$ is*

$$\{\chi(S) : \chi \text{ is a real character}\} \cup \{\pm|\chi(S)| : \chi \text{ is not a real character}\}.$$

For an elementary abelian 2-group \mathbb{Z}_2^n , let $S = \{s_1, \dots, s_t\}$ be a subset of $\mathbb{Z}_2^n \setminus \{e\}$, where e is the identity element. Then $S = (\langle s_1 \rangle \setminus \{e\}) \cup \dots \cup (\langle s_t \rangle \setminus \{e\}) \in \mathbb{B}(\mathbb{Z}_2^n)$. It has been shown in [14] that for any character χ of a finite abelian group G , every set in $\mathbb{B}(G)$ is χ -integral. Thus by Proposition 4, we have the next.

Proposition 5. *Let G be a finite abelian group and S a square-free subset of G . If $S \in \mathbb{B}(G)$, then $\text{CayS}(G, S)$ is integral. In particular, $\text{CayS}(\mathbb{Z}_2^n, S)$ is integral if and only if S does not contain the identity element of \mathbb{Z}_2^n .*

Lemma 6 (cf. [6, p. 9]). *An n -cycle C_n is integral only for $n = 3, 4$, or 6 .*

Lemma 7 [8, Proposition 2.3]. *Let G be an abelian group and S a square-free subset of G . Then $\text{CayS}(G, S)$ is connected if and only if $\langle S \rangle = G$ and $|\langle S' \rangle| \geq |G|/2$, where $S' = \{a - b : a, b \in S\}$.*

3. PROOFS OF THE MAIN RESULTS

Denote by $\pi_e(G)$ the set of all orders of elements of a group G . For a graph Γ and a positive integer n , $n\Gamma$ denotes the graph union of n copies of Γ .

Lemma 8. $\mathbb{Z}_2 \times \mathbb{Z}_3^n \in \mathcal{A}_2$ for each integer $n \geq 1$.

Proof. Write $G = \mathbb{Z}_2 \times \mathbb{Z}_3^n$. Then G has a unique involution, and $\pi_e(G) = \{1, 2, 3, 6\}$. Let $S := \{a, b\}$ be a square-free subset of size 2 of G .

Case 1. S has an involution. Without loss of generality, let $O(a) = 2$, where $O(a)$ is the order of a . Then $O(b) = 6$ and $a = 3b$. Take any element x in G ; one gets that

$$(4) \quad x \sim b - x \sim a - b + x \sim 2b - a - x \sim -2b + x \sim a - x \sim x$$

is a cycle of length 6 in $\text{CayS}(G, S)$. Since $\text{CayS}(G, S)$ is 2-regular, (4) is a connected component of $\text{CayS}(G, S)$. It follows that $\text{CayS}(G, S) \cong 3^{n-1}C_6$. Consequently $\text{CayS}(G, S)$ is integral.

Case 2. S has no involutions. In this case, $O(a) = O(b) = 6$ and $3a = 3b$. For any $x \in G$,

$$x \sim a - x \sim b - a + x \sim 2a - b - x \sim 2b - 2a + x \sim b - x \sim x$$

is a 6-cycle. Similarly to Case 1, we conclude that $\text{CayS}(G, S)$ is integral.

Note that for any subgroup H of G with a square-free subset of size 2, we see that H is isomorphic to a group $\mathbb{Z}_2 \times \mathbb{Z}_3^m$ for some $m \geq 1$. Thus, we have $G \in \mathcal{A}_2$. ■

Lemma 9. *If $G \in \mathcal{A}_2$, then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$.*

Proof. Let g be an element of even order in G . If g is not an involution, then the cycle $C_{O(g)}$ is integral, and so $O(g) = 4$ or 6 by Lemma 6. Suppose, towards a contradiction, that G has an element b with $O(b) \notin \{1, 2, 3, 4, 6\}$. Then $O(b)$ is odd and $O(b) \geq 5$. For an involution a of G , one has $O(a + b) = 2O(b)$, a contradiction. ■

Proof of Theorem 1. Note that there exists precisely one Cayley sum graph $\text{CayS}(\mathbb{Z}_4, \{1, 3\})$ of valency 2 on \mathbb{Z}_4 , which is integral. Note that each subgroup of \mathbb{Z}_2^n is elementary abelian. Then by Lemma 8 and Proposition 5, all groups in (1) belong to \mathcal{A}_2 .

Suppose that $G \in \mathcal{A}_2$. Since an abelian group is a direct product of some cyclic groups of prime power order, according to Lemma 9, G is isomorphic to one of the following groups:

$$\mathbb{Z}_2^n, \mathbb{Z}_4^n, \mathbb{Z}_2^n \times \mathbb{Z}_3^m, \mathbb{Z}_2^n \times \mathbb{Z}_4^m, \quad m \geq 1, \quad n \geq 1.$$

Case 1. $G \cong \mathbb{Z}_4^n$. Suppose that $n \geq 2$. Then G has a subgroup isomorphic to \mathbb{Z}_4^2 . It follows that $\mathbb{Z}_4^2 \in \mathcal{A}_2$. On the other hand, $\text{CayS}(\mathbb{Z}_4^2, \{(1, 0), (0, 1)\}) \cong 2C_8$, contrary to Lemma 6. Therefore, in this case we conclude $G \cong \mathbb{Z}_4$.

Case 2. $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_3^m$. Suppose that $n \geq 2$. Note that $\text{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1, 0), (0, 1)\}) \cong C_{12}$. Similarly to Case 1, we get a contradiction.

Case 3. $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_4^m$. Note that G has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\text{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_4, \{(1, 0), (0, 1)\}) \cong C_8$. Similarly to Case 1, we get a contradiction. ■

Proposition 10. *Let G be an abelian group. Then there is a connected cubic integral Cayley sum graph on G if and only if G is one the following groups:*

$$\mathbb{Z}_2^2, \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6.$$

In particular, there are exactly five connected cubic integral Cayley sum graphs.

Proof. Let $\text{CayS}(G, S)$ be a connected cubic integral graph. By Schwenk’s result [16], independently by Bussemaker and Cvetković [7], there are exactly thirteen cubic connected integral graphs. By checking the list of these thirteen graphs, it follows that

$$|G| \in \{4, 6, 8, 10, 12, 20, 24, 30\}.$$

For each group G of the mentioned orders, finding all 3-element subsets S of G such that all $\text{CayS}(G, S)$ are pairwise non-isomorphic connected integral graphs, we get Table 1.

Note that

$$\begin{aligned} \text{CayS}(\mathbb{Z}_8, \{1, 3, 5\}) &\cong \text{CayS}(\mathbb{Z}_2^3, \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}) \\ &\cong \text{CayS}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1, 0), (0, 1), (2, 1)\}) \end{aligned}$$

and

$$\text{CayS}(\mathbb{Z}_{12}, \{1, 3, 5\}) \cong \text{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1, 2), (1, 4), (0, 3)\}).$$

We get the desired result. ■

Table 1. All cubic connected integral Cayley sum graphs.

G	S
\mathbb{Z}_2^2	$\{(1, 0), (0, 1), (1, 1)\}$
\mathbb{Z}_6	$\{1, 3, 5\}$
\mathbb{Z}_8	$\{1, 3, 5\}$
\mathbb{Z}_2^3	$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\{(1, 0), (0, 1), (2, 1)\}$
\mathbb{Z}_{12}	$\{1, 3, 5\}$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\{(1, 0), (1, 1), (0, 5)\}, \{(1, 2), (1, 4), (0, 3)\}$

Lemma 11. *If a group G belongs to \mathcal{A}_3 , then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6, 8\}$.*

Proof. Assume that a is a non-identity element of G . We consider two cases.

Case 1. $O(a)$ is odd. Then there exists an element b in G such that $O(b) = 2O(a)$. Note that $O(b) \geq 6$. According to Lemma 7, $\text{CayS}(\langle b \rangle, \{b, 3b, 5b\})$ is a cubic connected graph. It follows from Proposition 10 that $O(a) = 3$, as desired.

Case 2. $O(a)$ is even. Suppose, to derive a contradiction, that $O(a) \notin \{2, 4, 6, 8\}$. By Lemma 7, one gets that $\text{CayS}(\langle a \rangle, \{a, 3a, 5a\})$ is cubic connected. By Proposition 10, one has $O(a) = 12$. It is straightforward to check that $\text{CayS}(\langle a \rangle, \{a, 5a, 11a\})$ is not integral, a contradiction. ■

Proof of Theorem 2. Firstly, it is easy to check that $\mathbb{Z}_6, \mathbb{Z}_8 \in \mathcal{A}_3$. Thus, by Proposition 5 all groups in (2) belong to \mathcal{A}_3 .

Suppose that $G \in \mathcal{A}_3$. Then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6, 8\}$ by Lemma 11.

Case 1. G has an element of order 3. For elements $x, y \in G$, if $O(y)$ and $O(x)$ are relatively prime, then $O(x + y) = O(x)O(y)$. It follows that $\pi_e(G) = \{1, 2, 3, 6\}$. Therefore, $G \cong \mathbb{Z}_6$, or G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_3 \times \mathbb{Z}_6$. If $\mathbb{Z}_3 \times \mathbb{Z}_6$ is a subgroup of G , then by Lemma 7 $\text{CayS}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(0, 1), (1, 1), (2, 3)\})$ is a connected cubic integral graph, contrary to Proposition 10. If $\mathbb{Z}_2 \times \mathbb{Z}_6$ is a subgroup of G , then $\text{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1, 5), (1, 4), (0, 1)\})$ is not integral, also a contradiction.

Case 2. G has no elements of order 3. In this case $\pi_e(G) \subseteq \{1, 2, 4, 8\}$. Suppose that G has an element of order 8. Then $G \cong \mathbb{Z}_8$, or G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_8$. Note that $\text{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_8, \{(0, 1), (1, 1), (0, 3)\})$ is not integral. Similarly to Case 1, we get the desired result.

Suppose now that G has no elements of order 8. Then $G \cong \mathbb{Z}_2^n$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$ is a subgroup of G , where $n \geq 2$. Note that $\text{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_8, \{(0, 1), (1, 1), (1, 0)\})$ is not integral. Similarly to Case 1, we end the proof. ■

Proof of Theorem 3. Clearly, both \mathbb{Z}_4 and \mathbb{Z}_6 are Cayley sum integral. Thus by Proposition 5, every group in (3) is Cayley sum integral.

Now let G be a finite Cayley sum integral group. Suppose that G has a unique square-free element. Since any element with maximal even order is square-free, every non-identity element is an involution. Then G is an elementary abelian 2-group. This implies that G is isomorphic to \mathbb{Z}_2 .

Suppose that G has precisely two square-free elements. Then G belongs to \mathcal{A}_2 . By Theorem 1, one has $G \cong \mathbb{Z}_4$.

Now suppose that the number of square-free elements of G is greater than 2. Then G belongs to $\mathcal{A}_2 \cap \mathcal{A}_3$. In view of Theorems 1 and 2, G is \mathbb{Z}_2^n or \mathbb{Z}_6 , where $n \geq 2$. ■

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REFERENCES

- [1] A. Abdollahi and M. Jazaeri, *Groups all of whose undirected Cayley graphs are integral*, European J. Combin. **38** (2014) 102–109.
doi:10.1016/j.ejc.2013.11.007
- [2] A. Abdollahi and E. Vatandoost, *Which Cayley graphs are integral?*, Electron. J. Combin. **16** (2009) #R122.
- [3] A. Ahmady, J.P. Bell and B. Mohar, *Integral Cayley graphs and groups*, SIAM J. Discrete Math. **28** (2014) 685–701.
doi:10.1137/130925487
- [4] N. Alon, *Large sets in finite fields are sumsets*, J. Number Theory **126** (2007) 110–118.
doi:10.1016/j.jnt.2006.11.007
- [5] M. Amooshahi and B. Taeri, *On Cayley sum graphs of non-Abelian groups*, Graphs Combin. **32** (2016) 17–29.
doi:10.1007/s00373-015-1535-4
- [6] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs* (Springer-Verlag, New York, 2012).
doi:10.1007/978-1-4614-1939-6
- [7] F.C. Bussemaker and D.M. Cvetković, *There are exactly 13 connected, cubic, integral graphs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **544–576** (1976) 43–48.
- [8] B. Cheyne, V. Gupta and C. Wheeler, *Hamilton cycles in addition graphs*, Rose-Hulman Undergrad. Math J. **4** (2003) 1–17.

- [9] F.R.K. Chung, *Diameters and eigenvalues*, J. Amer. Math. Soc. **2** (1989) 187–196.
doi:10.1090/S0894-0347-1989-0965008-X
- [10] M. DeVos, L. Goddyn, B. Mohar and R. Šámal, *Cayley sum graphs and eigenvalues of (3, 6)-fullerenes*, J. Combin. Theory Ser. B **99** (2009) 358–369.
doi:10.1016/j.jctb.2008.08.005
- [11] I. Estélyi and I. Kovács, *On groups all of whose undirected Cayley graphs of bounded valency are integral*, Electron. J. Combin. **21** (2014) #P4.45.
- [12] B.J. Green, *Counting sets with small sumset, and the clique number of random Cayley graphs*, Combinatorica **25** (2005) 307–326.
doi:s00493-005-0018-2
- [13] F. Harary and A.J. Schwenk, *Which graphs have integral spectra?* in: R.A. Bari and F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Math. **406** (Springer, 1974) 45–51.
doi:10.1007/bfb0066434
- [14] W. Klotz and T. Sander, *Integral Cayley graphs over abelian groups*, Electron. J. Combin. **17** (2010) #R81.
- [15] X. Ma and K. Wang, *On finite groups all of whose cubic Cayley graphs are integral*, J. Algebra Appl. **15** (2016) 1650105, 10 pp.
doi:10.1142/S021949881650105X
- [16] A.J. Schwenk, *Exactly thirteen connected cubic graphs have integral spectra*, in: Y. Alavi and D.R. Lick (Eds.), Theory and Applications of Graphs, Lecture Notes in Math. **642** (Springer, 1978) 516–533.
doi:10.1007/BFb0070407
- [17] W. So, *Integral circulant graphs*, Discrete Math. **306** (2006) 153–158.
doi:10.1016/j.disc.2005.11.006

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