Discussiones Mathematicae
Graph Theory 35 (2015) 261–270
doi:10.7151/dmgt.1797

THE *k*-RAINBOW BONDAGE NUMBER OF A DIGRAPH

Jafar Amjadi, Negar Mohammadi

Seyed Mahmoud Sheikholeslami

Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran

e-mail: {j-amjadi;s.m.sheikholeslami}@azaruniv.edu

and

Lutz Volkmann

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany

e-mail: volkm@math2.rwth-aachen.de

Abstract

Let $D = (V, A)$ be a finite and simple digraph. A *k-rainbow dominating function* (kRDF) of a digraph D is a function f from the vertex set V to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^-(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled, where $N^-(v)$ is the set of in-neighbors of v. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *k-rainbow domination number* of a digraph D, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of D. The *k-rainbow bondage number* $b_{rk}(D)$ of a digraph D with maximum in-degree at least two, is the minimum cardinality of all sets $A' \subseteq A$ for which $\gamma_{rk}(D - A') > \gamma_{rk}(D)$. In this paper, we establish some bounds for the k-rainbow bondage number and determine the k-rainbow bondage number of several classes of digraphs.

Keywords: k-rainbow dominating function, k-rainbow domination number, k-rainbow bondage number, digraph.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let D be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices. We write $\deg_D^+(v) = \deg^+(v)$ for the *outdegree* of a vertex v and $\deg_D^-(v) = \deg^-(v)$ for its *indegree*. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If (u, v) is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. The *underlying graph* $G[D]$ of a digraph D is the graph obtained by replacing each arc uv by an edge uv. Note that $G[D]$ has two parallel edges uv when D contains the arcs (u, v) and (v, u). A digraph D is called *connected*, if the underlying graph $G[D]$ is connected. For the notation and terminology not defined here, we refer the reader to [11].

Let k be a positive integer. A *k-rainbow dominating function* (kRDF) of a digraph D is a function f from the vertex set $V(D)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^-(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a digraph D, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of D. A *$\gamma_{rk}(D)$-function* is a k-rainbow dominating function of D with weight $\gamma_{rk}(D)$. Note that $\gamma_{r1}(D)$ is the classical domination number $\gamma(D)$. The k-rainbow domination numbers in digraphs were investigated by Amjadi *et al.* in [1]. A 2-rainbow dominating function (briefly, rainbow dominating function) $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$ and $V_{1,2} = \{v \in V \mid f(v) = \{1, 2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

Proposition A [1]**.** *Let D be a digraph of order n. Then $\gamma_{r2}(D) < n$ if and only if $\Delta^+(D) \geq 2$ or $\Delta^-(D) \geq 2$.*

Proposition B [1]**.** *Let $k \geq 1$ be an integer. If D is a digraph of order n, then*

$$\min\{k, n\} \leq \gamma_{rk}(D) \leq n.$$

Proposition C [1]**.** *Let D be a digraph of order $n \geq 2$. Then $\gamma_{r2}(D) = 2$ if and only if $n = 2$ or $n \geq 3$ and $\Delta^+(D) = n - 1$ or there exist two different vertices u and v such that $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$.*

Proposition D [1]. *Let $k \geq 1$ be an integer. If D is a digraph of order n, then*

$$\gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1.$$

The definition of the k-rainbow dominating function for undirected graphs was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 4, 5, 9, 10, 12, 13]).

Following the ideas in [7], we initiate the study of k-rainbow bondage number on digraphs D. The k-rainbow bondage number $b_{rk}(D)$ of a digraph D is the cardinality of a smallest set of arcs $A' \subseteq A(D)$ for which $\gamma_{rk}(D - A') > \gamma_{rk}(D)$. An edge set B with $\gamma_{rk}(D - B) > \gamma_{rk}(D)$ is called the *k-rainbow bondage set.* A $b_{rk}(D)$-set is a k-rainbow bondage set of D of size $b_{rk}(D)$. If B is a $b_{rk}(D)$-set, then clearly

$$\gamma_{rk}(D - B) = \gamma_{rk}(D) + 1. \tag{1}$$

By Proposition A, we note that if D is a digraph with $\Delta^+(D) \leq 1$ and $\Delta^-(D) \leq 1$, then $\gamma_{r2}(D) = n$ and hence if $A' \subseteq A(D)$, then $\gamma_{r2}(D - A') = \gamma_{r2}(D)$. Therefore the 2-rainbow bondage number is only defined for a digraph with maximum in-degree or maximum out-degree at least two.

The definition of the k-rainbow bondage number for undirected graphs was given by Dehgardi, Sheikholeslami and Volkmann [6].

The purpose of this paper is to establish some bounds for the k-rainbow bondage number of a digraph.

Observation 1. *Let D be a digraph of order n with $\gamma_{rk}(D) < n$. Assume that H is a spanning subdigraph of D with $\gamma_{rk}(H) = \gamma_{rk}(D)$. If $K = A(D) - A(H)$, then $b_{rk}(H) \leq b_{rk}(D) \leq b_{rk}(H) + |K|$.*

Proof. Let $F \subseteq A(D)$ be a $b_{rk}(D)$-set. It follows that $\gamma_{rk}(H-F) \geq \gamma_{rk}(D-F) > \gamma_{rk}(D) = \gamma_{rk}(H)$ and hence $b_{rk}(H) \leq |F| = b_{rk}(D)$.

Now let $F' \subseteq A(H)$ be a $b_{rk}(H)$-set. We deduce that $\gamma_{rk}(D - (K \cup F')) = \gamma_{rk}(H - F') > \gamma_{rk}(H) = \gamma_{rk}(D)$ and thus $b_{rk}(D) \leq b_{rk}(H) + |K|$. ∎

Observation 2. *If a digraph D has a vertex v such that every $\gamma_{rk}(D)$-function assigns a set of size at least 2 to v, then $b_{rk}(D) \leq \deg^+(v) \leq \Delta^+$.*

Proof. Assume that A_v^+ is the set of arcs in D with tail v and let f be a $\gamma_{rk}(D - A_v^+)$-function. Since $N^+_{D-A_v^+}(v) = \emptyset$, we deduce that $|f(v)| \leq 1$ and hence f is not a $\gamma_{rk}(D)$-function. Thus $\gamma_{rk}(D - A_v^+) > \gamma_{rk}(D)$, and the proof is complete. ∎

Theorem 3. *Let k be a positive integer and let D be a digraph of order $n \geq k+1$. If the underlying graph of D is connected, then*

$$b_{rk}(D) \leq (\gamma_{rk}(D) - k + 1)\Delta(G[D]).$$

Proof. By Proposition B, $\gamma_{rk}(D) \geq k$. We proceed by induction on $\gamma_{rk}(D)$. If $\gamma_{rk}(D) = k$, then let u be a vertex in D, and let A_u denote the set of arcs incident with u. Since $n \geq k+1$, we deduce from Proposition B that $\gamma_{rk}(D - A_u) = 1 + \gamma_{rk}(D-u) \geq k+1 > \gamma_{rk}(D)$. This implies that $b_{rk}(D) \leq |A_u| = \deg_{G[D]}(u)$ and hence $b_{rk}(D) \leq \Delta(G[D])$.

Now assume that the statement is true for any digraph of order $n \geq k+1$ with k-rainbow domination number $k \leq \gamma_{rk}(D) \leq s$. Assume that D is a digraph of order $n \geq k+1$ with $\gamma_{rk}(D) = s+1$. Suppose to the contrary that $b_{rk}(D) > (\gamma_{rk}(D) - k + 1)\Delta(G[D]) > \Delta(G[D])$. Let u be an arbitrary vertex of D, and let A_u denote the set of arcs incident with u. Then $\gamma_{rk}(D) = \gamma_{rk}(D - A_u)$, because $\deg_{G[D]}(u) < b_{rk}(D)$. Let f be a $\gamma_{rk}(D - A_u)$-function. Obviously, $|f(u)| = 1$ and the function f restricted to $D-u$ is a $\gamma_{rk}(D-u)$-function. This yields $\gamma_{rk}(D-u) = \gamma_{rk}(D) - 1$. It follows from Observation 1 that $b_{rk}(D) \leq b_{rk}(D-u) + \deg_{G[D]}(u)$, and by the induction hypothesis we obtain

$$\begin{aligned} b_{rk}(D) &\leq b_{rk}(D-u) + \deg_{G[D]}(u) \\ &\leq (s-k+1)\Delta(G[D-u]) + \deg_{G[D]}(u) \\ &\leq (s-k+1)\Delta(G[D]) + \Delta(G[D]) \\ &= ((s+1)-k+1)\Delta(G[D]) = (\gamma_{rk}(D)-k+1)\Delta(G[D]). \end{aligned}$$

This contradiction completes the proof. ■

2. Upper Bounds on the 2-Rainbow Bondage Number

In this section we mainly present bounds on the 2-rainbow bondage number of a digraph.

Theorem 4. *If D is a digraph, and xyz a path of length 2 in $G[D]$ such that $(y,x),(y,z) \in A(D)$, then*

(2) $\quad b_{r2}(D) \leq \deg_{G[D]}(x) + \deg_{G[D]}(y) + \deg_{G[D]}(z) - 3 - |N^-(x) \cap N^-(y)|.$

Moreover, if x and z are adjacent in $G[D]$, then

(3) $\quad b_{r2}(D) \leq \deg_{G[D]}(x) + \deg_{G[D]}(y) + \deg_{G[D]}(z) - 4 - |N^-(x) \cap N^-(y)|.$

Proof. Let A_1 be the set of arcs incident with x, y or z with the exception of (y,z) and all arcs going from $N^-(x) \cap N^-(y)$ to y. Then

$$|A_1| \leq \deg_{G[D]}(x) + \deg_{G[D]}(y) + \deg_{G[D]}(z) - 3 - |N^-(x) \cap N^-(y)|$$

and

$$|A_1| \leq \deg_{G[D]}(x) + \deg_{G[D]}(y) + \deg_{G[D]}(z) - 4 - |N^-(x) \cap N^-(y)|$$

when x and z are adjacent. Now let $D_1 = D - A_1$. Obviously in D_1, the vertex x is isolated, z is a vertex with indegree 1, y is an in-neighbor of z, and all in-neighbors of y in D_1, if any, are contained in $N^-(x)$. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D_1)$-function. Then $|f(x)| = 1$ and $|f(z)| \leq 1$.

If $f(z) = \emptyset$, then $f(y) = \{1, 2\}$ and therefore $(V_0 \cup \{x\}, V_1 - \{x\}, V_2 - \{x\}, V_{1,2})$ is a 2RDF on D of weight less than $\omega(f)$, and consequently (2) as well as (3) are proved.

Now assume that $|f(z)| = 1$. If $|f(y)| = 1$, then $(V_0 \cup \{z\}, V_1 - \{y, z\}, V_2 - \{y, z\}, V_{1,2} \cup \{y\})$ is also a $\gamma_{r2}(D_1)$-function, and we are in the situation discussed in the previous case. However, if $f(y) = \emptyset$, then there exists a vertex $w \in N^-(x) \cap N^-(y)$ such that $f(w) = \{1, 2\}$ or there exist two vertices $w_1, w_2 \in N^-(x) \cap N^-(y)$ such that $f(w_1) = \{1\}$ and $f(w_2) = \{2\}$. Since w, w_1 and w_2 are in-neighbors of x in D, $(V_0 \cup \{x\}, V_1 - \{x\}, V_2 - \{x\}, V_{1,2})$ is a 2RDF on D of weight less than f, and the proof is complete. ■

Theorem 5. *If D is a digraph, and xyz a path of length 2 in $G[D]$ such that $(y, x), (y, z) \in A(D)$, then*

$$b_{r2}(D) \leq \deg_{G[D]}(x) + \deg^-(y) + \deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|. \tag{4}$$

Moreover, if x and z are adjacent in $G[D]$, then

$$b_{r2}(D) \leq \deg_{G[D]}(x) + \deg^-(y) + \deg_{G[D]}(z) - 1 - |N^-(x) \cap N^-(y) \cap N^-(z)|. \tag{5}$$

Proof. Let F be the set of arcs incident with x or z and all arcs terminating in y except the arcs $w \to y$ for which the arcs $w \to x$ and $w \to z$ also occur in D. Then

$$|F| \leq \deg_{G[D]}(x) + \deg^-(y) + \deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|$$

and

$$|F| \leq \deg_{G[D]}(x) + \deg^-(y) + \deg_{G[D]}(z) - 1 - |N^-(x) \cap N^-(y) \cap N^-(z)|$$

when x and z are adjacent. Let now $D' = D - F$. In D', the vertices x, z are isolated, and all in-neighbors of y in D', if any, are contained in $N^-(x) \cap N^-(z)$. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D')$-function. Then $|f(x)| = |f(z)| = 1$ and we may assume, without loss of generality, that $f(x) = f(z) = \{1\}$.

If $f(y) = \{1, 2\}$, then $(V_0 \cup \{x, z\}, V_1 - \{x, z\}, V_2, V_{1,2})$ is a 2RDF on D of weight less than $\omega(f)$, and therefore (4) and (5) are proved.

If $|f(y)| = 1$, then $(V_0 \cup \{x, z\}, V_1 - \{x, y, z\}, V_2 - \{y\}, V_{1,2} \cup \{y\})$ is a 2RDF on D of weight less than $\omega(f)$, and the desired bounds are proved.

However, if $f(y) = \emptyset$, then there exists a vertex $w \in N^-(x) \cap N^-(y) \cap N^-(z)$ such that $f(w) = \{1, 2\}$ or there exist two vertices $w_1, w_2 \in N^-(x) \cap N^-(y) \cap$

$N^-(z)$ such that $f(w_1) = \{1\}$ and $f(w_2) = \{2\}$. Since w, w_1 and w_2 are in-neighbors of x and z in D, $(V_0 \cup \{x,z\}, V_1 - \{x,z\}, V_2, V_{1,2})$ is a 2RDF on D of weight less than f, and the proof is complete. ∎

Corollary 6. *If D is a digraph with $\delta^+(D) \geq 2$, then $b_{r2}(D) \leq 2\Delta(G[D]) + \delta^-(D)$.*

Proof. Let $y \in V(D)$ be a vertex with $\deg^-(y) = \delta^-(D)$. Since $\delta^+(D) \geq 2$, there exist two different vertices $x, z \in N^+(y)$. Thus $G[D]$ contains a path xyz such that $(y,x), (y,z) \in A(D)$. Now the result follows from Theorem 5. ∎

Since $\sum_{v\in V(D)} \deg^+(v) = \sum_{v\in V(D)} \deg^-(v)$ and $\sum_{v\in V(D)}(\deg^+(v)+\deg^-(v)) \leq n\Delta(G[D])$, we have $\delta^-(D) \leq \frac{1}{2}\Delta(G[D])$. Now, Corollary 6 leads to the next result.

Corollary 7. *If D is a digraph with $\delta^+(D) \geq 2$, then $b_{r2}(D) \leq \frac{5}{2}\Delta(G[D])$.*

For every graph G, the expression $\deg_a(G) = \sum_{v\in V(G)} \deg(v)/|V(G)|$ is called the *average degree* of G.

Lemma 8. *For any digraph D with $\delta^-(D) \geq 1$, there exists a pair of vertices, say u and v, that are either adjacent or at distance two in $G[D]$ with a common in-neighbor in D, with the property that*

$$\deg_{G[D]}(u) + \deg_{G[D]}(v) \leq 2\deg_a(G[D]).$$

Proof. Suppose that the lemma is false, and let D be a connected digraph where the result does not hold. Let the vertices of degree less than or equal to $\deg_a(G[D])$ in $G[D]$ be $S = \{u_1, u_2, \ldots, u_m\}$ and the vertices of degree greater than $\deg_a(G[D])$ be $T = \{v_1, v_2, \ldots, v_n\}$.

Observe that no pair of vertices of S can be joined by an arc. Hence, each $u_i \in S$ has only vertices in T as in-neighbors or out-neighbors. Also note that each v_j has at most one out-neighbor in S, for otherwise if there were two, they would contradict our assumption.

Now we proceed to sum the degrees of all vertices in the underlying graph $G[D]$ as follows. For each $u_i \in S$ we consider an in-neighbor $v_j \in T$ of u_i and take $\deg_{G[D]}(u_i) + \deg_{G[D]}(v_j)$. By assumption, we observe that $\deg_{G[D]}(u_i) + \deg_{G[D]}(v_j) > 2\deg_a(G[D])$. Furthermore, by the above remarks, these in-neighbors in T must be distinct. After adding m such pairs (to exhaust S), the degree of any remaining members of T are included. But the total sum of the degrees is greater than $|V(G[D])|\deg_a(G[D])$ which is impossible. This completes the proof. ∎

Next we present an upper bound on the size of a digraph with given rainbow domination number and rainbow bondage number.

Theorem 9. *Let D be a digraph of order n with $\delta^-(D) \geq 1$, $\delta^+(D) \geq 2$ and rainbow bondage number $b_{r2}(D)$. If $\deg_a(G[D])$ is the average degree of the underlying graph of D, then $b_{r2}(D) \leq 2\deg_a(G[D]) + \Delta(G[D]) - 3$ and $|A(D)| \geq (n/4)(b_{r2}(D) - \Delta(G[D]) + 3)$.*

Proof. Let D be a digraph satisfying the assumptions of the theorem. By Lemma 8, there is at least one pair of vertices, say u and v, that are either adjacent or at distance 2 from each other with a common in-neighbor, and with the property that $\deg_{G[D]}(u) + \deg_{G[D]}(v) \leq 2\deg_a(G[D])$. Since $\delta^+(D) \geq 2$, there is a path uvw in $G[D]$ such that $(v,u),(v,w) \in A(D)$, a path vuw in $G[D]$ such that $(u,v),(u,w) \in A(D)$, or a path uwv in $G[D]$ such that $(w,u),(w,v) \in A(D)$. Since these cases are symmetrical, we only consider the first. Applying Theorem 4, we obtain

$$\begin{aligned} b_{r2}(D) &\leq \deg_{G[D]}(u) + \deg_{G[D]}(v) + \deg_{G[D]}(w) - 3 \\ &\leq 2\deg_a(G[D]) + \Delta(G[D]) - 3. \end{aligned}$$

Since $2|E(G[D])| = n\deg_a(G[D])$, we have

$$4|E(G[D])| = 2n\deg_a(G[D]) \geq n(b_{r2}(D) - \Delta(G[D]) + 3).$$

Hence

$$|A(D)| = |E(G[D])| \geq (n/4)(b_{r2}(D) - \Delta(G[D]) + 3).$$

■

3. Some Classes of Digraphs

In this section we investigate complete digraphs, complete bipartite digraphs and transitive tournaments.

Lemma 10. *If $K^*_{p,q}$ is the complete bipartite digraph such that $q \geq p \geq 2k$, then $\gamma_{rk}(K^*_{p,q}) = 2k$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be the partite sets of $K^*_{p,q}$. It is easy to see that the function f defined by $f(x_i) = f(y_i) = \{i\}$ for $1 \leq i \leq k$ and $f(x) = \emptyset$ otherwise, is a k-rainbow dominating function of $K^*_{p,q}$ of weight $2k$ and hence $\gamma_{rk}(K^*_{p,q}) \leq 2k$.

Let now f be a $\gamma_{rk}(K^*_{p,q})$-function. If $f(x_i) \neq \emptyset$ for each i, then $\gamma_{rk}(K^*_{p,q}) = \omega(f) \geq 2k$. So assume $f(x_i) = \emptyset$ for some i, say $i = 1$. Similarly, we may assume $f(y_1) = \emptyset$. This implies that $\bigcup_{i=1}^{p} f(x_i) = \bigcup_{i=1}^{q} f(y_i) = \{1, 2, \ldots, k\}$. Hence $\gamma_{rk}(K^*_{p,q}) = \omega(f) \geq 2k$ and the proof is complete. ■

Theorem 11. *Let $k \geq 2$ be an integer and let $K^*_{p,q}$ be the complete bipartite digraph such that $2k+1 \leq p \leq q$. Then $p + 1 \leq b_{rk}(K^*_{p,q}) \leq 2p$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be the partite sets of $K^*_{p,q}$. If B is an arc set of $K^*_{p,q}$, then define $D = K^*_{p,q} - B$. If D contains a vertex $x \in X$ and a vertex $y \in Y$ such that $\deg^+_D(x) = q$ and $\deg^+_D(y) = p$, then it follows from Lemma 10 that $2k = \gamma_{rk}(K^*_{p,q}) \leq \gamma_{rk}(D) \leq 2k$ and therefore $\gamma_{rk}(D) = 2k$. Hence $b_{rk}(K^*_{p,q}) \geq p$. Now let $|B| = p$ and $D = K^*_{p,q} - B$ such that, without loss of generality, $\deg^+_D(x) \neq q$ for all $x \in X$. Then $B = \{x_1y_{i_1}, x_2y_{i_2}, \ldots, x_py_{i_p}\}$ with $y_{i_j} \in Y$ for $1 \leq j \leq p$. Assume that $y_{i_1} = y_1$. Define the function f by $f(x_1) = f(y_1) = \{1, 2, \ldots, k\}$ and $f(u) = \emptyset$ for $u \in V(K^*_{p,q}) - \{x_1, y_1\}$. It is easy to see that f is a k-rainbow dominating function of D of weight $2k$. Lemma 10 implies that $2k = \gamma_{rk}(K^*_{p,q}) \leq \gamma_{rk}(D) \leq 2k$ and thus $\gamma_{rk}(D) = 2k$. Consequently, $b_{rk}(K^*_{p,q}) \geq p + 1$.

Let now B_1 be the set of all arcs incident with the vertex y_1, and let $H = K^*_{p,q} - B_1$. Then y_1 is an isolated vertex in H and thus $\gamma_{rk}(H) = \gamma_{rk}(K^*_{p,q-1}) + 1$. Since $q \geq p \geq 2k+1$, Lemma 10 shows that $\gamma_{rk}(K^*_{p,q-1}) = 2k$ and thus $\gamma_{rk}(H) = 2k+1$. Since $|B_1| = 2p$, it follows that $b_{rk}(K^*_{p,q}) \leq 2p$, and the proof is complete. ■

Conjecture 12. *For integers $k \geq 2$ and $q \geq p \geq 2k + 1$, $b_{rk}(K^*_{p,q}) = 2p$.*

Theorem 13. *Let $k \geq 2$ be an integer. If K^*_n is the complete digraph of order $n \geq k + 1$, then $n \leq b_{rk}(K^*_n) \leq n + k - 1$.*

Proof. According to Propositions B and D, we deduce that $\gamma_{rk}(K^*_n) = k$. If B is an arc set of K^*_n, then define $D = K^*_n - B$. If D contains a vertex x such that $\deg^+_D(x) = n - 1$, then it follows from Propositions B and D that $\gamma_{rk}(D) = k$. This implies that $b_{rk}(K^*_n) \geq n$.

Now let $\{x_1, x_2, \ldots, x_n\}$ be the vertex set of the complete digraph K^*_n. Define the arc sets $B_1 = \{x_1x_n, x_2x_n, \ldots, x_{n-1}x_n\}$ and $B_2 = \{x_nx_1, x_nx_2, \ldots, x_nx_k\}$, and let $D = K^*_n - (B_1 \cup B_2)$. Then it is easy to see that $b_{rk}(D) = b_{rk}(K^*_{n-1}) + 1 = k + 1$. Since $\gamma_{rk}(K^*_n) = k$, we obtain $b_{rk}(K^*_n) \leq |B_1| + |B_2| = n - 1 + k$, and this is the desired upper bound. ■

Theorem 14. *If K^*_n is the complete digraph of order $n \geq 3$, then $b_{rk}(D) = b_{rk}(K^*_{n-1}) + 1 = k + 1$.*

Proof. By Theorem 13, we have $b_{r2}(K^*_n) \geq n$.

Now let $\{x_1, x_2, \ldots, x_n\}$ be the vertex set of K^*_n. We define the arc set B of K^*_n by $B = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. If $D = K^*_n - B$, then we observe that $\Delta^+(D) = n - 2$. In addition, we see that there do not exist two different vertices u and v in D such that $V(D) - \{u, v\} \subseteq N^+_D(u)$ and $V(D) - \{u, v\} \subseteq N^+_D(v)$. This can be seen as follows:

Suppose on the contrary that there exist two different vertices u and v in D such that $V(D) - \{u, v\} \subseteq N^+_D(u)$ and $V(D) - \{u, v\} \subseteq N^+_D(v)$. If, without loss of generality, $u = x_1$, then $x_2 \notin N_D^+(x_1)$. Therefore $v = x_2$. However, now $x_3 \notin N_D^+(x_2)$, a contradiction.

Applying Proposition C, we conclude that $\gamma_{r2}(D) \geq 3$. Since $\gamma_{r2}(K_n^*) = 2$, we deduce that $b_{r2}(K_n^*) \leq n$, and the proof is complete. ■

A *tournament* $T = (V, E)$ is an orientation of a complete graph. A tournament is called *transitive* if $p \to q$ and $q \to r$ imply that $p \to r$.

Theorem 15. *Let $k \geq 2$ be an integer. If T_n is the transitive tournament of order $n \geq k + 1$, then $b_{rk}(T_n) = 1$.*

Proof. Let $x_1 x_2 \cdots x_n$ be the unique directed Hamiltonian path of T_n. Then $\deg^+_{T_n}(x_1) = n - 1$, and therefore Propositions B and D imply that $\gamma_{rk}(T_n) = k$. Now let $D = T_n - \{x_1 x_n\}$, and let f be a $\gamma_{rk}(D)$-function.

Assume first that $f(x_n) = \emptyset$. This implies that $\bigcup_{u \in N_D^-(x_n)} f(u) = \{1, 2, \ldots, k\}$. Since $|f(x_1)| \geq 1$ and $x_1 \notin N_D^-(x_n)$, we obtain $\omega(f) \geq k + 1$.

Next, assume that $|f(x_n)| \geq 1$. If $|f(x_i)| \geq 1$ for each $1 \leq i \leq n - 1$, then $\omega(f) \geq n \geq k + 1$. So assume that $f(x_i = \emptyset$ for an index $i \in \{1, 2, \ldots, n - 1\}$. Then $\bigcup_{u \in N_D^-(x_i)} f(u) = \{1, 2, \ldots, k\}$. Since $x_n \notin N_D^-(x_i)$, we obtain $\omega(f) \geq k + 1$ again.

Therefore $\gamma_{rk}(D) \geq k + 1$. Since $\gamma_{rk}(T_n) = k$, we deduce that $b_{rk}(T_n) = 1$, and the proof is complete. ■

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Received 10 February 2014
Revised 12 June 2014
Accepted 12 June 2014