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Dedicated to Mietek Borowiecki on the occasion of his seventieth birthday.

ON MAXIMUM WEIGHT OF A BIPARTITE GRAPH OF GIVEN ORDER AND SIZE [1]

Mirko Horňák

Stanislav Jendrol'

Institute of Mathematics
P.J. Šafárik University
Jesenná 5, 04001 Košice, Slovakia

e-mail: mirko.hornak@upjs.sk
stanislav.jendrol@upjs.sk

and

Ingo Schiermeyer

Institut für Diskrete Mathematik und Algebra
Technische Universität Bergakademie Freiberg
09596 Freiberg, Germany

e-mail: ingo.schiermeyer@tu-freiberg.de

Abstract

The weight of an edge xy of a graph is defined to be the sum of degrees of the vertices x and y. The weight of a graph G is the minimum of weights of edges of G. More than twenty years ago Erdős was interested in finding the maximum weight of a graph with n vertices and m edges. This paper presents a complete solution of a modification of the above problem in which a graph is required to be bipartite. It is shown that there is a function $w^*(n,m)$ such that the optimum weight is either $w^*(n,m)$ or $w^*(n,m)+1$.

Keywords: weight of an edge, weight of a graph, bipartite graph.

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Let G be a finite simple nonoriented graph. The *weight* $w_G(e)$ of an edge $e = xy \in E(G)$ is defined to be $\deg_G(x) + \deg_G(y)$. The concept of the weight of an edge was introduced by Kotzig [10] who proved that every planar 3-connected graph contains an edge of the weight not exceeding 13.

The mentioned result was further developed in various directions. Grünbaum [4], Jucovič [7], Borodin [1], Fabrici and Jendrol' [3] studied inequalities for the number of edges having weight at most 13 in planar 3-connected graphs. Ivančo [5] found an analogue of Kotzig's result for graphs with minimum degree at least 3 and embedded on orientable 2-manifolds. Another analogue of Kotzig's result, this time for triangulations of orientable 2-manifolds, can be found in Zaks [11]. The case of graphs embedded on nonorientable 2-manifolds was investigated by Jendrol' et al. [9].

In [3] it is proved that each 3-connected planar graph of maximum degree at least k contains a path on k vertices such that each of its vertices has degree at most $5k$; moreover, the bound $5k$ is the best possible. Enomoto and Ota [2] proved that each planar 3-connected graph of order at least k contains a connected subgraph on k vertices such that the degree sum of the vertices of this subgraph is at most $8k - 1$.

Let $p, q \in \mathbb{Z}$. Throughout the paper we shall use the notation

$$[p, q] := \{z \in \mathbb{Z} : p \le z \le q\},$$
$$[p, \infty) := \{z \in \mathbb{Z} : p \le z\}$$

(for *integer* intervals).

Let the *weight* of a graph G, in symbols $w(G)$, be the minimum of weights of edges of G. At the Fourth Czechoslovak Symposium on Combinatorics held in Prachatice in 1990, Erdős posed the question: What is the maximum weight of an (n, m)-graph (having n vertices and m edges)? If $\mathcal{P}$ is a *graph property*, i.e., a set of (isomorphism classes of) finite simple nonoriented graphs, $n \in [2, \infty)$ and $m \in [1, \binom{n}{2}]$ is such that

$$\mathcal{P}(n, m) := \{G \in \mathcal{P} : |V(G)| = n, |E(G)| = m\} \neq \emptyset,$$

then the above problem can be naturally generalised:

Problem 1. Determine $w(\mathcal{P}, n, m) := \max\{w(G) : G \in \mathcal{P}(n, m)\}$.

Thus, Erdős was interested in finding $w(\mathcal{I}, n, m)$, where $\mathcal{I}$ is the set of all finite simple nonoriented graphs, $n \in [2, \infty)$ and $m \in [1, \binom{n}{2}]$. In [6] Ivančo and Jendrol' obtained some partial results. They observed that the weight of any edge e of a graph $G \in \mathcal{I}(n, m)$ cannot be larger than $m + 1$.

Proposition 2. *If $n \in [2, \infty)$ and $m \in [1, n - 1]$, then $w(\mathcal{I}, n, m) = m + 1$ and the bound is attained by the graph $K_{1,m} \cup (n - m - 1)K_1$.*

The case of very dense graphs is solved by the following theorem of [6].

Theorem 3. *If* $n \in [2, \infty)$ *and* $m = \binom{n}{2} - r$ *with* $r \in [0, n-2]$*, then*

$$w(\mathcal{I}, n, m) = \begin{cases} 2n-2, & \text{if } r = 0, \\ 2n-3, & \text{if } r = 1, \\ 2n-4, & \text{if } r \in [2, \lfloor \frac{n}{2} \rfloor] \text{ or } r = 3, \\ 2n-5, & \text{if } r \in [\lfloor \frac{n}{2} \rfloor + 1, \lceil \frac{n+2}{2} \rceil] \text{ or } r = 6, \\ 2n-6, & \text{otherwise.} \end{cases}$$

Graphs that attain the extremal value can be obtained by taking K_n and removing from it r independent edges or edges of a triangle (if $r = 3$) in the cases when $w(\mathcal{I}, n, m) \in [2n-2, 2n-4]$. In the case of $w(\mathcal{I}, n, m) = 2n-5$ take K_n and remove from it either $r-3$ independent edges and edges of an independent triangle or edges of a K_4 (if $r = 6$). Finally, in the case of $w(\mathcal{I}, n, m) = 2n-6$, edges of a cycle of length r are deleted from K_n.

In [6] there was also found a lower bound for $w(\mathcal{I}, n, m)$. The result reads as follows:

Theorem 4. *Let* $n \in [2, \infty)$, $m \in [1, \binom{n}{2}]$, $a = \lceil \frac{1}{2}(1 + \sqrt{1+8m}) \rceil$, $b = \frac{1}{2}(a^2 - a - 2m)$, $h = \left\lceil \frac{1}{2}(2n - 1 - \sqrt{(2n-1)^2 - 8m}) \right\rceil$ *and let* p, k *be integers such that* $hk + p = m$, $h + k \le n$ *and* $h(h-3) < 2p \le h(h-1)$*. Let* $f(n,m) = h + k + \lfloor \frac{2p}{h} \rfloor$ *and let* $g(n,m)$ *be defined by*

$$g(n,m) = \begin{cases} 2a-2, & \text{if } b = 0, \\ 2a-3, & \text{if } b = 1, \\ 2a-4, & \text{if } 2 \le b \le \lfloor \frac{a}{2} \rfloor \text{ or } b = 3, \\ 2a-5, & \text{if either } \lfloor \frac{a}{2} \rfloor + 1 \le b \le \lceil \frac{a+2}{2} \rceil \text{ or } a = 8 \text{ and } b = 6, \\ 2a-6, & \text{otherwise.} \end{cases}$$

Then $w(\mathcal{I}, n, m) \ge \max\{f(n,m), g(n,m)\}$.

The authors of [6] conjectured that the lower bound of Theorem 4 is in fact equal to $w(\mathcal{I}, n, m)$. The conjecture was proved by Jendrol' and Schiermeyer in [8].

Theorem 5. *If* $n \in [2, \infty)$, $m \in [1, \binom{n}{2}]$ *and* $f(n,m)$, $g(n,m)$ *are functions defined in Theorem* 4, then $w(\mathcal{I}, n, m) = max\{f(n,m), g(n,m)\}$.

In this paper we are dealing with the graph property

$$\mathcal{B} := \{G \in \mathcal{I} : G \text{ is bipartite}\}$$

and we solve completely the corresponding "portion" of Problem 1. Namely, we prove that there is $w^*(n,m) \in [2,n]$ such that $w^*(n,m) \le w(\mathcal{B}, n, m) \le$

$w^*(n,m)+1$. Moreover, $w(\mathcal{B},n,m) \le n$ and $w(\mathcal{B},n,m) = w^*(n,m)+1$ implies $w(\mathcal{B},n,m) = n-1$.

It is well known that $\mathcal{B}(n,m) \ne \emptyset$ if and only if $n \in [2,\infty)$ and $m \in [1, \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil]$. Henceforth we shall suppose implicitly that n and m are fixed and $\mathcal{B}(n,m) \ne \emptyset$. Then $1 \le m \le \frac{n^2}{4}$ and $m = \frac{n^2-4k}{4}$ for some $k \in [0, \frac{n^2-4}{4}]$ provided that $n \equiv 0 \pmod 2$, while $n \equiv 1 \pmod 2$ means that $m = \frac{n^2-4k-1}{4}$ for some $k \in [0, \frac{n^2-5}{4}]$.

Let G be a bipartite graph with a bipartition $\{X,Y\}$. An edge $xy \in E(G)$, $x \in X$, $y \in Y$, is *universal in* G provided that $\deg_G(x) = |Y|$ and $\deg_G(y) = |X|$ (or, equivalently, if $N_G(x) = Y$ and $N_G(y) = X$).

Lemma 6. *If* $G \in \mathcal{B}(n,m)$ *and* $e \in E(G)$*, then* $w_G(e) \in [2,n]$*. Moreover,* $w_G(e) = n$ *if and only if* e *is universal in* G*.*

Proof. Suppose that $\{X,Y\}$ is a bipartition of G and $e = xy$ with $x \in X$ and $y \in Y$. Then $1 \le \deg_G(x) \le |Y|$, $1 \le \deg_G(y) \le |X|$ and $2 \le w_G(e) = \deg_G(x) + \deg_G(y) \le |Y| + |X| = n$. Moreover, $w_G(e) = n$ is equivalent to $\deg_G(x) = |Y|$ and $\deg_G(y) = |X|$. ■

Corollary 7. $w(\mathcal{B},n,m) \in [2,n]$.

Lemma 8. *Suppose that* $n \in [2,\infty)$ *and* $l \in [1, \lfloor \frac{n^2}{4} \rfloor]$*. Then* $\sqrt{n^2-4l}$ *is an integer if and only if there is* $k \in [1, \lfloor \frac{n}{2} \rfloor]$ *such that* $l = k(n-k)$*.*

Proof. If $\sqrt{n^2-4l}$ is an integer, then $\sqrt{n^2-4l} = n-j$ for some $j \in [1,n]$, $4l = j(2n-j)$, hence j is even, $j = 2k$ with $k \in [1, \lfloor \frac{n}{2} \rfloor]$ and $l = k(n-k)$.

If $l = k(n-k)$, where $k \in [1, \lfloor \frac{n}{2} \rfloor]$, then $n-2k \ge 0$, $n^2-4l = (n-2k)^2$ and $\sqrt{n^2-4l} = n-2k$ is an integer. ■

Proposition 9. $w(\mathcal{B},n,m) = n$ *if and only if* $\sqrt{n^2-4m}$ *is an integer.*

Proof. Suppose that $w(\mathcal{B},n,m) = n = w(G)$ for some $G \in \mathcal{B}(n,m)$ with a bipartition $\{X,Y\}$. By Lemma 6 then each edge of G is universal in G and $E(G)$ consists of all edges joining X to Y. Therefore, $G \cong K_{k,n-k}$, where $k = |X|$, $m = |E(G)| = k(n-k)$ and $k^2 - nk + m = 0$. Thus k, as a root of the quadratic equation $x^2 - nx + m = 0$, is either $\frac{1}{2}(n - \sqrt{n^2-4m})$ or $\frac{1}{2}(n + \sqrt{n^2-4m})$, from which it follows that $\sqrt{n^2-4m}$ is an integer.

If $\sqrt{n^2-4m}$ is an integer, then, by Lemma 8, $m = k(n-k)$ with $k \in [1, \lfloor \frac{n}{2} \rfloor]$, $K_{k,n-k} \in \mathcal{B}(n,m)$ and, since $w(K_{k,n-k}) = n$, using Corollary 7 we obtain $w(\mathcal{B},n,m) = n$. ■

Proposition 10. *The following two statements are equivalent:*

(1) $w(\mathcal{B},n,m) = n-1$.

(2) *The number* $\sqrt{n^2-4m}$ *is not an integer, while* (*exactly*) *one of the numbers* $\sqrt{(n-1)^2-4m}$ *and* $\sqrt{n^2-4m-4}$ *is.*

Proof. (1) $\Rightarrow$ (2): The fact that $\sqrt{n^2-4m}$ is not an integer follows from Proposition 9.

To prove the rest consider a pair (n,m) with $w(\mathcal{B},n,m) = n-1 = w(G)$, where $G \in \mathcal{B}(n,m)$ has a bipartition $\{X,Y\}$. Without loss of generality we may suppose that X does not contain isolated vertices of G. Let $d := \min\{\deg_G(x) : x \in X\}$ and pick $x \in X$ so that $\deg_G(x) = d$. Clearly, $d < |Y|$, because $d = |Y|$ means that G is a complete bipartite graph with $w(G) = n$. On the other hand, $d > |Y|-2$, since $d = |Y|-i$ with $i \geq 2$ yields $w_G(xy) \leq |Y|-i+|X| = n-i < n-1$ for any edge $xy \in E(G)$. Thus, $d = |Y|-1$.

Now let y be the unique vertex of Y with $xy \notin E(G)$. If y is isolated in G, then $G-y \in \mathcal{B}(n-1,m)$ and $w(G-y) = w(G) = n-1$ so that, by Proposition 9, $\sqrt{(n-1)^2-4m}$ is an integer.

If y is not isolated in G, then $\deg_G(y) = |X|-1$, since from $\deg_G(y) = |X|-j$ with $j \geq 2$ we obtain $w_G(x'y) \leq |Y|+|X|-j = n-j < n-1$ for any edge $x'y \in E(G)$. Further, if $x_1y_1 \neq xy$, $x_1 \in X$ and $y_1 \in Y$, then $x_1y_1 \in E(G)$. Indeed, if $x_1y_1 \notin E(G)$, then $y_1 \neq y$ and $w_G(xy_1) \leq |Y|-1+|X|-1 = n-2$. So with $k := |X|$ we have $G = K_{k,n-k} - e$, $m = k(n-k)-1$, $k^2-nk+m+1 = 0$ and $\sqrt{n^2-4m-4}$ is an integer.

(2) $\Rightarrow$ (1): As a consequence of Proposition 9 and Corollary 7 we obtain $w(\mathcal{B},n,m) \leq n-1$.

If $\sqrt{(n-1)^2-4m}$ is an integer, then, by Lemma 8, $m = k(n-1-k)$ for some $k \in [1, \lfloor\frac{n-1}{2}\rfloor]$, hence $K_{k,n-1-k} \cup K_1 \in \mathcal{B}(n,m)$ and $w(\mathcal{B},n,m) \geq w(K_{k,n-1-k} \cup K_1) = n-1$.

If $\sqrt{n^2-4m-4}$ is an integer, then, again by Lemma 8, $m+1 = k(n-k)$, where $k \in [1, \lfloor\frac{n}{2}\rfloor]$, $K_{k,n-k} - e \in \mathcal{B}(n,m)$ and $w(\mathcal{B},n,m) \geq w(K_{k,n-k} - e) = n-1$. ■

If $G \in \mathcal{B}(n,m)$, there are $i_1 \in [1, \lfloor\frac{n}{2}\rfloor]$ and $i_2 \in [i_1, n-i_1]$ such that $G \subseteq K_{i_1,i_2} \cup (n-i_1-i_2)K_1$. In general, the pair (i_1,i_2) is not necessarily unique; it is said to be *standard for* G if it is lexicographically minimal from among all such pairs. Clearly, if (i_1,i_2) is standard for G, then no vertex of G belonging to K_{i_1,i_2} is isolated.

Let us define some numbers that will be important in our analysis:

$$i_{\min} := \left\lceil \frac{n-\sqrt{n^2-4m}}{2} \right\rceil,\ i_{\mathrm{mid}} := \lceil \sqrt{m}\,\rceil,\ i_{\max} := \left\lfloor \frac{n+\sqrt{n^2-4m}}{2} \right\rfloor;$$

it is easily seen that $i_{\min} \leq \lfloor \frac{n}{2} \rfloor$ and $i_{\min} \leq i_{\max}$. Further, for $i \in [1, n-1]$ let

$$a_i := i, \quad b_i := \lceil m/a_i \rceil, \quad s_i := a_i b_i - m, \; p_i := \min\{s_i, 2\}, \; w_i := a_i + b_i - p_i,$$
$$a^* := i_{\min}, \; b^* := \lceil m/a^* \rceil, \; s^* := a^* b^* - m, \; p^* := \min\{s^*, 2\}, \; w^* := a^* + b^* - p^*.$$

Clearly, $w^* = w^*(n, m)$ is an integer depending on n and m.

Proposition 11. *If $G \in \mathcal{B}(n, m)$ and (i_1, i_2) is the standard pair for G, then $i_{\min} \leq i_1 \leq i_2 \leq i_{\max}$.*

Proof. For both $l = 1, 2$, the graph G is a subgraph of the graph $K_{i_l, n-i_l}$. Therefore, $m = |E(G)| \leq i_l(n - i_l)$, $i_l^2 - ni_l + m \leq 0$, and so $i_l \in [\lceil x_1 \rceil, \lfloor x_2 \rfloor] = [i_{\min}, i_{\max}]$, where $x_{1,2} := \frac{n \mp \sqrt{n^2-4m}}{2}$ are solutions of the quadratic equation $x^2 - nx + m = 0$. ■

Proposition 12. *For every $i \in [1, n-1]$ the following hold:*

1. *$i + b_i \leq n$ if and only if $i \in [i_{\min}, i_{\max}]$.*
2. *If $i + b_i \leq n$ and $i \leq b_i + 1$, then $w(\mathcal{B}, n, m) \geq w_i$.*

Proof. 1. If $i + b_i \leq n$, then $i + \frac{m}{i} \leq n$, $i^2 - ni + m \leq 0$ and (as in the proof of Proposition 11) $i \in [\lceil x_1 \rceil, \lfloor x_2 \rfloor]$. To show that $i \in [\lceil x_1 \rceil, \lfloor x_2 \rfloor]$ implies $i + b_i \leq n$ we prove an equivalent assertion $i + b_i > n \Rightarrow (i < \lceil x_1 \rceil \vee i > \lfloor x_2 \rfloor)$. For that purpose notice that $i + \frac{m}{i} + 1 > i + b_i \geq n + 1$, $i^2 - ni + m > 0$, and then either $i < \lceil x_1 \rceil$ or $i > \lfloor x_2 \rfloor$, as required.

2. We have $0 \leq s_i = i \lceil \frac{m}{i} \rceil - m \leq i \frac{m+i-1}{i} - m = i - 1$. If $i - 1 \leq b_i$, then the graph $K_{i,b_i} \cup (n - i - b_i)K_1$ has a matching of size s_i, and so $G_i := (K_{i,b_i} - s_i K_2) \cup (n - i - b_i)K_1$ is a bipartite graph of order n and size $ib_i - s_i = m$. If $p_i = 0$, then $s_i = 0$ and all edges of G_i are of weight $i + b_i = w_i$. If $p_i = 1$, then $s_i = 1$ and the weight of G_i is attained on any edge sharing a vertex with the unique non-edge of G_i so that $w(G_i) = i + b_i - 1 = w_i$. Finally, $p_i = 2$ implies $s_i \geq 2$ and the weight of G_i is attained on any edge joining a vertex of a non-edge of G_i to a vertex of another non-edge of G_i, which yields $w(G_i) = i + b_i - 2 = w_i$. Thus $w(\mathcal{B}, n, m) \geq w(G_i) = w_i$. ■

Lemma 13. *The following statements are equivalent:*

(1) $a^* = k$.

(2) $(k-1)(n-k+1) + 1 \leq m \leq k(n-k)$.

(3) $\lceil \frac{m}{k} \rceil + k \leq n \leq \left\lfloor \frac{m+k(k-2)}{k-1} \right\rfloor$.

Proof. The equivalence of (1) and (2) follows from the defining inequalities for $a^* = \left\lceil \frac{n-\sqrt{n^2-4m}}{2} \right\rceil$, i.e., $\frac{n-\sqrt{n^2-4m}}{2} \leq a^* < \frac{n-\sqrt{n^2-4m}}{2} + 1$, and from the fact that m is an integer.

The equivalence of (2) and (3) is an obvious consequence of the fact that n is an integer. (For $k = 1$ the righthand side of (3) can be formally set to ∞ indicating that n is not bounded from above.) ■

Corollary 14. *If $a^* = k$, then $m \geq k^2$.*

Proof. The assumption $a^* = k$ by Lemma 13 means that $\frac{m}{k} + k \leq \lceil \frac{m}{k} \rceil + k \leq n \leq \left\lfloor \frac{m+k(k-2)}{k-1} \right\rfloor \leq \frac{m+k(k-2)}{k-1}$. Standard manipulations applied to the inequality $\frac{m}{k} + k \leq \frac{m+k(k-2)}{k-1}$ yield the desired result. ■

Theorem 15. $w(\mathcal{B}, n, m) = \max\{w_i : i \in [i_{\min}, i_{\text{mid}}]\}$.

Proof. Let us first show that i_{mid} (in the role of i) satisfies the assumptions of Proposition 12.2. We have $i_{\text{mid}} \leq \left\lceil \sqrt{n^2/4} \right\rceil \leq \frac{n+1}{2}$, and so $i_{\text{mid}} = \frac{n+k}{2}$ with $k \in [2-n, 1]$ and $k \equiv n \pmod 2$. From $\left(\frac{n+k-2}{2}\right)^2 < m \leq \left(\frac{n+k}{2}\right)^2$ it follows that $\left\lceil \frac{m}{i_{\text{mid}}} \right\rceil \leq \lceil (\frac{n+k}{2})^2/(\frac{n+k}{2}) \rceil = \frac{n+k}{2}$ and $i_{\text{mid}} + \left\lceil \frac{m}{i_{\text{mid}}} \right\rceil \leq n+k$. If $k \leq 0$, then $i_{\text{mid}} + \left\lceil \frac{m}{i_{\text{mid}}} \right\rceil \leq n+k \leq n$. On the other hand, the assumption $i_{\text{mid}} = \frac{n+1}{2}$ yields $n \equiv 1 \pmod 2$, $m \leq \frac{n^2-1}{4}$, $\left\lceil \frac{m}{i_{\text{mid}}} \right\rceil \leq \left\lceil (\frac{n^2-1}{4})/(\frac{n+1}{2}) \right\rceil = \frac{n-1}{2}$ and $i_{\text{mid}} + \left\lceil \frac{m}{i_{\text{mid}}} \right\rceil \leq n$. Thus, by Proposition 12.1, $i_{\text{mid}} \in [i_{\min}, i_{\max}]$, and $i + b_i \leq n$ for any $i \in [i_{\min}, i_{\text{mid}}]$.

Moreover, $\frac{m}{i_{\text{mid}}} > \left(\frac{n+k-2}{2}\right)^2 / \left(\frac{n+k}{2}\right) > \frac{n+k-4}{2}$, and hence $\left\lceil \frac{m}{i_{\text{mid}}} \right\rceil \geq \frac{n+k-2}{2} = i_{\text{mid}} - 1$. Let us prove by descending induction that $\lceil \frac{m}{i} \rceil \geq i - 1$ for every $i \in [i_{\min}, i_{\text{mid}}]$. The first step has been performed above. So, suppose that $i \in [i_{\min} + 1, i_{\text{mid}}]$ and $\lceil \frac{m}{i} \rceil \geq i - 1$. If the inequality $\left\lceil \frac{m}{i-1} \right\rceil \geq i - 2$ is not true, then $\frac{m}{i-1} \leq i - 3$, $m \leq (i-1)(i-3) < (i-2)^2$, $i > \sqrt{m} + 2$ and $i \geq \lceil \sqrt{m} \rceil + 2 > i_{\text{mid}}$, a contradiction. By Proposition 12.1 we know that $i + b_i \leq n$ for any $i \in [i_{\min}, i_{\text{mid}}]$. Therefore, with help of Proposition 12.2, we see that $w(\mathcal{B}, n, m) \geq M := \max\{w_i : i \in [i_{\min}, i_{\text{mid}}]\}$.

To prove the inequality $w(\mathcal{B}, n, m) \leq M$ consider an arbitrary graph $G \in \mathcal{B}(n, m)$. Let (i_1, i_2) be the standard pair for G and let U_1, U_2 be partite sets of the graph K_{i_1,i_2} with $E(K_{i_1,i_2}) \supseteq E(G)$ satisfying $|U_l| = i_l$, $l = 1, 2$. Then $m = |E(G)| \leq i_1 i_2$, $i_2 \geq \left\lceil \frac{m}{i_1} \right\rceil$, $i_1 + \left\lceil \frac{m}{i_1} \right\rceil \leq i_1 + i_2 \leq n$, and so, by Proposition 12.1, $i_1 \geq i_{\min}$.

If $i_1 \leq i_{\text{mid}}$, we can show that $w(G) \leq w_{i_1}$. Suppose first that there is a vertex $u_2 \in U_2$ such that $\deg_G(u_2) \in [1, i_1 - 1]$, say $\deg_G(u_2) = i_1 - t$ for some $t \in [1, i_1 - 1]$. If $w(G) \geq w_{i_1} + 1 = i_1 + b_{i_1} - p_{i_1} + 1$, it follows that $\deg_G(u_1) \geq b_{i_1} + t + 1 - p_{i_1}$ for all vertices $u_1 \in N_G(u_2) \subseteq U_1$. Further, $\deg_G(u_1) \geq b_{i_1} + 1 - p_{i_1}$ for all vertices $u_1 \in U_1 - N_G(u_2)$. Since $\min\{x(i_1 - x) : x \in \langle 1, i_1 - 1 \rangle\} = i_1 - 1$

and $i_1(2-p_{i_1}) > 1-p_{i_1}$ (which is a consequence of $p_{i_1} \in [0,2]$), we have

$$\begin{aligned} m = |E(G)| &\geq t(b_{i_1}+1-p_{i_1}) + (i_1-t)(b_{i_1}+t+1-p_{i_1}) \\ &= i_1(b_{i_1}+1-p_{i_1}) + t(i_1-t) \geq i_1(b_{i_1}+1-p_{i_1}) + i_1 - 1 \\ &= i_1 b_{i_1} - 1 + i_1(2-p_{i_1}) > i_1 b_{i_1} - p_{i_1} \geq i_1 b_{i_1} - s_{i_1} = m, \end{aligned}$$

a contradiction.

Now we may assume that $\deg_G(u_2) = i_1$ for every $u_2 \in U_2$. In such a case $m = i_1 i_2$, $i_2 = \frac{m}{i_1} = b_{i_1}$, $p_{i_1} = 0$, $G = K_{i_1,i_2} \cup (n-i_1-i_2)K_1$ and $w(G) = i_1 + i_2 = i_1 + b_{i_1} - p_{i_1} = w_{i_1}$.

In the remaining part of the proof we suppose that $i_1 \geq i_{\text{mid}}+1 \geq \sqrt{m}+1$. We have $\lceil\sqrt{m}\,\rceil(\lceil\sqrt{m}\,\rceil-2) < (\sqrt{m}+1)(\sqrt{m}-1) < m$, hence $m/\lceil\sqrt{m}\,\rceil > \lceil\sqrt{m}\,\rceil - 2$ and $\lceil m/\lceil\sqrt{m}\,\rceil\rceil \geq \lceil\sqrt{m}\,\rceil - 1$; on the other hand, $m/\lceil\sqrt{m}\,\rceil \leq m/\sqrt{m} = \sqrt{m}$, which implies $\lceil m/\lceil\sqrt{m}\,\rceil\rceil \leq \lceil\sqrt{m}\,\rceil$. So,

$$\lceil\sqrt{m}\,\rceil - 1 \leq b_{i_{\text{mid}}} = \left\lceil \frac{m}{\lceil\sqrt{m}\,\rceil} \right\rceil \leq \lceil\sqrt{m}\,\rceil. \tag{1}$$

Choose $u_l \in U_l$ so as to satisfy $\deg_G(u_l) = \min\{\deg_G(u) : u \in U_l\}$, choose $v_{3-l} \in N_G(u_l) \subseteq U_{3-l}$ and put $d_l := \deg_G(u_l)$. Let us prove the inequality

$$d_l \leq \lfloor\sqrt{m}\rfloor - 1, \ \ l = 1, 2. \tag{2}$$

First, a weaker (in general) inequality $d_l < \sqrt{m}$ is evident, since with $d_l \geq \sqrt{m}$ we would obtain $m \geq i_l d_l \geq (\sqrt{m}+1)\sqrt{m} > m$, a contradiction.

To show (2), admit that $d_l \geq \lfloor\sqrt{m}\rfloor$ for some $l \in [1,2]$. From the above weaker inequality we see that then $\sqrt{m} \notin \mathbb{Z}$ and

$$\begin{aligned} m = \sum\nolimits_{u \in U_l} \deg_G(u) &\geq i_l d_l \geq (\lceil\sqrt{m}\,\rceil+1)(\lceil\sqrt{m}\,\rceil-1) \\ &= \lceil\sqrt{m}\,\rceil^2 - 1 \geq m + 1 - 1 = m, \end{aligned}$$

hence

$$m = (\lceil\sqrt{m}\,\rceil+1)(\lceil\sqrt{m}\,\rceil-1), \tag{3}$$

every vertex in U_l is of degree $\lceil\sqrt{m}\,\rceil - 1$ and

$$w(G) = \lceil\sqrt{m}\,\rceil - 1 + d_{3-l} \leq \lceil\sqrt{m}\,\rceil - 1 + \lfloor\sqrt{m}\rfloor = 2\lceil\sqrt{m}\,\rceil - 2. \tag{4}$$

Because of (1), there are two cases to be considered.

If $b_{i_{\text{mid}}} = \lceil\sqrt{m}\,\rceil$, then, by (4), $M \geq w_{i_{\text{mid}}} = 2\lceil\sqrt{m}\,\rceil - p_{\lceil\sqrt{m}\,\rceil} \geq 2\lceil\sqrt{m}\,\rceil - 2 \geq w(G)$, which contradicts our assumption.

If, however, $\lceil\sqrt{m}\,\rceil - 1 = b_{i_{\text{mid}}} = \lceil m/\lceil\sqrt{m}\,\rceil\rceil$, then $m/\lceil\sqrt{m}\,\rceil \le \lceil\sqrt{m}\,\rceil - 1$, so that (3) yields $\lceil\sqrt{m}\,\rceil - 1 = m/(\lceil\sqrt{m}\,\rceil + 1) < m/\lceil\sqrt{m}\,\rceil \le \lceil\sqrt{m}\,\rceil - 1$, a contradiction.

Let us prove by the way of contradiction that $w(G) \le M$. So, suppose that $a^* = k$ and

$$(5)\qquad e \in E(G) \Rightarrow w_G(e) \ge M + 1 \ge \max\{w^* + 1, w_{i_{\text{mid}}} + 1\}.$$

If $k = 1$, then $b^* = m$, $w^* = m + 1$ and, by (2), $\deg_G(v_{3-l}) \ge w^* + 1 - d_l \ge m + 3 - \sqrt{m}$, $l = 1, 2$. We have $d_1 = d_2 = 1$, since $d_l \ge 2$ for some $l \in [1, 2]$ yields $m = \sum_{u \in U_{3-l}} \deg_G(u) \ge \sum_{u \in N_G(u_l)} \deg_G(u) \ge 2(m + 3 - \sqrt{m}) > m$, a contradiction. Thus, $w_G(u_1 v_2) \le 1 + m = w^*$ in contradiction to (5).

If $k = 2$, then $b^* = \lceil \frac{m}{2} \rceil$, $s^* = 2\lceil \frac{m}{2} \rceil - m \le 1$, $p^* = s^*$ and $w^* = 2 + \lceil \frac{m}{2} \rceil - p^* \ge \frac{m+2}{2}$. Further, Corollary 14 yields $m \ge 4$, hence $i_2 \ge i_1 \ge \lceil\sqrt{m}\,\rceil + 1 \ge 3$. If $l \in [1, 2]$, then, by (5) and (2), $w_G(u_l v_{3-l}) \ge w^* + 1 \ge \frac{m+4}{2}$ and $\deg_G(u) \ge \frac{m+4}{2} - d_l \ge \frac{m+6}{2} - \sqrt{m}$. Now $d_l \le 2$, for otherwise

$$m \ge \sum\nolimits_{u \in N_G(u_l)} \deg_G(u) \ge 3\left(\frac{m+6}{2} - \sqrt{m}\right) > m,$$

a contradiction. Therefore, $\deg_G(v_{3-l}) \ge \frac{m+4}{2} - 2 = \frac{m}{2}$. In the case $d_l = 2$ we obtain (having in mind that $i_{3-l} \ge 3 > d_l$) $m = \sum_{u \in U_{3-l}} \deg_G(u) > \sum_{u \in N_G(u_l)} \deg_G(u) \ge 2 \cdot \frac{m}{2} = m$, a contradiction. If $d_1 = d_2 = 1$, then $\deg_G(v_{3-l}) \ge \frac{m+4}{2} - d_l = \frac{m+2}{2}$, $l = 1, 2$, and $m \ge \deg_G(v_1) + \deg_G(v_2) - 1 \ge 2 \cdot \frac{m+2}{2} - 1 > m$, a contradiction.

Henceforth we may suppose that $k \ge 3$, and, consequently, by Corollary 14, $m \ge k^2 \ge 9$.

If $k = 3$, then $b^* = \lceil \frac{m}{3} \rceil$, $s^* = 3\lceil \frac{m}{3} \rceil - m \le 2$, $p^* = s^*$ and $w^* = m + 3 - 2\lceil \frac{m}{3} \rceil \ge \frac{m+5}{3}$. If $l \in [1, 2]$ and $u \in N_G(u_l)$, then, by (5), $w_G(u_l u) \ge w^* + 1 \ge \frac{m+8}{3}$ and $\deg_G(u) \ge \frac{m+8}{3} - d_l$. Since $v_{3-l} \in N_G(u_l)$, $l = 1, 2$, the assumption $d_1 + d_2 \le 5$ leads to

$$\begin{aligned} n &\ge \sum\nolimits_{l=1}^{2} i_l \ge \sum\nolimits_{l=1}^{2} \deg_G(v_{3-l}) \ge \sum\nolimits_{l=1}^{2} \left(\frac{m+8}{3} - d_l\right) \\ &= \frac{2m+16}{3} - (d_1 + d_2) \ge \frac{2m+1}{3} > \left\lfloor \frac{m+3}{2} \right\rfloor, \end{aligned}$$

which contradicts Lemma 13. The above assumption is fulfilled if $9 \le m \le 15$, because then, by (2), $d_l \le 2$, $l = 1, 2$.

So we may assume that $d_1 + d_2 \ge 6$ and $m \ge 16$. Pick $l \in [1, 2]$. Since

$$m \ge \sum\nolimits_{u \in N_G(u_l)} \deg_G(u) \ge d_l \left(\frac{m+8}{3} - d_l\right),$$

the inequality $d_l(\frac{m+8}{3} - d_l) > m$ equivalent to $3d_l^2 - (m+8)d_l + 3m < 0$ suffices for obtaining a contradiction. The discriminant of the quadratic equation

$$3x^2 - (m+8)x + 3m = 0$$

is $D_1(m) = m^2 - 20m + 64 \geq 0$ and $\sqrt{D_1(m)} \geq m - 16$. Thus, a contradiction will appear as soon as there is $l \in [1,2]$ with

$$d_l \in \left(\frac{m+8-\sqrt{D_1(m)}}{6}, \frac{m+8+\sqrt{D_1(m)}}{6}\right) \supseteq \left(4, \frac{m-4}{3}\right).$$

Therefore, for the rest of our analysis of the case $k = 3$ we may suppose that either $d_l \leq 4$ or $d_l \geq \frac{m-4}{3}$ for both $l = 1,2$. However, the latter possibility does not apply at all, for otherwise, by (2), we would obtain $\frac{m-4}{3} \leq d_l \leq \lfloor\sqrt{m}\rfloor - 1 \leq \sqrt{m} - 1$, which yields $m \leq 10$, a contradiction; thus, $3 \leq \max\{d_1, d_2\} \leq 4$.

If there is $l \in [1,2]$ with $d_l = 4$, then $4 = d_l \leq \sqrt{m} - 1$, $m \geq 25$, $\deg_G(u) \geq \frac{m-4}{3}$ for each $u \in N_G(u_l)$ and $m \geq 4 \cdot \frac{m-4}{3} \geq m + 3$, a contradiction.

Finally, if $d_1 = d_2 = 3$, then

$$\sum\nolimits_{u \in N_G(u_1)} \deg_G(u) \geq 3\left(\frac{m+8}{3} - 3\right) = m - 1,$$

hence $i_2 = 3$ (as a consequence of $d_2 = 3$). Thus, in U_2 there are two vertices of degree $\frac{m-1}{3}$ and one vertex of degree $\frac{m+2}{3}$, so that $3 = d_1 = \frac{m-1}{3}$ and $m = 10$, a contradiction.

From now on suppose $k \geq 4$, so that $n \geq 2k \geq 8$, and, by Lemma 13, $m \geq 3n - 8 \geq 16$. Putting

$$j_l := \lfloor\sqrt{m}\rfloor - d_l$$

we see from (2) that $j_l \in [1, \lfloor\sqrt{m}\rfloor - 1]$, $l = 1, 2$.

The following assertion will be important for the rest of the proof of our theorem.

Claim. *If $l \in [1,2]$, then*

(i) $\deg_G(u) \geq \lceil\sqrt{m}\,\rceil$ *for every* $u \in N_G(u_l)$,

(ii) $N_G(u_l) \subsetneq U_{3-l}$,

(iii) $j_l + j_{3-l} \geq \frac{\sqrt{m}}{2}$.

Proof. Consider the distance

$$\alpha := \lceil\sqrt{m}\,\rceil - \sqrt{m} \in \langle 0, 1)$$

between $\sqrt{m}$ and $\lceil\sqrt{m}\,\rceil$. First notice that Claim (ii) is a direct consequence of Claim (i); indeed, if Claim (i) is true, then the assumption $N_G(u_l) = U_{3-l}$ would mean

$$m = \sum\nolimits_{u \in U_{3-l}} \deg_G(u) \geq i_{3-l}\lceil\sqrt{m}\,\rceil \geq (\lceil\sqrt{m}\,\rceil + 1)\lceil\sqrt{m}\,\rceil > m,$$

a contradiction.

Let $u \in N_G(u_l)$. Using (5) we have $w_G(u_l u) \geq w_{i_{\rm mid}} + 1$ and

$$\deg_G(u) \geq \lceil\sqrt{m}\,\rceil + \left\lceil \frac{m}{\lceil\sqrt{m}\,\rceil} \right\rceil - p_{\lceil\sqrt{m}\,\rceil} + 1 - \lfloor\sqrt{m}\rfloor + j_l. \tag{6}$$

Suppose first that $\sqrt{m} \notin \mathbb{Z}$ (which implies $\alpha > 0$ and $\lceil\sqrt{m}\,\rceil = \lfloor\sqrt{m}\rfloor + 1$). By (1) there are two cases to be considered.

If $\lceil m/\lceil\sqrt{m}\,\rceil\rceil = \lceil\sqrt{m}\,\rceil$, then (6) is transformed into

$$\deg_G(u) \geq (\lceil\sqrt{m}\,\rceil - \lfloor\sqrt{m}\rfloor + 1 - p_{\lceil\sqrt{m}\,\rceil}) + \lceil\sqrt{m}\,\rceil + j_l \geq \lceil\sqrt{m}\,\rceil + j_l,$$

so that

$$\begin{aligned} \sum\nolimits_{u\in N_G(u_l)} \deg_G(u) &\geq (\lfloor\sqrt{m}\rfloor - j_l)(\lceil\sqrt{m}\,\rceil + j_l) = (\lfloor\sqrt{m}\rfloor - j_l)(\lfloor\sqrt{m}\rfloor + j_l + 1) \\ &= \lfloor\sqrt{m}\rfloor^2 - j_l^2 + \lfloor\sqrt{m}\rfloor - j_l \end{aligned} \tag{7}$$

and $N_G(u_l) \subsetneq U_{3-l}$. Therefore, (7) yields

$$\begin{aligned} \lfloor\sqrt{m}\rfloor - j_{3-l} = d_{3-l} &\leq \frac{\sum_{u\in U_{3-l}-N_G(u_l)} \deg_G(u)}{|U_{3-l} - N_G(u_l)|} = \frac{m - \sum_{u\in N_G(u_l)} \deg_G(u)}{i_{3-l} - (\lfloor\sqrt{m}\rfloor - j_l)} \\ &\leq \frac{m - \lfloor\sqrt{m}\rfloor^2 + j_l^2 + j_l - \lfloor\sqrt{m}\rfloor}{j_l + (i_{3-l} - \lfloor\sqrt{m}\rfloor)}. \end{aligned} \tag{8}$$

Since $i_{3-l} - \lfloor\sqrt{m}\rfloor \geq \lceil\sqrt{m}\,\rceil + 1 - \lfloor\sqrt{m}\rfloor = 2$ and $\frac{j_l^2+j_l}{j_l+2} \leq j_l - \frac{1}{3}$ (as a consequence of $j_l \geq 1$), from (8) it follows

$$\lfloor\sqrt{m}\rfloor - j_{3-l} \leq \frac{m - \lfloor\sqrt{m}\rfloor^2 - \lfloor\sqrt{m}\rfloor}{3} + \frac{j_l^2 + j_l}{j_l + 2} \leq \frac{m - \lfloor\sqrt{m}\rfloor^2 - \lfloor\sqrt{m}\rfloor - 1}{3} + j_l,$$

and

$$\begin{aligned} j_l + j_{3-l} &\geq \frac{4\lfloor\sqrt{m}\rfloor + \lfloor\sqrt{m}\rfloor^2 - m + 1}{3} \\ &= \frac{4(\sqrt{m} + \alpha - 1) + (\sqrt{m} + \alpha - 1)^2 - m + 1}{3} \\ &= \frac{\sqrt{m}(2 + 2\alpha) + \alpha^2 + 2\alpha - 2}{3} > \frac{2\sqrt{m} - 2}{3} \geq \frac{\sqrt{m}}{2} \end{aligned}$$

(where the last inequality comes from $m \geq 16$).

If $\lceil m/\lceil\sqrt{m}\,\rceil\rceil = \lceil\sqrt{m}\,\rceil - 1$, then $m/(\sqrt{m} + \alpha) = m/\lceil\sqrt{m}\,\rceil \leq \lceil\sqrt{m}\,\rceil - 1 = \sqrt{m} + \alpha - 1$ and $m \leq m + \sqrt{m}(2\alpha - 1) + \alpha(\alpha - 1)$, so that necessarily $\alpha > \frac{1}{2}$. From (6) we have

$$\begin{aligned} \deg_G(u) &\geq (\lceil\sqrt{m}\,\rceil - \lfloor\sqrt{m}\rfloor + 1 - p_{\lceil\sqrt{m}\,\rceil}) + \lceil\sqrt{m}\,\rceil - 1 + j_l \\ &\geq \lceil\sqrt{m}\,\rceil - 1 + j_l = \lfloor\sqrt{m}\rfloor + j_l, \end{aligned}$$

so that

$$\sum_{u\in N_G(u_l)} \deg_G(u) \ge (\lfloor\sqrt{m}\rfloor - j_l)(\lfloor\sqrt{m}\rfloor + j_l) = \lfloor\sqrt{m}\rfloor^2 - j_l^2$$

and $N_G(u_l) \subsetneqq U_{3-l}$. Since $\frac{j_l^2}{j_l+2} \le j_l - \frac{2}{3}$, similarly as above we obtain

$$\lfloor\sqrt{m}\rfloor - j_{3-l} = d_{3-l} \le \frac{m - \lfloor\sqrt{m}\rfloor^2 + j_l^2}{j_l+2} \le \frac{m - \lfloor\sqrt{m}\rfloor^2 - 2}{3} + j_l,$$

$$j_l + j_{3-l} \ge \frac{3\lfloor\sqrt{m}\rfloor + \lfloor\sqrt{m}\rfloor^2 - m + 2}{3} = \frac{\sqrt{m}(2\alpha+1) + \alpha(\alpha+1)}{3} > \frac{2\sqrt{m}}{3} > \frac{\sqrt{m}}{2}.$$

Finally, suppose that $\sqrt{m} \in \mathbb{Z}$, which yields $w_{i_{\text{mid}}} = 2\sqrt{m}$. Then (6) reads as $\deg_G(u) \ge \lceil\sqrt{m}\,\rceil + j_l + 1 \ge \lceil\sqrt{m}\,\rceil + 2$, hence

$$\sum_{u\in N_G(u_l)} \deg_G(u) \ge (\sqrt{m} - j_l)(\sqrt{m} + j_l + 1) = m + \sqrt{m} - j_l^2 - j_l$$

and $N_G(u_l) \subsetneqq U_{3-l}$. As $|U_{3-l} - N_G(u_l)| = j_l + (i_{3-l} - \sqrt{m}) \ge j_l + 1$, proceeding analogously as above we obtain $\sqrt{m} - j_{3-l} = d_{3-l} \le \frac{j_l^2 + j_l - \sqrt{m}}{j_l+1} \le j_l - \frac{\sqrt{m}}{2}$ and $j_l + j_{3-l} \ge \frac{\sqrt{m}}{2}$. □

Since $v_{3-l} \in N_G(u_l)$, $l = 1, 2$, using (5) and Claim (iii) we get

$$\begin{aligned}
(9)\qquad n &\ge \sum_{l=1}^{2} i_l \ge \sum_{l=1}^{2} |N_G(v_{3-l})| \ge \sum_{l=1}^{2} (w^* + 1 - d_l)\\
&= \sum_{l=1}^{2} (a^* + b^* - p^* + 1 - \lfloor\sqrt{m}\rfloor + j_l)\\
&= 2\left(k + \left\lceil\frac{m}{k}\right\rceil - p^* + 1 - \lfloor\sqrt{m}\rfloor\right) + (j_1 + j_2)\\
&\ge 2\left(k + \frac{m}{k} - 1 - \sqrt{m}\right) + \frac{\sqrt{m}}{2} = 2\left(k + \frac{m}{k} - 1 - \frac{3\sqrt{m}}{4}\right).
\end{aligned}$$

From (9) it is clear that to obtain a contradiction it suffices to show that $k + \frac{m}{k} - 1 - \frac{3\sqrt{m}}{4} > \frac{n}{2}$. The function $f_1(x) = k + \frac{x}{k} - 1 - \frac{3\sqrt{x}}{4}$ is nondecreasing in the interval $\langle\frac{9k^2}{64}, \infty)$. If $a^* = k$, then, by Lemma 13, $m \ge (k-1)(n-k+1)+1$. We have $[(k-1)(n-k+1)+1, \infty) \subseteq \langle\frac{9k^2}{64}, \infty)$; indeed, from $k = i_{\min} \le \frac{n}{2}$ it follows that $(k-1)(n-k+1) \ge (k-1)(2k-k+1)+1 = k^2 > \frac{9k^2}{64}$. Therefore, in order to obtain a contradiction mentioned above, it is sufficient to check that

$$\frac{n}{2} < f_1((k-1)(n-k+1)+1) = n + 1 - \frac{n}{k} - \frac{3\sqrt{(k-1)(n-k+1)+1}}{4},$$

or, equivalently,

$$(10)\qquad n(2k-4) + 4k > 3k\sqrt{(k-1)(n-k+1)+1},$$

or either (after squaring both sides of (10))

$$n^2(2k-4)^2 + n(-9k^3 + 25k^2 - 32k) + 7k^2 + 9k^2(k-1)^2 > 0. \tag{11}$$

The discriminant of the quadratic equation

$$x^2(2k-4)^2 + x(-9k^3 + 25k^2 - 32k) + 7k^2 + 9k^2(k-1)^2 = 0$$

is $D_2(k) = k^3 D_3(k)$ with $D_3(k) := -63k^3 + 414k^2 - 783k + 576$. The function $D_3(x)$ is nonincreasing in the interval $\langle 3, \infty)$. Since $D_3(5) = -864$, it is clear that $D_2(k) < 0$ for every $k \in [5, \lfloor \frac{n}{2} \rfloor]$, which confirms the validity of (11) yielding a contradiction.

If $k = 4$, then (11) is equivalent to $n^2 - 19n + 88 > 0$. The last inequality is true whenever $n \geq 12$. On the other hand, the assumption $n \in [8, 11]$ (recall that we have $n \geq 8$) together with the inequality $m \geq 2n - 8$ (Lemma 13) lead to $n \geq i_1 + i_2 \geq 2(\lceil \sqrt{m} \rceil + 1) \geq 2(\sqrt{m} + 1) \geq 2(\sqrt{3n-8} + 1) > n$, a final contradiction. ■

Lemma 16. *If $i \in [i_{\min}, i_{\text{mid}} - 1]$, then the following hold:*

1. $w_{i+1} \leq w_i + 1$.
2. *If $w_{i+1} = w_i + 1$, then $b_{i+1} = b_i - 2$, $s_i \geq 2$ and $s_{i+1} = 0$.*
3. *If $w_{i+1} = w_i$ and $s_{i+1} \geq 2$, then $b_{i+1} = b_i - 1$.*
4. *If $s_i \leq 1$ and $i \leq i_{\text{mid}} - 2$, then $w_{i+1} \leq w_i - 1$.*

Proof. We have $b_{i+1} = \lceil \frac{m}{i+1} \rceil \leq \lceil \frac{m}{i} \rceil = b_i$. Let us prove that $b_{i+1} < b_i$. If $i \leq i_{\text{mid}} - 2$, then $i(i+1) \leq (\lceil \sqrt{m} \rceil - 2)(\lceil \sqrt{m} \rceil - 1) < (\sqrt{m} - 1)\sqrt{m} < m$, $\frac{m}{i} - \frac{m}{i+1} = \frac{m}{i(i+1)} > 1$ and the desired inequality follows. It remains to be shown that $b_{\lceil \sqrt{m} \rceil - 1} \neq b_{\lceil \sqrt{m} \rceil}$. Since $m/\lceil \sqrt{m} \rceil \leq m/\sqrt{m} = \sqrt{m} < m/(\lceil \sqrt{m} \rceil - 1)$, we see that $\lceil m/\lceil \sqrt{m} \rceil \rceil$ can be equal to $\lceil m/(\lceil \sqrt{m} \rceil - 1) \rceil$ only if each of those two numbers is $\lceil \sqrt{m} \rceil$. In such a case, however, both $m/(\lceil \sqrt{m} \rceil - 1)$ and $m/\lceil \sqrt{m} \rceil$ are in the interval $(\lceil \sqrt{m} \rceil - 1, \lceil \sqrt{m} \rceil \rangle$, and then $(\lceil \sqrt{m} \rceil - 1)\lceil \sqrt{m} \rceil < m \leq (\lceil \sqrt{m} \rceil - 1)\lceil \sqrt{m} \rceil$, a contradiction.

If $b_{i+1} \leq b_i - 4$, then $w_{i+1} \leq i + 1 + b_i - 4 - p_{i+1} \leq i + b_i - 3 < i + b_i - p_i = w_i$.

If $b_{i+1} = b_i - 3$, then $w_{i+1} = i + 1 + b_i - 3 - p_{i+1} \leq i + b_i - 2 \leq i + b_i - p_i = w_i$ and $w_{i+1} = w_i$ implies $p_i = 2$ and $p_{i+1} = 0$, hence $s_i \geq 2$ and $s_{i+1} = 0$.

If $b_{i+1} = b_i - 2$, then $w_{i+1} = i + 1 + b_i - 2 - p_{i+1} \leq i + b_i - 1 \leq i + b_i - p_i + 1 = w_i + 1$. Moreover, $w_{i+1} = w_i + 1$ yields $p_i = 2$ and $p_{i+1} = 0$ (and, consequently, $s_i \geq 2$ and $s_{i+1} = 0$), while $w_{i+1} = w_i$ implies either $p_i = 1$ and $p_{i+1} = 0$ ($s_i = 1$ and $s_{i+1} = 0$) or $p_i = 2$ and $p_{i+1} = 1$ ($s_i \geq 2$ and $s_{i+1} = 1$).

Finally, if $b_{i+1} = b_i - 1$, then $m = ib_i - s_i = (i+1)(b_i - 1) - s_{i+1}$. From $b_i - (i+1) \geq \lceil m/(\lceil \sqrt{m} \rceil - 1) \rceil - \lceil \sqrt{m} \rceil \geq \lceil m/\sqrt{m} \rceil - \lceil \sqrt{m} \rceil = 0$ and $(i+1)(b_i - 1) = ib_i + b_i - (i+1) \geq ib_i$ it follows that $s_{i+1} \geq s_i$, $p_{i+1} \geq p_i$ and

$w_{i+1} = i+1+b_i-1-p_{i+1} \le i+b_i-p_i = w_i$. Besides that, from the assumption $i \le i_{\rm mid}-2$ we obtain $b_i-(i+1) \ge \lceil m/(\lceil\sqrt{m}\,\rceil-2)\rceil - \lceil\sqrt{m}\,\rceil + 1 \ge \lceil m/(\sqrt{m}-1)\rceil - \lceil\sqrt{m}\,\rceil + 1 \ge \lceil\sqrt{m}+1\rceil - \lceil\sqrt{m}\,\rceil + 1 = 2$, $s_{i+1} \ge s_i+2$, and then w_{i+1} can be equal to w_i only if $s_i \ge 2 = p_i = p_{i+1}$.

The statements of lemma follow by inspecting the above assertions. ■

Lemma 17. *If* $i \in [i_{\rm min}, i_{\rm mid}-1]$ *and* $j \in [i+1, i_{\rm mid}]$, *then* $w_j \le w_i + 1$.

Proof. If there is $l \in [i+1, i_{\rm mid}]$ with $w_l \ge w_i+1$, then, by Lemma 16.1, $J := \{j \in [i+1, i_{\rm mid}] : w_j = w_{j-1}+1\} \ne \emptyset$. Moreover, $s_{j-1} \ge 2$ and $s_j = 0$ for every $j \in J$ (Lemma 16.2) and $w_{j+1} \le w_j - 1$ for every $j \in J - \{i_{\rm mid}-1, i_{\rm mid}\}$ (Lemma 16.4). Let $r := |J|$ and let $J = \{j_k : k \in [1,r]\}$, where the sequence $(j_1, \ldots, j_r)$ is increasing. (Notice that $j_{k+1} \ge j_k+2$ for every $k \in [1, r-1]$.) Then $w_j \le w_i$ for every $j \in [i+1, j_1-1]$ and $w_{j_1} \le w_i+1$. Further, if $k \in [1, r-1]$, then (by induction one can prove) $w_j \le w_{j_k} - 1 \le w_i$ for every $j \in [j_k+1, j_{k+1}-1]$ and $w_{j_{k+1}} \le w_{j_k} \le w_i+1$. Finally, if $j_r = i_{\rm mid}$, then $w_j \le w_{j_r} - 1 \le w_i$ for every $j \in [j_r+1, i_{\rm mid}]$. If $j_r = i_{\rm mid}-1$, then $w_{i_{\rm mid}} \le w_{j_r} \le w_i+1$ (the first inequality follows from the fact that $i_{\rm mid} \notin J$). ■

Theorem 18. $w(\mathcal{B}, n, m)$ *is either* w^* *or* w^*+1 *and in the latter case there is a positive integer* l *such that* $a^*+l \le i_{\rm mid}$, $m = (a^*+l)(b^*-l-1)$, $b^* \le 2a^*$, $s^* \ge 2$ *and* $p^* = 2$.

Proof. By Theorem 15 and by Lemma 17 with $i = i_{\rm min} = a^*$ we have $w^* = w_{i_{\rm min}} \le w(\mathcal{B}, n, m) \le w^*+1$.

If $w(\mathcal{B}, n, m) = w^*+1$, by Theorem 15 there is $j \in [1, i_{\rm mid} - i_{\rm min}]$ such that $w_{a^*+j} = w^*+1$. With $l := \min\{j \in [1, i_{\rm mid}-i_{\rm min}] : w_{a^*+j} = w^*+1\}$ Lemma 17 yields $w_{a^*+j} = w^*$ for every $j \in [1, l-1]$ ($w_{a^*+j} \le w^*-1$ for some $j \in [1, l-1]$ would imply $w_{a^*+l} \le w_{a^*+j}+1 \le w^*$, a contradiction).

Then, by Lemma 16.2, $s_{a^*+l} = 0$ and $s_{a^*+l-1} \ge 2$. If $s_{a^*+j} \le 1$ for some $j \in [0, l-2]$, then by taking j to be maximum, we have $s_{a^*+j+1} \ge 2$. Since $a^*+j \le a^*+l-2 \le i_{\rm mid}-2$, by using Lemma 16.4, we have $w_{a^*+j+1} \le w_{a^*+j}-1$, a contradiction. Thus $s_{a^*+j} \ge 2$ for every $j \in [0, l-1]$, in particular $s^* \ge 2$ and $p^* = 2$. Moreover, by Lemma 16.3, $b_{a^*+j} = b_{a^*} - j = b^* - j$ for each $j \in [0, l-1]$, and by Lemma 16.2, $b_{a^*+l} = b_{a^*+l-1} - 2 = b^*-l-1$ and $s_{a^*+l} = 0 = p_{a^*+l}$. Consequently,

$$m = (a^*+l)b_{a^*+l} - p_{a^*+l} = (a^*+l)(b^*-l-1), \tag{12}$$

where $a^*+l \le a^* + i_{\rm mid} - i_{\rm min} = i_{\rm mid}$.
Let us show that $b^* \le 2a^*$. Since $a^*+1 \le a^*+l \le i_{\rm mid}$,

$$m = a^*b^* - s^* = (a^*+1)b_{a^*+1} - s_{a^*+1}. \tag{13}$$

If $l = 1$, then $b_{a^*+1} = b^* - 2$ and $s_{a^*+1} = 0$. Thus, by (12) and (13), $2a^* - b^* = s^* - 2 \geq 0$ as required. If $l \geq 2$, then $b_{a^*+1} = b^* - 1$. Since $s_{a^*+1} \leq a^*$, from (12) and (13) we obtain $2a^* - b^* = a^* - s_{a^*+1} + s^* - 1 > 0$ and the proof follows. ■

Theorem 19. *If $w(\mathcal{B}, n, m) = w^* + 1$, then $a^* + b^* = n$ and $w(\mathcal{B}, n, m) = n - 1$.*

Proof. The assumption $w(\mathcal{B}, n, m) = w^* + 1$ gives us $a^* \geq 2$, because $a^* = 1$ yields $b^* = m$ and $s^* = 0 = p^*$ so that, by Theorem 18, $w(\mathcal{B}, n, m) = w^*$.

From Theorem 18 we know that $2a^* \geq b^*$, $p^* = 2$ and $s^* \geq 2$, hence, by Proposition 12.1, $w(\mathcal{B}, n, m) = a^* + b^* - p^* + 1 = a^* + b^* - 1 = a_{i_{\min}} + b_{i_{\min}} - 1 \leq n - 1$, $a^* + b^* \leq n$ and $a^* + b^* = n - r$ with $r \geq 0$. Suppose that $r \geq 1$. The complete bipartite graph K_{a^*-1, b^*+1+r} is of order $a^* + b^* + r = n$ and (as $m = a^*b^* - s^*$) of size $(a^* - 1)(b^* + 1 + r) = m + (s^* - 2) + (r - 1)(a^* - 1) + (2a^* - b^*) \geq m$. Consider an arbitrary subgraph G of K_{a^*-1, b^*+1+r} belonging to $\mathcal{B}(n, m)$. Then the standard pair (i_1, i_2) for G satisfies $i_1 \leq a^* - 1 = i_{\min} - 1$ in contradiction to Proposition 11. Therefore, $r = 0$, $a^* + b^* = n$ and $w(\mathcal{B}, n, m) = n - 1$. ■

Theorem 20. *Suppose that $r_0 = \sqrt{n^2 - 4m}$, $r_1 = \sqrt{(n-1)^2 - 4m}$ and $r_1' = \sqrt{n^2 - 4m - 4}$.*

1. *If r_0 is an integer, then $w(\mathcal{B}, n, m) = n$.*
2. *If r_0 is not an integer and (exactly) one of r_1, r_1' is, then $w(\mathcal{B}, n, m) = n - 1$.*
3. *If r_0, r_1, r_1' are not integers, then $w(\mathcal{B}, n, m) = w^*$.*

Proof. The theorem is a direct consequence of Propositions 9 and 10, and of Theorems 18 and 19. ■

The rest of the paper is devoted to showing that there are parameters n, m such that $w(\mathcal{B}, n, m) = w^* + 1$.

Lemma 21. *Suppose that $w(\mathcal{B}, n, m) = w^* + 1$.*

1. *If $n \equiv 0 \pmod 2$, then $a^* \leq \frac{n-4}{2}$.*
2. *If $n \equiv 1 \pmod 2$, then $a^* \leq \frac{n-3}{2}$.*

Proof. The lemma will be proved by the way of contradiction with the help of Theorem 18. Namely, we shall show that if the inequalities for a^* are invalid, then $w(\mathcal{B}, n, m) = w^*$. This will be done mostly by exhibiting that $s^* \in [0, 1]$.

1. Assume that n is even and $a^* \geq \frac{n-2}{2}$. Then $\frac{n - \sqrt{n^2 - 4m}}{2} > \frac{n-4}{2}$, $n^2 - 4m < 16$ and $m \in \{\frac{n^2 - 4i}{2} : i \in [0, 3]\}$. If $m = \frac{n^2}{4}$, then $a^* = \frac{n}{2} = b^*$ and $s^* = a^*b^* - m = 0$. Let $m = \frac{n^2 - 4i}{4}$, $i \in [1, 3]$, so that $n \geq 4$ and $a^* = \left\lceil \frac{n - \sqrt{4i}}{2} \right\rceil = \frac{n-2}{2}$. By Theorem 19, $b^* = n - a^* = \frac{n+2}{2}$ and $s^* = \frac{n^2 - 4}{4} - \frac{n^2 - 4i}{4} = i - 1$ so that with $i \in [1, 2]$ the mentioned contradiction follows. If $i = 3$, then $s^* = 2$, $w^* = n - 2$, $i_{\mathrm{mid}} =$

$\left\lceil \sqrt{(n^2-12)/4} \right\rceil \le \frac{n}{2}$, $\frac{n-2}{2} < \frac{n^2-12}{4} \le \frac{n^2}{4}$, hence $b_{\frac{n}{2}} = \frac{n}{2}$, $s_{\frac{n}{2}} = \frac{n^2}{4} - \frac{n^2-12}{4} = 3$, $p_{\frac{n}{2}} = 2$ and $w_{\frac{n}{2}} = n-2 = w^*$ so that, by Theorem 15, $w(\mathcal{B}, n, m) = w^*$.

2. Provided that n is odd and $a^* \ge \frac{n-1}{2}$, we have $\frac{n-\sqrt{n^2-4m}}{2} > \frac{n-3}{2}$, $n^2 - 4m < 9$ and $m = \frac{n^2-1-4i}{4}$ with $i \in [0,1]$ and $a^* = \left\lceil \frac{n-\sqrt{4i+1}}{2} \right\rceil = \frac{n-1}{2}$. By Theorem 19, $b^* = n - a^* = \frac{n+1}{2}$ and $s^* = \frac{n^2-1}{4} - \frac{n^2-1-4i}{4} = i \in [0,1]$. ■

Theorem 22. *If* $w(\mathcal{B}, n, m) = w^* + 1$, *then* $m \le \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$ *and there is* $i \in [0, \infty)$ *such that one of the following three series of conditions is satisfied:*

$n \equiv 0 \,(\mathrm{mod}\ 3)$, $n \ge 9$ *and* $m = \left(\frac{n+3}{3} + i\right)\left(\frac{2n-6}{3} - i\right) \ge \frac{n+3}{3} \cdot \frac{2n-6}{3} = \frac{2n^2-18}{9}$;

$n \equiv 2 \,(\mathrm{mod}\ 3)$, $n \ge 11$ *and* $m = \left(\frac{n+4}{3} + i\right)\left(\frac{2n-7}{3} - i\right) \ge \frac{n+4}{3} \cdot \frac{2n-7}{3} = \frac{2n^2+n-28}{9}$;

$n \equiv 1 \,(\mathrm{mod}\ 3)$, $n \ge 16$ *and* $m = \left(\frac{n+5}{3} + i\right)\left(\frac{2n-8}{3} - i\right) \ge \frac{n+5}{3} \cdot \frac{2n-8}{3} = \frac{2n^2+2n-40}{9}$.

Proof. Let us first show that with $w(\mathcal{B}, n, m) = w^* + 1$ we cannot have $n \le 8$ or $n \in \{10, 13\}$.

If $n \le 8$, then, by Lemma 21, $a^* \le \frac{n-3}{2} < 3$, $a^* \le 2$, $s^* \le 1$ and so, by Theorem 18, $w(\mathcal{B}, n, m) = w^*$.

Suppose $n = 10$ and $w(\mathcal{B}, n, m) = w^* + 1$. By Theorem 18 and Lemma 21 then $2 \le s^* \le a^* - 1 \le 2$, $s^* = 2$ and $a^* = 3$ so that Theorem 19 yields $b^* = 10 - a^* = 7$, which contradicts the inequality $b^* \le 2a^*$ of Theorem 18.

Suppose $n = 13$ and $w(\mathcal{B}, n, m) = w^* + 1$. By Lemma 21, $a^* \le \frac{n-3}{2} = 5$, while Theorems 18 and 19 imply $b^* = 13 - a^* \le 2a^*$, which yields $a^* > 4$. Thus $a^* = 5$ and $b^* = 8$. By Theorem 18, $s^* \ge 2$, and then $m = a^* b^* - s^* = 40 - s^* \le 38$. Since $\left\lceil \frac{13-\sqrt{169-4m}}{2} \right\rceil = a^* = 5$, we have $m > 36$, thus $m \in [37, 38]$. Then, however, m cannot be expressed as $(a^* + l)(b^* - l - 1)$, where l is a positive integer with $a^* + l \le i_{\mathrm{mid}} = \lceil \sqrt{m}\, \rceil = 7$, a contradiction to Theorem 18.

So, in the sequel we suppose that $w(\mathcal{B}, n, m) = w^* + 1$, $n \ge 9$ and $n \notin \{10, 13\}$. By Theorem 19 and Theorem 18 then $n - 1 = w(\mathcal{B}, n, m) = w^* + 1 = a^* + b^* - 1$ and $n = a^* + b^* \le 3a^*$ so that $a^* \ge \lceil \frac{n}{3} \rceil$. Therefore, $a^* \ge \frac{n+c(n)}{3}$, where $c(n) \in [0, 2]$ is such that $n + c(n) \equiv 0 \pmod 3$. As a consequence, $a^* = \frac{n+c(n)}{3} + j$ and $b^* = \frac{2n-c(n)}{3} - j$ for some nonnegative integer j. By Theorem 18 there is a positive integer l such that

$$a^* + l \le i_{\mathrm{mid}} = \lceil \sqrt{m}\, \rceil \tag{14}$$

and

$$\begin{aligned} m &= \left(\frac{n+c(n)}{3} + j + l\right)\left(\frac{2n-c(n)}{3} - j - 1 - l\right) \\ &= \left(\frac{n+c(n)+3}{3} + i\right)\left(\frac{2n-c(n)-6}{3} - i\right) =: f_4(i) \end{aligned}$$

with $i := j+l-1 \in [0,\infty)$. Thus we know that $m = k_1k_2 = k_1(n-1-k_1) \leq \left(\frac{n-1}{2}\right)^2$ and $m \leq \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$. Moreover, it is easy to check that $f_4(x) = f_4(\frac{n-2c(n)-9}{3} - x)$ and that

$$\text{(15)} \qquad \min\left\{ f_4(x) : x \in \left\langle 0, \frac{n-2c(n)-9}{3} \right\rangle \right\} = f_4(0) = f_4\left(\frac{n-2c(n)-9}{3}\right).$$

If n is even, then $i_{\text{mid}} = \lceil \sqrt{m} \rceil \leq \left\lceil \sqrt{n^2}/4 \right\rceil = \frac{n}{2}$, hence, by (14), $\frac{n+c(n)+3}{3} + i = a^* + l \leq \frac{n}{2}$, and

$$0 \leq i \leq \frac{n-2c(n)-6}{6} \leq \frac{n-2c(n)-9}{3}$$

(where the last inequality immediately follows from our assumptions on n).

If n is odd, then $i_{\text{mid}} \leq \frac{n-1}{2}$, $\frac{n+c(n)+3}{3} + i \leq \frac{n-1}{2}$, and

$$0 \leq i \leq \frac{n-2c(n)-9}{6} \leq \frac{n-2c(n)-9}{3}.$$

Thus, independently from the parity of n, because of (15) we have $m = f_4(i) \geq f_4(0)$. So, the statement of our theorem follows from the fact that $f_4(0) = \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3}$ is exactly the claimed lower bound for m depending on the congruence class modulo 3 containing n. ■

Let us prove now the tightness of the bounds for m in Theorem 22. Recall that $c(n) \in [0,2]$ is such that $n + c(n) \equiv 0 \pmod 3$.

Proposition 23. 1. *If $n \geq 9$, $n \notin \{10, 13\}$ and $m = \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3}$, then $w(\mathcal{B}, n, m) = w^* + 1$.*

2. *If $n = 2^{2q+1} + 1$ with $q \in \mathbb{Z}^+$ and $m = \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$, then $w(\mathcal{B}, n, m) = w^* + 1$.*

Proof. 1. If $m = \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3}$, then $n^2 - 4m = \frac{1}{9}[n^2 - 4nc(n) + 4(c(n) + 3)(c(n) + 6)]$ and $a^* = \left\lceil \frac{1}{2}(n - \sqrt{n^2-4m}) \right\rceil = \frac{n+c(n)}{3}$, because a necessary and sufficient pair of inequalities is

$$\frac{n+c(n)-3}{3} < \frac{1}{2}\left[n - \frac{1}{3}\sqrt{n^2-4nc(n)+4(c(n)+3)(c(n)+6)}\right] \leq \frac{n+c(n)}{3};$$

the first inequality is equivalent to $5c(n) + 3 < n$ and the second one is obvious.

Therefore, we have

$$
\begin{aligned}
b^* &= \left\lceil \frac{m}{a^*} \right\rceil = \left\lceil \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3} \cdot \frac{3}{n+c(n)} \right\rceil = \frac{2n-c(n)}{3}, \\
s^* &= \frac{n+c(n)}{3} \cdot \frac{2n-c(n)}{3} - \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3} = c(n)+2 \\
&\geq 2 = p^*, \\
w^* &= \frac{n+c(n)}{3} + \frac{2n-c(n)}{3} - 2 = n-2.
\end{aligned}
$$

On the other hand, $a^*+1 = \frac{n+c(n)+3}{3} \leq \frac{2n-c(n)-6}{3}$, hence $(a^*+1)^2 \leq \frac{n+c(n)+3}{3} \cdot \frac{2n-c(n)-6}{3} = m$ and $i_{\text{min}} \leq a^*+1 \leq \sqrt{m} \leq i_{\text{mid}}$. By Theorems 15 and 18 then $w^*+1 \geq w(\mathcal{B},n,m) \geq w_{a^*+1} = \frac{n+c(n)+3}{3} + \frac{2n-c(n)-6}{3} - 0 = n-1 = w^*+1$ and $w(\mathcal{B},n,m) = w^*+1$.

2. If $n = 2^{2q+1}+1$ and $m = \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor = 2^{4q}$, then

$$
\begin{aligned}
a^* &= \left\lceil (2^{2q+1}+1-\sqrt{2^{2q+2}+1})/2 \right\rceil = 2^{2q}-2^q+1, \\
b^* &= \lceil 2^{4q}/(2^{2q}-2^q+1) \rceil = 2^{2q}+2^q, \\
s^* &= (2^{2q}-2^q+1)(2^{2q}+2^q) - 2^{4q} = 2^q \geq 2 = p^*, \\
w^* &= (2^{2q}-2^q+1)+(2^{2q}+2^q)-2 = 2^{2q+1}-1 = n-2.
\end{aligned}
$$

Besides that, $i_{\text{min}} = a^* \leq 2^{2q} = \sqrt{m} = i_{\text{mid}}$, and, since $w_{2^{2q}} = 2^{2q}+2^{2q}-0 = 2^{2q+1} = w^*+1$, as above we obtain $w(\mathcal{B},n,m) = w^*+1$. ■

Note that there are n's such that the maximum m satisfying $w(\mathcal{B},n,m) = w^*+1$ is smaller than $\left\lfloor \frac{n^2-2n+1}{4} \right\rfloor$. Indeed, if $n = 2q^2$, $q \in \mathbb{Z}^+$, then with $m = \left\lfloor \frac{n^2-2n+1}{4} \right\rfloor = q^2(q^2-1)$ we have $a^* = q(q-1)$, $b^* = q(q+1)$, $s^* = 0 = p^*$ and $w^* = q(q-1)+q(q+1) = n$ so that $w(\mathcal{B},n,m) = w^*$.

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References

[1] O.V. Borodin, *Computing light edges in planar graphs*, in: Topics in Combinatorics and Graph Theory, R. Bodendiek and R. Henn (Ed(s)), (Physica-Verlag, Heidelberg, 1990) 137–144.

[2] H. Enomoto and K. Ota, *Connected subgraphs with small degree sum in* 3*-connected planar graphs*, J. Graph Theory **30** (1999) 191–203.
doi:10.1002/(SICI)1097-0118(199903)30:3⟨191::AID-JGT4⟩3.0.CO;2-X

[3] I. Fabrici and S. Jendroľ, *Subgraphs with restricted degrees of their vertices in planar* 3*-connected graphs*, Graphs Combin. **13** (1997) 245–250.

[4] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973) 390–408.
doi:10.1007/BF02764716

[5] J. Ivančo, *The weight of a graph*, in: Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity, J. Nešetřil and M. Fiedler (Ed(s)), (North-Holland, Amsterdam, 1992) 113–116.

[6] J. Ivančo and S. Jendroľ, *On extremal problems concerning weights of edges of graphs*, in: Sets, Graphs and Numbers, G. Halász, L. Lovász, D. Miklós and T. Szőnyi (Ed(s)), (North-Holland, Amsterdam, 1992) 399–410.

[7] E. Jucovič, *Strengthening of a theorem about* 3*-polytopes*, Geom. Dedicata **13** (1974) 233–237.
doi:10.1007/BF00183214

[8] S. Jendroľ and I. Schiermeyer, *On a max-min problem concerning weights of edges*, Combinatorica **21** (2001) 351–359.
doi:10.1007/s004930100001

[9] S. Jendroľ, M. Tuhársky and H.-J. Voss, *A Kotzig type theorem for large maps on surfaces*, Tatra Mt. Math. Publ. **27** (2003) 153–162.

[10] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955) 111–113.

[11] J. Zaks, *Extending Kotzig's theorem*, Israel J. Math. **45** (1983) 281–296.
doi:10.1007/BF02804013

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