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Dedicated to Mieczysław Borowiecki on the occasion of his 70th birthday

SUMS OF POWERED CHARACTERISTIC ROOTS COUNT DISTANCE-INDEPENDENT CIRCULAR SETS

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Abstract

Significant values of a combinatorial count need not fit the recurrence for the count. Consequently, initial values of the count can much outnumber those for the recurrence. So is the case of the count, $G_l(n)$, of distance-l independent sets on the cycle C_n, studied by Comtet for $l \geq 0$ and $n \geq 1$ [sic]. We prove that values of $G_l(n)$ are nth power sums of the characteristic roots of the corresponding recurrence unless $2 \leq n \leq l$. Lucas numbers $L(n)$ are thus generalized since $L(n)$ is the count in question if $l = 1$. Asymptotics of the count for $1 \leq l \leq 4$ involves the golden ratio (if $l = 1$) and three of the four smallest Pisot numbers inclusive of the smallest of them, plastic number, if $l = 4$. It is shown that the transition from a recurrence to an OGF, or back, is best presented in terms of mutually reciprocal (shortly: co-reciprocal) polynomials. Also the power sums of roots (i.e., moments) of a polynomial have the OGF expressed in terms of the co-reciprocal polynomial.

Keywords: distance independent set, Lucas numbers, Pisot numbers, power sums, generating functions, (co-) reciprocal polynomials.

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1. Introduction

In what follows we restrict our study to connected n-vertex graphs, the path P_n with $n \geq 1$ and the cycle C_n with $n \geq 2$, which are simple, with exception that C_2 will stand for the 2-vertex multigraph ${}^2K_2$, the 2-cycle. Additionally, C_1 will stand for the (1-vertex) loop-graph. The letter l stands for a nonnegative integer.

Our aim is to study the numbers, say $F_l(n)$ and $G_l(n)$, of l-independent sets (inclusive of the empty set) on the path P_n and the cycle C_n, respectively.

The *distance* between any two vertices x and y in a graph G is the length of a shortest x–y path of G. A set S (possibly empty) is called *l-independent* in G if S comprises vertices of G and any two elements of S are distance at least $l+1$ apart. In other words, if an l-independent set S includes distinct vertices x and y then every x–y path of G includes l or more vertices which do not belong to S. Consequently, each vertex subset of G is 0-independent. Moreover, 1-independent coincides with *independent*.

The numbers $F_l(n)$ and $G_l(n)$, denoted respectively by $F(n+l,l)$ and $G(n,l)$, appear in Comtet [4, p. 46]. Their OGFs (*ordinary generating functions*) are presented, too, though the case of G_l for any $l>1$ is questionable, see Remark 4 in Section 5 below. Moreover, closed formulas for the corresponding numbers, $f_l(n,p)$ and $g_l(n,p)$, of l-independent sets of cardinality p are presented in Comtet [4, pp. 21,24], namely

$$f_l(n,p)=\tbinom{n-(p-1)l}{p} \text{ and } g_l(n,p)=\tfrac{n}{n-pl}\tbinom{n-pl}{p}.$$

The formula for f_l is credited to Gergonne (1812) and Muir (1902) and that for g_l to Kaplansky (1943), but the parameter l therein is due to Comtet since independent sets only, i.e. for $l=1$, (on P_n and C_n) are counted by Kaplansky. All the four sequences of numbers and the two formulas in question, though for $l=1$ only, appeared earlier in Berge's book, see [2, pp. 31–32]. Clearly,

$$F_l(n)=\textstyle\sum_{p\geq 0} f_l(n,p) \text{ and } G_l(n)=\sum_{p\geq 0} g_l(n,p).$$

Note that for $l=0$ the four numbers are pairwise 2^n $(=F_0(n)=G_0(n))$ and $\binom{n}{p}$ $(=f_0(n,p)=g_0(n,p))$. It is known that for $l=1$ the number $F_1(n)$ is the shifted Fibonacci number F_{n+2}, as in Sloane [15], under the assumption that Fibonacci numbers F_n begin at 0, 1 (with $F_0=0$). On the other hand, $G_1(n)=L(n)$, which is the nth Lucas number (as noted in [10], but not in Comtet [4], and called a *corrected Fibonacci number* in Berge [2]), with two initial values 2 $(=L(0))$, 1. All the four (sequences of) numbers (but with distance bound l expressed in terms of $k=l+1$) are presented in [7]. Also the linear recurrence for $F(k,n)$ $(=F_l(n)$ in our notation) appears in [7].

Our main objective is the study of the numbers $G_l(n)$ via the corresponding recurrence and its characteristic roots. The known recurrence for $F_l(n)$ is recalled (with a simplified proof) because it considerably simplifies our reasoning. We show that both $F_l(n)$ and $G_l(n)$, on denoting them by $a(n)$, satisfy the same 3-term linear homogeneous recurrence

$$u(n)=u(n-1)+u(n-l-1). \tag{1}$$

In fact, $G_l(n)$ satisfies the recurrence (but only for $n\geq 2l+2$ if $l\geq 2$) and generalizes (includes) integer sequences: powers of 2 $(l=0)$ and Lucas numbers

$L(n)$ $(l = 1)$, where $L(n)$ is the sum of nth powers of the two characteristic roots (including the *golden ratio*) of the recurrence (1) with $l = 1$. Our main result is a simple proof that in the remaining case of $l \geq 2$, $G_l(n)$ is the sum of nth powers of all $l+1$ characteristic roots unless $2 \leq n \leq l$. Hence we derive both the asymptotic equivalent of $G_l(n)$ for any l and, for small l only, a simple formula in terms of nearest integer function $\lfloor \cdot \rceil$. Moreover, the related recent formula for the number of Hamilton cycles in the square of a cycle is discussed. Rational OGF for the sequence of *moments* (defined to be power sums of roots) of any polynomial is announced.

2. DISTANCE-INDEPENDENT SETS

We shall use classical setting for the problem in question. Namely, as in Comtet, the path P_n is represented by the integer interval $[n] := \{1, 2, \ldots, n\}$ for $n \geq 1$ and the cycle C_n by the cyclic group $\mathbb{Z}_n =: [\tilde{n}]$, with elements $0, 1, \ldots, n-1$, for $n \geq 1$, too.

Theorem 1. *For any nonnegative integer $l \geq 0$, $F_l(n)$ and $G_l(n)$ stand for the counts of l-independent vertex subsets on the path P_n and the cycle C_n, respectively. Then*

(2) $\quad F_l(n) = F_l(n-1) + F_l(n-l-1)$ *for $n \geq l+1$, with initial conditions*

(3) $\quad F_l(n) = n+1$ *for $n = 1, \ldots, l$, extended to $n = 0$ by $F_l(0) := 1$;*

(4) $\quad G_l(n) = G_l(n-1) + G_l(n-l-1)$ *for $n \geq 2l+2$ if $l \geq 2$, and $n \geq l+1$ if $l = 0, 1$, with initial conditions*

(5) $\quad G_l(0) := l+1$ *for $l \geq 0$,* $\quad G_l(1) := 1$ *for $l \geq 1$,*

(6) $\quad G_l(n) = n+1$ *for* $n = 2, 3, \ldots, 2l+1$ *if $l \geq 1$.*

Remark 2. $G_l(1) := 1$ counts the empty subset only. This reflects the convention that the vertex (as well as the edge) of the loop graph is self-adjacent and therefore self-dependent.

Proof. Definitions concerning $n = 0, 1$ in (3) and (5) conveniently extend validity of the corresponding recurrence (2) and (4), though (4) for $l = 0, 1$ only. For $l = 0$, all equalities are clear, also in (2) and (4). Consequently, $F_0(n) = 2^n = G_0(n)$ for any admissible n.

Therefore we assume that $l \geq 1$. Initial conditions (3) and (6) are easily seen.

Let us determine the number $F_l(n)$ of l-independent subsets X of $[n]$ for $n \geq l+1 \geq 2$. The subsets X containing n do not contain any of l integers $n-1, n-2, \ldots, n-l$, and hence there are $F_l(n-l-1)$ of the sets X; those not containing n amount to $F_l(n-1)$, whence (2) follows. Hence

$$F_l(n) = F_l(n-1) + F_l(n-l-1) \quad \text{for } n \geq l+1 \text{ (since } F_l(0) = 1). \tag{7}$$

Assume that $l = 1$. Then the recurrence (4) holds for $n = 2, 3$ due to (5) since $G_1(n) = n+1$ for $n = 2, 3$, see (6). It remains to determine the number $G_l(n)$ of l-independent subsets of $[\tilde{n}]$ for any $l \geq 1$ and $n > 2l+1$. Then the subsets which contain 0 do not contain any of $2l$ integers $1, 2, \ldots, l$ and $n-1, n-2, \ldots, n-l$, whence there are $F_l(n-2l-1)$ of the subsets. Similar statement is true if subsets contain any integer $m \in [\tilde{n}]$. Therefore subsets, Y, which contain any of l consecutive integers $n-l+1, n-l+2, \ldots, n$ $(=0)$, contain exactly one of them. Hence the class of sets Y splits into l parts of cardinality $F(n-2l-1)$ each. On the other hand, remaining l-independent subsets contain none of those l integers. Hence there are $F_l(n-l)$ of such subsets. Consequently,

$$G_l(n) = F_l(n-l) + l \cdot F_l(n-2l-1) \qquad \text{for } n \geq 2l+2,$$

where, by (7) with n replaced by $n-l$,

$$F_l(n-2l-1) = F_l(n-l) - F_l(n-l-1) \qquad \text{for } n \geq 2l+1.$$

On substituting,

$$G_l(n) = (l+1)F_l(n-l) - l \cdot F_l(n-l-1), \tag{8}$$

which holds not only for $n \geq 2l+2$ but also for $l+1 \leq n \leq 2l+1$ due to the stated initial values of G_l and F_l. Hence, first by (8) for $n \geq 2l+2$,

$$\begin{aligned}
&G_l(n-1) + G_l(n-l-1)\\
&= (l+1)(F_l(n-l-1) + F_l(n-2l-1)) - l\,(F_l(n-l-2) + F_l(n-2l-2))\\
&= (l+1)F_l(n-l) - l \cdot F_l(n-l-1) \text{(by (7))},\\
&= G_l(n) \text{ (by (8))},
\end{aligned}$$

which completes the proof. ■

3. Cyclic Strong Independence

Note that significant values of the count $G_l(n)$, namely exactly those on short n-cycles with $2 \leq n \leq l$, do not fit the recurrence (4) (in case $l \geq 2$ only). We now modify those values so that the recurrence could hold for $n \geq l+1$ with

$l \geq 0$. We next show that the modified count comprises nth power sums of the $l+1$ characteristic roots of the recurrence for all $n \geq 0$ and $l \geq 0$. Let

$$G_l^*(n) = \begin{cases} 1 & \text{for } n = 2, \ldots, l \text{ with } l \geq 2, \\ G_l(n) & \text{otherwise.} \end{cases} \tag{9}$$

Proposition 3. *The sequence $G_l^*(n)$ satisfies recurrence* (1) *for $n \geq l+1$, with initial values as above.*

Proof. In view of Theorem 1 it is enough to see the following. Assume that $l \geq 2$. Then for $l+2 \leq n \leq 2l+1$, due to (9) and (6), we have

$$G_l^*(n-1) + G_l^*(n-l-1) = G_l(n-1) + 1 = n + 1 = G_l^*(n),$$

as required. For $n = l+1$, we have $G_l^*(n) = (l+1) + 1 = G_l^*(0) + G_l^*(n-1)$, as required, too. ■

Hence and in regard to Remark 2 the following definition is motivated. A vertex subset S of a (general) graph (or a cycle) G is l-**independent* (or cyclically strong l-independent) in G if S is l-independent unless $l \geq 1$, the graph G is a short cycle, $G = \mathbb{Z}_n$ with $1 \leq n \leq l$, and $|S| > 0$. Thus only the empty set is l-*independent on a short cycle if $l \geq 1$. Therefore $G_l^*(n)$ is the count of such l-*independent subsets on the n-cycle.

For other information on sequences $G_l^*(n)$, see sequence A000204 (Lucas numbers beginning with $L(1) = 1$) in [15] and comments therein on generalizations.

4. Recurrence-OGF and Co-reciprocal Polynomials

It is a good opportunity now to show how the notion of mutually reciprocal polynomials simplifies the procedure which leads from a given recurrence which is LinHomConst (*linear homogeneous with constant coefficients*) and complete (i.e., with initial values) to the corresponding OGF (and/or *vice versa*). Let

$$g(z) = \sum_{j=0}^{r} c_j z^j \in \mathbb{C}[z] \quad \text{with constant term } c_0 \neq 0 \tag{10}$$

be a complex polynomial of positive degree r and with nonzero roots only, possibly multiple. Then we say that the polynomial $f(z) := z^r\, g(z^{-1})$ is *co-reciprocal for* (or the *reciprocal polynomial of*) $g(z)$, and that polynomials $f(z)$ and $g(z)$ are *co-reciprocal* (or mutually reciprocal). These notions are not well-established in literature yet; e.g., 'reciprocal' in Andrews' [1] means 'self-reciprocal'. A self-reciprocal polynomial is invariant under reciprocation of the set of roots and so invariant is the set of roots itself. By the way, the minimal polynomial of the golden ratio, $h_1(x) := x^2 - x - 1$ (see (13) with $l = 1$), is not so invariant, but the reciprocation of its roots results in negating both of roots.

A polynomial $f(x) \in \mathbb{C}[x]$ is said to be *characteristic* or *in characteristic form* if $f(x)$ is monic, of positive degree, say r, with nonzero roots, and with coefficient at x^{r-j} denoted by a_j:

$$f(x) = \sum_{j=0}^{r} a_j x^{r-j} \quad \text{with positive } r,\ a_r \neq 0 \text{ and } a_0 = 1. \tag{11}$$

A polynomial $Q(x) = \sum_{j=0}^{r} c_j z^j$ is said to be *co-characteristic* or *in co-characteristic form* if $Q(x)$ is the reciprocal polynomial of a characteristic polynomial, that is, the co-reciprocal polynomial $x^{\deg Q(x)} Q(1/x)$ is a characteristic polynomial. Then the constant term of $Q(x)$, $c_0 = 1$. We say that a recurrence is a *characteristic recurrence* or *is in the characteristic form* if the recurrence is LinHomConst, with highest argument n, the highest coefficient, say, $c_0 = 1$, and is as in (12) below.

Note that given a characteristic (order-r) recurrence (12), substitutions $u(n-j) \leftarrow x^j$ in the left-hand side therein produce a polynomial, say $Q(x)$, in co-characteristic form, and reciprocation of $Q(x)$ gives a characteristic (degree-r) polynomial, $f(x)$, which is characteristic polynomial of the recurrence, too. Therefore $Q(x)$ is said to be the *co-characteristic polynomial of* the recurrence. On the other hand, $f(x)$ is obtained straightforwardly by the substitutions $u(n-j) \leftarrow x^{r-j}$ (instead of the former ones) provided that r is the order of the recurrence. Going backwards from $f(x)$ we arrive at the corresponding characteristic recurrence with $f(x)$ as a characteristic polynomial of the recurrence. Passing on to the intermediate stage, the polynomial $Q(x)$, simplifies hand calculations.

In this section it is assumed that a count/sequence $u(n)$ is defined for $n \geq n_1 \geq 0$ where n_1 is an *initial argument.* Then $u(j) := 0$ for all integers $j < n_1$.

procedure LinHomConstR-OGF.
Input [A complete characteristic recurrence of order r]:

$$\sum_{j=0}^{r} c_j u(n-j) = 0 \quad \text{for } n \geq k \text{ where a certain } k \geq r, \tag{12}$$

with at least r initial values (of which last r ones are initial for the recurrence):

$$u(n_1),\ u(n_1+1), \ldots, u(k-r), \ldots, u(k-1)$$

for some $n_1 \leq k-r$, provided that c_j are constant coefficients, $c_0 = 1$ and $c_r \neq 0$.

Output [The OGF (possibly reducible), say]:

$\phi(x) = \frac{P(x)}{Q(x)}$, where $Q(x)$ is the co-characteristic polynomial of the OGF,

$Q(x) = \sum_{j=0}^{r} c_j x^j$, with coefficients c_j taken from the recurrence,

$P(x) := Q(x) \cdot \phi(x) = Q(x) \sum_{j=n_1}^{k-1} u(j) x^j \bmod x^k$, a polynomial of degree less than k.

Note that reducing the OGF (if possible) leads to an equivalent simpler recurrence, by using what follows.

The following converse procedure includes a recursive generation, see Stanley [16], of initial values of the count.

procedure OGF-LinHomRec.
Input [A rational function $\Phi(x) := P(x)/Q(x)$ which is the irreducible OGF for $u(n)$ where $n \geq n_1 \geq 0$. Let $r = \deg Q(x)$, $Q(x) = \sum_{j=0}^{r} c_j x^j$ with $c_0 = Q(0) = 1$, as above. Let b_j be coefficients of the numerator polynomial $P(x)$, $P(x) = \sum_{j=0}^{s} b_j x^j$ with deg $P(x) = s$.]
Output [The recurrence (LinHomConst and of the smallest possible order r) is obtainable from the co-characteristic polynomial $Q(x)$:

$$u(n) + \sum_{j=1}^{r} c_j\, u(n-j) = 0 \quad \text{for } n \geq \max(r + n_1, 1 + s).$$

The resulting recurrence is valid for $n \geq \max\left(\deg Q(x) + n_1, 1 + \deg P(x)\right)$. Initial (and any) terms $u(m)$ of the sequence $u(n)$ can be found recursively on equating coefficients of x^m in the identity

$$Q(x) \cdot \sum_{m \geq 0} u(m) x^m = P(x).$$

Consequently, values of $u(n)$ (inclusive of the initial ones, for $n_1 \leq n \leq \max(r + n_1 - 1, s)$, are found recursively for consecutive $m = 0, 1, \ldots$ from

$$u(m) + \sum_{j=1}^{\min(m,r)} c_j\, u(m-j) = b_m$$

where $b_m = 0$ for $m < n_1$ and for $m > s = \deg P(x)$.]

5. OGF and Power Sums of Roots

The recurrences (2), (4), and (1) are LinHomConst (linear homogeneous, with constant coefficients) and of order $l + 1$ and are essentially the same. Their characteristic polynomial, say $h_l(x)$, for $x = z \in \mathbb{C}$, is

$$h_l(z) = z^{l+1} - z^l - 1, \tag{13}$$

with all characteristic roots being nonzero.

We now find an OGF, say $\Phi(x) = \Phi_F(x), \Phi_G(x), \Phi_G^*(x)$, for each of the corresponding counts $F_l(n)$, $G_l(n)$, $G_l^*(n)$. Then $\Phi(x) = \frac{P(x)}{Q(x)}$ where $Q(x)$ is the co-characteristic polynomial, that is,

$$Q(x) = x^{l+1}h_l(1/x) = 1 - x - x^{l+1},$$

and the numerator $P(x) = Q(x)\,\Phi(x)$ depends on the respective initial values presented in Theorem 1 and Proposition 3. Thus we get

$$\Phi_F(x) := \sum_{n\geq 0} F_l(n)x^n = \frac{1 + x + \cdots + x^l}{1 - x - x^{l+1}}, \tag{14}$$

$$\Phi_G^*(x) := \sum_{n\geq 0} G_l^*(n)x^n = \frac{l + 1 - lx}{1 - x - x^{l+1}}, \tag{15}$$

$$\Phi_G(x) := \sum_{n\geq 0} G_l(n)x^n = \Phi_G^*(x) + \sum_{n=2}^{l} nx^n. \tag{16}$$

Remark 4. In Comtet's valuable book [4, p. 46] the OGF for the sequence $G(n, l)$, namely, $(t + (l + 1)t^{l+1})(1 - t - t^{l+1})^{-1}$ which equals $\Phi_G^*(t) - (l + 1)$, should be replaced by

$$\Phi_G(t) - l - 1 = (t + (l + 1)t^{l+1})(1 - t - t^{l+1})^{-1} + \textstyle\sum_{n=2}^{l} nt^n.$$

Proposition 5. *The characteristic roots, roots of $h_l(z)$, are nonzero and simple.*

Proof. The constant term of $h_l(z)$ is nonzero and the only nonzero root of the derivative $h_l'(z) = (l + 1)z^{l-1}(z - l/(l + 1))$ does not nullify $h_l(z)$. ∎

Let $z_1, z_2, \ldots, z_{l+1}$ be all roots of the characteristic polynomial $h_l(z)$. Define

$$\sigma_n(l) = \sum_{j=1}^{l+1} {z_j}^n, \tag{17}$$

which is the nth power sum of characteristic roots.

Theorem 6. *For integers $l \geq 0$ and $n \geq 1$, each count $G_l^*(n)$ of l-*independent subsets of the cycle $\mathbb{Z}_n$ equals the nth power sum of roots of the characteristic polynomial, i.e., $G_l^*(n) = \sigma_n(l)$. Additionally, for $n = 0$, $\sigma_0(l) = l + 1 =: G_l^*(0)$.*

Proof. Let $P(x) = l + 1 - lx$, $Q(x) = 1 - x - x^{l+1}$, and let t_j, $j = 1, \ldots, l + 1$, be all roots of $Q(x)$. Hence, by (15), the OGF for $G_l^*(n)$ is $\Phi_G^*(x) = P(x)/Q(x)$. Moreover, the reciprocals $1/t_j$ are characteristic roots z_j. Due to Proposition 5, we use the following standard expansion into partial fractions,

$$\Phi_G^*(x) = \sum_{j=1}^{l+1} \frac{P(t_j)}{Q'(t_j)} \cdot \frac{1}{x - t_j} = \sum_{j=1}^{l+1} \frac{P(t_j)}{Q'(t_j)} \cdot \frac{1}{-t_j \cdot (1 - xz_j)}$$
$$= \sum_{n=0}^{\infty} x^n \sum_j c_j \cdot (z_j)^n$$

where

$$c_j := \frac{P(t_j)}{-t_j\, Q'(t_j)} = \frac{1 + l \cdot (1 - t_j)}{(t_j + {t_j}^{l+1}) + l{t_j}^{l+1}} = 1, \quad j = 1, \ldots, l+1,$$

because $Q(t_j) = 0$, i.e., ${t_j}^{l+1} = 1 - t_j$ for each root t_j. Thus $G_l^*(n) = [x^n]\, \Phi_G^*(x) = \sigma_n(l)$, which completes the proof. ∎

Corollary 7. *The count $G_l(n)$ of l-independent subsets of the cycle C_n is the nth power sum $\sigma_n(l)$, i.e., $G_l(n) = G_l^*(n)$, unless $l \geq 2$ and $2 \leq n \leq l$.*

This corollary gives rise to closed formulas for $G_l(n)$ if l is small, $l \leq 4$. The formulas are known for $l = 0, 1$ and $n \geq 0$. Namely,

$G_0(n) = 2^n$, the number of all subsets of an n-set, and

$G_1(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = L(n)$, the nth Lucas number.

For $l = 2, 3, 4$ the formulas for roots due to Cardano-del Ferro-Tartaglia (l=2,4; since $h_4(z) = (z^3 - z - 1)(z^2 - z + 1)$) on one hand and Ferrari ($l = 3$) on the other hand and the de Moivre formula are helpful, see the result in [12, formula (11)] for $G_2(n)$ with $n > 2$ only.

6. Main Result Via Newton's Formulas

Given a degree-r characteristic polynomial $f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$, its nth moment, S_n, being the nth power sum of roots of $f(x)$, satisfies the order-r recurrence corresponding to $f(x)$, namely, $S_n + a_1 S_{n-1} + \cdots + a_r S_{n-r} = 0$ for each $n \geq r$. It is so because the general solution includes S_n as a particular solution. Initial values S_k for $k = 0, 1, \ldots, r-1$ ($S_0 = r$, $S_1 = -a_1$) can be obtained for $k \geq 1$ recursively from the following Newton formulas: $-na_n = S_n + a_1 S_{n-1} + \cdots + a_{n-1} S_1$ where $n = 1, 2 \ldots,$ with $a_k = 0$ for $k > \deg f(x) = r$.

***Alternative proof of Theorem* 6.** The moment $\sigma_n(l)$ and the count $G_l^*(n)$ satisfy the same recurrence with characteristic polynomial $h_l(z)$ of degree $r := l + 1$ and with only two nonzero coefficients a_j, namely $a_1 = -1 = a_r$. Hence, due to

Newton's formulas, the r initial values of $\sigma_n(l)$, for $n = 0, 1, \dots, r-1 = l$, are $l+1, 1, \dots, 1$, and these are initial values of $G_l^*(n)$ due to (9) and (5). ■

For the case $l = 2$ only, a similar proof in [12, Lemma 10 and Remark 3.2] uses the Viète formulas (instead of Newton's).

7. Asymptotics

The following celebrated result is of basic importance in asymptotic analysis of combinatorial counting sequences, see [5].

Theorem 8 (Pringsheim's Theorem). *Let $f(z)$ be a power series analytic at the origin $z = 0$, with nonnegative coefficients and with finite radius of convergence R. Then the point $z = R$ is a dominant pole* (*of least magnitude*) *of the function $f(z)$.*

A polynomial $Q(x) \in \mathbb{Z}[x]$ is called a *multi-composition polynomial* if $Q(x) = 1 - \sum_{j=1}^{\nu} m_j x^{a_j}$ where all $\nu \geq 2$, m_js and $1 \leq a_1 < a_2 < \cdots < a_\nu$ are natural numbers of which a_is are relatively prime, $\gcd\{a_1, \dots, a_\nu\} = 1$. Then the co-reciprocal polynomial of $Q(x)$, say $h(x) := x^{a_\nu} Q(1/x)$, is the characteristic polynomial of a 'compositional' recurrence (for a 'compositional' count $u(n)$), $u(n) = \sum_{j=1}^{\nu} m_j u(n - a_j)$, generated by $Q(x)$ via the above LinHomConstR-OGF. Elementary reasoning gives the following result.

Lemma 9 (Skupień [13]). *Any multi-composition polynomial has a simple positive root, τ, which is smaller than the minimum magnitude among remaining roots, if any, and $\tau < 1$.*

Corollary 10. *If $u(n)$ is a compositional count with nontrivial natural initial terms and λ is a characteristic root of largest magnitude then λ is a simple positive root, $\lambda > 1$, and $u(n) = \Theta(\lambda^n)$, the exact asymptotic order of growth.*

This result applies to our counts due to Theorems 1 and 6, and Corollary 7. Hence,

Proposition 11. *If $\lambda(l)$ stands for the dominant root of the characteristic polynomial $h_l(z) = z^{l+1} - z^l - 1$ then $F_l(n) = \Theta(\lambda(l)^n)$, both $G_l^*(n)$, $G_l(n) \sim \lambda(l)^n$, and $G_l(n) = \lfloor \lambda(l)^n \rceil$ for $n \geq 2$ if $l = 1$, $n \geq 6$ if $l = 2$, and $n \geq 22$ if $l = 3$.*

Remark 12. It can be seen, for $l \leq 3$ only, that magnitudes of remaining characteristic roots are less than 1 and therefore nearest integer function is applicable.

Moreover, the initial $\lambda(l)$s are important in the subclass of algebraic integers which comprises Pisot numbers [3, 17]: golden ratio ($l = 1$) and next the 4th ($l = 2$), 2nd ($l = 3$), and 1st ($l = 4$) of the smallest Pisot numbers, the smallest being called the plastic number, and its minimal polynomial is the degree-3 factor of $h_4(z)$, $h_4(z) = (z^3 - z - 1)(z^2 - z + 1)$.

l	1	2	3	4
$\lambda(l)$	1.61803^+	1.46557^+	1.38028^-	1.32472^-

Table 1. Pisot numbers.

8. Hamilton Cycles in a Squared Cycle

Investigations into distance-independent circular sets, presented above, have been inspired by the problem of counting Hamilton cycles (i.e., connected 2-factors) in the square of a cycle [11, 12]. Recall that the square of the n-cycle C_n, in symbols ${C_n}^2$, is the graph C_n together with all n shortest chords (all chords of length two). One of the main results in [12] is the following closed formula which gives the number, $h({C_n}^2)$, of Hamilton cycles in ${C_n}^2$ for $n \geq 5$ in terms of the number, $G_2(n) = G_2^*(n)$, of 2-independent sets on the n-cycle. Namely, if

$$h_n := G_2^*(n) + 2\lceil n/2 \rceil, \tag{18}$$

then $h({C_n}^2) = h_n$ for $n \geq 5$.

n	0	1	2	3	4	5	6	7	8	9	10
$G_2^*(n)$	3	1	1	4	5	6	10	15	21	31	46
h_n	3	3	3	8	9	12	16	23	29	41	56

Table 2

Values of the extended h_n such that (18) holds for arguments $n \geq 0$ are presented in Table 2. Note that the result $h_n = h({C_n}^2)$ does not extend to $n = 4$ because $h({C_4}^2) = h(K_4) = 3 \neq h_4 = 9$. (In general, $h(K_n) = \lfloor (n-1)!/2 \rfloor$. That is why $h_5 = h(K_5) = 12$.)

Proposition 13. *For the extended sequence h_n,* OGF: $\frac{3-2x}{1-x-x^3} + \frac{x}{(1-x)^2} + \frac{x}{1-x^2}$, $h_n = 2h_{n-1} - h_{n-3} - h_{n-5} + h_{n-6}$ *for $n \geq 6$, with initial conditions included in Table* 2.

Proof. Due to (15) with $l = 2$, it is easily seen that the above OGF is the sum of three OGFs one each for three summands in $h_n = G_2^*(n) + n + (1 - (-1)^n)/2$. Therefore l.c.m., say $Q(x)$, of denominators of the three partial OGFs is the denominator of the above main OGF,

$$Q(x) = (1 - x - x^3)(1 - x^2)(1 + x) = 1 - 2x + x^3 + x^5 - x^6.$$

Hence the above Procedure OGF-LinHomRec gives the stated recurrence (of order six). ■

9. Concluding Remarks

Inspired by the above study is the following recent theorem related to very old Girard-Newton-Waring's formulas for moments (power sums of roots) of a polynomial. The theorem seems to be unpublished yet, and this opinion agrees with comments in the introductory part of [8].

Theorem 14 [14]. *Let $f(z)$ be a polynomial of degree $r > 0$ and with nonzero roots only, whereas $g(z)$ the reciprocal polynomial of $f(z)$. Let $S_n(f)$ and $S_n(g)$ be the nth moments of f and g, resp. Then the OGF for moments of $f(x)$ is*

$$\frac{rg(z) - zg'(z)}{g(z)} = \sum_{n=0}^{\infty} S_n(f) z^n$$

and OGF for moments of $g(x)$ results on interchanging symbols $f \leftrightarrow g$ on both sides of the formula.

Procedure RootsPowerSums.
Input [$h(z)$, a polynomial with nonzero roots].
Output [The sequence of power sums of roots of $h(z)$, represented by the rational OGF $\frac{P(z)}{Q(z)}$ or by LinHomRec obtainable by Procedure OGF-LinHomRec, see Section 4].
Action
$Q(z) := z^{\deg h(z)} h(1/z)$, the co-reciprocal polynomial of $h(z)$;
$P(z) := -z\,Q'(z) \mod Q(z)$ so that $P(0) = \deg h(z)$;
Procedure OGF-LinHomRec;
STOP.

Another byproduct (which is useful when dealing with LinHomConst recurrences) is the notion of mutually reciprocal polynomials.

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